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FRANK MORLEY

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A. COHEN, CHARLOTTE A. SCOTT

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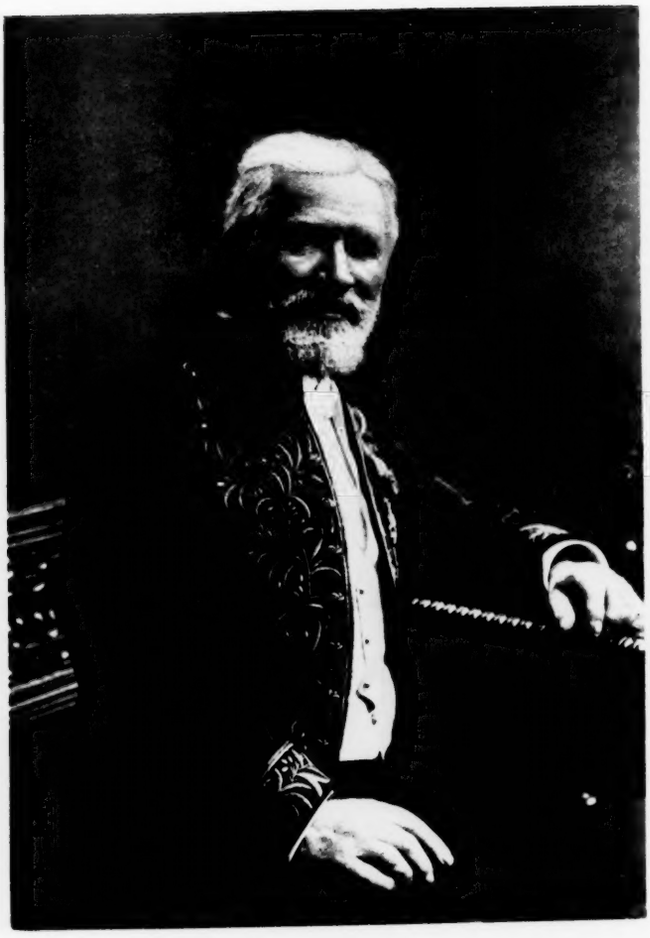
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C. Jordan

Groups Containing a Given Number of Operators Whose Orders Are Powers of the Same Prime Number.

By G. A. Miller.

§ 1. *Introduction.*

Let G be any group whose order g is divisible by p^m but not by p^{m+1}, p being a prime number. A necessary and sufficient condition that G contains only one subgroup of order p^m is that the number of its operators whose orders are powers of p is exactly p^m. According to a theorem proved by Frobenius, the number of the distinct operators whose orders are powers of p, which are contained in G, is always divisible by p^m.* It is, however, not possible to construct a group in which this number is an arbitrary multiple of p^m. In fact, this number is at least p^{m+1} whenever it exceeds p^m, and it is at least $2\,p^{m+1} - p^m$ whenever it exceeds p^{m+1}, in accord with the theorems which we shall establish in what follows.

Let $H_1, H_2, \ldots, H_\lambda$ represent the totality of the subgroups of order p^m which are contained in G. It may happen that these subgroups have the property that all the operators which are common to one pair of them are common to every pair. On the other hand, it may happen that the operators which are common to one pair of such Sylow subgroups are not common to every pair of them. In what follows we shall be mainly concerned with the latter of these two possible cases, and we shall prove, in particular, that the number of these Sylow subgroups in the latter case must be at least $(p+1)^2$. That is, we shall prove that every possible pair of subgroups of order p^m contained in G has the same common operators whenever the number of these subgroups is less than $(p+1)^2$.

Suppose that H_1, H_2 have operators in common which are not common to every pair of the subgroups $H_1, H_2, \ldots, H_\lambda$. Let p^s be the largest number of common operators which can be found in any one of the possible pairs of these subgroups. All these subgroups may then be divided into sets characterized by the fact that each set is composed of all those which have the same

* Frobenius, *Berlin Sitzungsberichte*, 1895, p. 984.

p^α operators in common. It is evident that each of these sets is composed of the same number $1+kp$ of subgroups, since all the Sylow subgroups are conjugate under G. Moreover, no two of these sets can have more than one common subgroup of order p^m, since no pair of these Sylow subgroups can have more than p^α common operators.

The case which will interest us mostly in what follows is the one in which $k=1$. We may evidently assume that the two subgroups H_1, H_2 have p^α common operators. It also results that the subgroup K_1 constituted by these common operators is transformed into itself by exactly $p^{\alpha+1}$ operators of H_1 and by the same number of operators of each of the other p subgroups of order p^m which have K_1 in common. These $p^{\alpha+1}$ operators of any one of these subgroups clearly transform the other p subgroups of the same set among themselves.

Let $K_1, K_2, \ldots, K_\gamma$ be the conjugates of K_1 under H_1. It is clear that H_1 will transform the set of $p+1$ subgroups determined by H_1, H_2 into $\gamma = p^{m-\alpha-1}$ such sets, which have H_1 in common, and hence no two of these γ sets have any other subgroup of order p^m in common. If S is an operator of any one of these sets, which has the property that it transforms the common operators of this set into themselves but does not belong to H_1, then S can not transform the common operators of any other of these sets into themselves. In fact, if we assume that S transforms into themselves the common operators of two of these sets, it results that S and these common operators must generate a group which has more than p^α operators in common with H_1. Hence it follows that if we add the operators of each of these sets which do not belong to H_1 but transform the common operators of the set into themselves, we obtain

$$p^{m-\alpha}(p^{\alpha+1}-p^\alpha) = p^{m+1}-p^m$$

distinct operators.

In particular, when $\alpha = m-1$ it results that $\gamma = 1$, and all the operators of the set to which K_1 belongs must transform K_1 into itself. The total number of the distinct operators of the set must therefore be p^{m+1}. As a special case of this we may observe that when G contains exactly $p+1$ subgroups of order p^m, the number of its operators whose orders are powers of p must be exactly p^{m+1}, as is also directly evident. If $\alpha < m-1$, each set of $p+1$ subgroups evidently contains more than p^{m+1} distinct operators, and if $\alpha = m-1$ and there is more than one such set, G must again contain more than p^{m+1} operators whose orders are powers of p. Hence we have, as a special case of the given developments, that *a necessary and sufficient condition that a group whose order is divisible by p^m but not by p^{m+1} contains exactly $p+1$ subgroups of order p^m is that it contains exactly p^{m+1} operators whose orders are powers of p.*

§ 2. *Groups in Which the Common Operators of Some One Pair of Sylow Subgroups Are Common to Every Pair of Such Subgroups.*

If some two Sylow subgroups H_1, H_2 of order p^m have operators in common which are not common to every pair of subgroups of order p^m contained in G, there are evidently two cases to be considered. In the first of these cases every pair of Sylow subgroups of order p^m contained in G has the same number of common operators but the operators which are common to some one pair are not common to every pair. In the second case, G contains some pair of subgroups of order p^m which has more operators in common than some other pair.

We shall first prove that if every pair of subgroups of order p^m in G has p^{m-1} operators in common, each of these pairs must contain the same p^{m-1} operators. To prove this theorem it will be convenient to consider the group of transformations T of the Sylow subgroups of G. Since every pair of subgroups of order p^m in G has p^{m-1} operators in common, all the transitive constituents of the Sylow subgroups of order p^m in T are of degree p. The subgroup T_1, which is composed of all of the substitutions of T which omit a given letter, must therefore contain an invariant subgroup of order p^{m_1} whose transitive constituents are all of degree p, and the degree of T_1 must be kp, $kp+1$ being the degree of T.*

If all the substitutions of T_1, excluding the identity, were of degree kp, all the Sylow subgroups of order p^m in G would involve the same p^{m-1} operators. Hence we may assume that the invariant abelian subgroup of order p^{m_1} in T_1 involves a subgroup of a degree which is less than kp and of order p^{m_1-1}, $m_1 > 1$. This subgroup must be contained invariantly in at least $p+1$ subgroups of order p^{m_1} contained in T. As these groups of order p^{m_1} are abelian, the group generated by them must have a central which includes the given subgroup of order p^{m_1-1}. As this group would have an abelian constituent whose order is a power of p, and another constituent of degree $1+kp$ involving non-invariant subgroups whose orders are powers of p, it evidently could not be constructed, and hence we have proved the theorem: *If every possible pair of Sylow subgroups of order p^m in a given group has p^{m-1} operators in common, then all of these subgroups must have the same p^{m-1} operators in common.*

From this theorem it results directly that G must contain at least one pair of subgroups of order p^m which has less than p^{m-1} common operators whenever all the pairs of subgroups of order p^m in G do not have the same largest common

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* American Journal of Mathematics, Vol. XXXII (1910), p. 299.

subgroup. The number of these subgroups in such a G must clearly exceed p^2. If this were not the case, these subgroups would be transformed according to a transitive substitution group T in which the subgroup T_1 would contain an invariant subgroup of order p^{m_1} whose transitive constituents would all be of degree p. As this subgroup would contain substitutions which would omit some of the letters of T_1, it results in exactly the same manner as in the preceding paragraph that such a T could not exist. This proves the lemma: *If any group contains less than p^2+1 Sylow subgroups of order p^m, every pair of these subgroups must contain the same common operators.*

When the Sylow subgroups of order p^m in G do not all contain the same largest common subgroup, they must occur in sets of $1+kp$ such that each set is composed of all those Sylow subgroups of order p^m which have the same largest subgroup in common. The order of the largest common subgroup for one set must be the same as the order of this subgroup in any other set, and k must have the same value for each of these sets since all the Sylow subgroups of G are conjugate under G. Any two of these sets have at most one subgroup of order p^m in common, for if two such sets would have two such subgroups in common these two subgroups would have more common operators than the order of the subgroups composed of all the operators which are common to such a set. As such a set can not be transformed into itself by any Sylow subgroup of order p^m unless this subgroup belongs to the set, it results that the number of these sets can not be less than $p+1$. Hence it results that the number of Sylow subgroups of order p^m in G can not be less than the sum of

$$p+1, \quad p, \quad p-1, \quad \ldots, \quad 1=\tfrac{1}{2}(p+1)(p+2)$$

unless all of these Sylow subgroups have the same largest subgroup in common. In the special case when $p=2$, this gives a larger limit than the one expressed in the preceding theorem, while it gives the same limit when $p=3$.

It is not difficult to prove that the lower limit for the number of Sylow subgroups, which was found in the preceding paragraph, is always too small. To prove this, it is only necessary to consider the cases when $p=2$ or 3, since the other cases are included in the theorem stated above. The case when $p=2$ may readily be disposed of, since the given formula gives an even number six, while the number of these subgroups is odd. In fact, the number of these subgroups could not be seven, since the groups of degree 7 which are groups of transformations of Sylow subgroups are contained in the matacyclic group of this degree, and hence the Sylow subgroup of order 2^{m_1} in T_1 is of order 2 and of degree 6. The case when $p=3$ is included in the more general lemma,

which we proceed to prove; viz., the number of Sylow subgroups of order p^m must exceed p^2+1 whenever some two of these subgroups have operators in common which are not common to all of them.

To prove this generalization of the lemma stated above, we shall prove that we arrive at an absurdity by assuming that G contains exactly p^2+1 Sylow subgroups of order p^m and that some two H_1, H_2 of these subgroups have operators in common which are not contained in all of them. We consider again the group of transformations T of these Sylow subgroups and observe that the only case which requires consideration is the one in which the invariant Sylow subgroup of order p^{m_1} in T_1 is transitive, T_1 being defined as before. This subgroup of order p^{m_1} must be non-regular, and hence it must contain an intransitive subgroup of order p^{m_1-2}, whose degree is less than p^2. This subgroup must occur in at least $p+1$ conjugates of T_1, and hence it is invariant in at least $p+1$ subgroups of order p^{m_1-1} contained in G. Just as in the cases considered above, it results that these subgroups of order p^{m_1-1} would generate a group which could not occur in T, and hence it results that *if a group contains no more than* p^2+1 *Sylow subgroups of order* p^m, *every pair of these subgroups must contain the same common operators.* This lemma will be useful to establish a more general result in what follows.

We are now in position to prove the theorem that the number of Sylow subgroups of order p^m in G must be at least equal to $(p+1)^2$, whenever the operators common to some two of these subgroups are not necessarily common to all of them. To prove this theorem by means of the two lemmas which have been established, it is only necessary to prove that the number of these subgroups could not be p^2+p+1. If this number were p^2+p+1, T_1 would have two transitive constituents of degrees p^2 and p respectively. If a transitive subgroup of the former would correspond to the identity of the latter, this transitive subgroup would be invariant under $p+1$ conjugates of T_1. These conjugates would generate an intransitive group with two transitive constituents of degrees p^2 and $p+1$ respectively. The latter would clearly be at least doubly transitive. Hence T would be imprimitive, since T_1 could not be maximal, and a system of imprimitivity would involve $p+1$ letters. This is impossible, since p^2+p+1 is not divisible by $p+1$.

If the subgroup which would correspond to the identity of the constituent of degree p in T_1 were intransitive, this would be the only subgroup of this order and degree contained in T_1, since the remaining substitutions of the constituent of degree p^2 would involve all the letters of this constituent. Hence the given subgroup would again be invariant under $p+1$ conjugates of T_1,

and T would be imprimitive. Just as in the preceding case, there would be a system of imprimitivity of degree $p+1$, while this is not a divisor of the degree of T. Hence we have established the theorem: *If any group contains less than $(p+1)^2$ Sylow subgroups of order p^m, then all the operators which are common to some two of these subgroups are necessarily common to all of them.*

To give an instance where G contains exactly $(p+1)^2$ Sylow subgroups of order p^m which are such that the operators common to some two of these subgroups are not common to all of them, we may consider the Sylow subgroups of order 4 in the direct product of two non-cylic groups of order 6. This direct product clearly contains nine subgroups of order 4; and one of these subgroups has two operators in common with each of four others, while it has only the identity in common with each of the remaining four. The total number of distinct operators which are contained in these nine subgroups is clearly sixteen. The transitive group of degree 6 and of order 72 furnishes another instance of a group which contains exactly $(p+1)^2$ Sylow subgroups of order p^m having the property that some two of these subgroups have operators in common which are not common to all of them.

§ 3. *Groups Which Contain a Small Number of Operators Whose Orders Are Powers of a Given Prime Number.*

We shall again assume that the order of G is divisible by p^m but not by p^{m+1}, p being a prime. For every pair of values for p and m, m being any positive integer, there is evidently at least one group G which contains exactly $p+1$ Sylow subgroups of order p^m. It has been observed that a necessary and sufficient condition that G contains exactly $p+1$ Sylow subgroups of order p^m is that the number of its operators whose orders are powers of p is exactly p^{m+1}. We proceed to find a lower limit for the number of such operators in G when G contains more than $p+1$ such subgroups.

If all the operators which are common to two subgroups of order p^m in G are necessarily common to all of its subgroups of this order, it is evident that the number of such operators contained in G can not be less than

$$(2p+1)(p^m - p^{m-1}) + p^{m-1} = 2p^{m+1} - p^m.$$

We proceed to consider the cases when G contains at least two Sylow subgroups H_1, H_2 of order p^m whose common operators do not occur in all of its other Sylow subgroups of this order, and we begin with the case when G contains at least one Sylow subgroup of order p^m which has p^{m-1} operators in

common with H_1. Hence G contains at least $p+1$ such subgroups. If it had more than $p+1$ such subgroups, they would contain at least $2p^{m+1}-p^m$ distinct operators. Hence we shall assume that G contains exactly $p+1$ Sylow subgroups having p^{m-1} operators in common with H_1.

The Sylow subgroups of order p^m contained in G must therefore occur in sets of $p+1$ such that each set is characterized by the fact that it is composed of all these subgroups which have a certain set of p^{m-1} operators in common. Two such sets have at most one subgroup in common. If they have no subgroup in common, the number of these sets is at least $p+1$, since G contains at least $(p+1)^2$ subgroups of order p^m. If they have a common subgroup, the number of these sets must be at least $2(p+1)$, since each of these subgroups must occur at least twice among the sets. We shall first consider the latter case.

The $p+1$ subgroups of order p^m which constitute such a set must generate a group K_1 which involves exactly p^{m+1} operators whose orders are powers of p in accord with the theorem proved above. Let K_2 be a conjugate of K_1 under G and suppose that H_1 is common to K_1 and K_2. If S is any operator of K_1 which is not included in H_1, then S can not occur in K_2. For, if S would also occur in K_2, it would transform into themselves two subgroups of order p^{m-1} contained in H_1, and hence it would transform H_1 into itself. It would therefore be contained in H_1, but this is contrary to the hypothesis. Hence K_2 contains exactly $p(p^m-p^{m-1})=p^{m+1}-p^m$ operators which are not found in K_1, and the total number of distinct operators in K_1 and K_2 is $2p^{m+1}-p^m$.

If K_3 is another conjugate of K_1, but is not identical with K_1 or K_2, it can not have more than p^m operators in common with either one of the two subgroups K_1, K_2. As it has the identity in common with both of these, and as it involves p^{m+1} distinct operators, it results that K_3 must involve at least one operator which is distinct from the operators contained in K_1 and K_2. Hence it follows that G must contain at least $2p^{m+1}-p^m+p^m=2p^{m+1}$ operators whose orders are powers of p whenever it satisfies the two conditions that the operators which are common to some two of its Sylow subgroups of order p^m are not common to all of these subgroups and that a Sylow subgroup is found in more than one set of $p+1$ subgroups having p^{m-1} common operators. This lower limit is exactly attained in the direct product of two symmetric groups of order 6.

If all the subgroups of order p^m in G are contained in sets of $p+1$, which have p^{m-1} operators in common, and if no such subgroup is common to two such sets, then each of these sets generates a subgroup which has one and only one

transitive constituent of degree p. Let K_1, K_2 represent two such subgroups. As the transitive constituents of degree $p+1$ in K_1, K_2 can not have any letter in common whenever G is the group of transformations of its Sylow subgroups of order p^m, it results that a subgroup of order p^m in K_1 can not have more than p^{m-2} operators in common with K_2, since these common operators would transform the transitive constituents of degrees $p+1$ in K_1 and K_2 into themselves, and hence they would omit a letter of a transitive constituent of degree p^α, $\alpha > 1$, in this Sylow subgroup of order p^m. Hence the $p+1$ Sylow subgroups of order p^m in K_1 can not have more than $(p+1)\ p^{m-2}$ operators in common with K_2. That is, G would again have at least $2\,p^{m+1}$ distinct operators whose orders are powers of p, since the number of these operators is divisible by p^m and must exceed $2\,p^m - p^m$. Since this result is true when G is the group of transformations of its own Sylow subgroups of order p^m, it evidently holds true in all cases.

We have now considered the possible cases when both of the following conditions are satisfied: G contains at least two Sylow subgroups of order p^m whose common operators are not common to all of its subgroups of this order, and there are at least two subgroups of order p^m in G which have p^{m-1} common operators. We proceed to consider the cases when the former but not the latter of these two conditions is fulfilled. Two of the Sylow subgroups of order p^m in G can therefore not have more than p^{m-2} common operators. We may divide all the subgroups of order p^m in G into sets which are such that each set is composed of all these subgroups which have a maximal number p^α of common operators.

Let H_1 be one of such a set of subgroups. If its subgroup of order p^α, which is common to all the subgroups of the set, is invariant under $p^{\alpha+\beta}$ operators of H_1, the set will be composed of $p^\beta+1$ subgroups of order p^m and these will involve $(p^\beta+1)\ (p^m - p^\alpha) + p^\alpha = p^{m+\beta} + p^m - p^{\alpha+\beta}$ distinct operators. This is at least as large as $2\,p^{m+1}$ when $\beta > 1$. Hence we may restrict our attention to the case when each of the given sets involves exactly $p+1$ subgroups of order p^m having p^α common operators, and hence we must assume $\beta = 1$ for all the possible sets.

Consider two of these sets of $p+1$ subgroups which have H_1 in common, and which have been so chosen that the two subgroups of order p^α, which are respectively common to all the Sylow subgroups of each of the two separate sets, have $p^{\alpha-1}$ common operators. Let S be any operator which occurs in each of these two sets but is not contained in H_1. The two subgroups of order p^m to which S belongs must therefore have S and the given $p^{\alpha-1}$ operators in

common. Hence the subgroup formed by these $p^{\alpha-1}$ operators must be invariant under S, and the group generated by S and this subgroup is of order p^{α}. This proves that all the operators of one of these subgroups of order p^m, besides H_1, which are common to both of the two sets under consideration, must form a group.

The order of this group is less than p^m; for, if it were p^m this Sylow group would involve an invariant subgroup of order p^{α} which would also appear in another Sylow subgroup of this order, since the given subgroup of order $p^{\alpha-1}$ would be invariant under the former of these two Sylow subgroups. Hence each Sylow subgroup, besides H_1, of one of these sets of $p+1$ subgroups must contain at least $p^m - p^{m-1}$ operators which do not appear in the other set. Hence the two sets contain at least

$$p^{m+1} - p^m + p^{m+1} + p^m - p^{\alpha+1} = 2\,p^{m+1} - p^{\alpha+1}$$

distinct operators. Hence G contains at least $2\,p^{m+1}$ distinct operators whose orders are powers of p in this case.

If two Sylow subgroups of order p^m have p^{α} common operators and these constitute a non-invariant subgroup, the set composed of all the subgroups which have these p^{α} operators in common is transformed by any subgroup of such a set into another set having another set of p^{α} operators in common, but having the transforming subgroup in common with the first set. Hence we have completed a proof of the theorem: *If a group whose order is divisible by p^m but not by p^{m+1} contains exactly $2\,p^{m+1} - p^m$ operators whose orders are powers of p, it involves exactly $2\,p+1$ subgroups of order p^m and all of these involve the same p^{m-1} operators.*

UNIVERSITY OF ILLINOIS.

*Normal Congruences Determined by Centers of Geodesic Curvature.**

By Frederick Wahn Beal.

Introduction.

In 1892 Caronnet announced the following theorem:† The necessary and sufficient condition that the lines which join the centers of geodesic curvature of the curves of an orthogonal system on a surface shall form a normal congruence, is that the corresponding radii of geodesic curvature be functions of one another or that the curves in one family or both have constant geodesic curvature. The present article is concerned with the determination of normal congruences of this kind and with an investigation of their properties. For the sake of brevity we shall refer to them as *associated normal congruences.*

In § 1, the theorem of Caronnet is proved and is extended to the case where one family of the orthogonal system is composed of geodesics, *i. e.*, when one of the radii of geodesic curvature is constantly infinite. In establishing the existence of associated normal congruences, there are three cases to be investigated: 1) the geodesic curvature is the same constant for the curves of each of the two families forming the orthogonal system; 2) it is constant and the same for the curves of only one family, and 3) the radii of geodesic curvature are functions of one another. The case in which both radii of geodesic curvature are given constants is discussed in §§ 2, 3 and 4. It is there shown that the surface on which such a system of curves exists must be pseudospherical and of total curvature equal to the negative of the sum of the squares of the constant geodesic curvatures. The associated normal congruence is found to be a congruence of normals of a Bianchi transform of the given surface. It is also proved that each of these congruences is determined by a single infinity of orthogonal systems or a double infinity of systems whose curves intersect under a constant angle and have constant geodesic curvature.

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* Read before the American Mathematical Society, September, 1911.

† "Sur les centres de courbure géodésique," *Comptes Rendus*, Vol. CXV, p. 589–592.

The general discussion of the subject occurs in §§ 5, 6, 7 and 8. It is shown in § 5 that associated normal congruences exist for any surface when the radii of geodesic curvature are given analytic functions of one another or when one radius of geodesic curvature is a constant. Characteristic properties of these congruences are discovered in §§ 6–7. The condition that a general normal congruence whose lines lie in the tangent planes of the given surface be an associated normal congruence is obtained. It is found that every system whose curves intersect under a constant angle, and for which the corresponding radii of geodesic curvature are functions of one another or for which one radius is a constant, determines an associated normal congruence. Further, the problem of determining these new systems is found to be the same as that of determining the orthogonal system. The restriction in the theorem of Caronnet that the curves be orthogonal may then be replaced by the condition that they intersect under a constant angle, but the more general theorem so obtained will not give any new congruences. In § 8, the condition that the developables of the associated normal congruence correspond to the lines of curvature on the given surface is investigated. The remainder of the paper deals with problems peculiar to associated normal congruences, determined by orthogonal systems, the curves of one family of which have a constant geodesic curvature. In the discussion we are led to consider surfaces on which one family of the lines of curvature have a constant geodesic curvature. Except in § 8 and most of § 9, the results obtained express deformation properties.

§ 1. *Theorem of Caronnet.*

We will give, for reference, a proof of the theorem of Caronnet. Let S be a surface referred to an orthogonal system of parametric curves $u = \text{const.}$, $v = \text{const.}$ Associate with S a set of moving axes, or trihedral, whose x- and y-axes are tangent to $v = \text{const.}$ and $u = \text{const.}$, respectively, and whose z-axis is normal to the surface. From the formula for the moving trihedral, this choice gives *

$$\xi_1 = \eta = 0, \quad r = -\frac{1}{\eta_1}\frac{\partial \xi}{\partial v}, \quad r_1 = \frac{1}{\xi}\frac{\partial \eta_1}{\partial u}.$$

Hence the radii of geodesic curvature R_1 and R_2 of the curves $v = \text{const.}$ and $u = \text{const.}$ respectively are given by the formulæ

$$R_1 = \xi/r \text{ and } R_2 = \eta_1/r_1.$$

The coordinates of the centers of geodesic curvature of the parametric curves,

* For notation used in this paper see Eisenhart's "Differential Geometry," §§ 57, 68–72, 76 and 119.

with respect to the moving trihedral, are $(0, R_1, 0)$ and $(-R_2, 0, 0)$. Consequently the coordinates of any point P on the line L joining these centers are $(R_2(\lambda - 1), \lambda R_1, 0)$, where $\lambda = \frac{t}{\sqrt{R_1^2 + R_2^2}}$ and t is the distance from $(-R_2, 0, 0)$ to P.

As M, the vertex of the moving trihedral, receives an infinitesimal displacement along the surface, the point P receives a displacement whose x and y components are:

$$\delta x = d[(\lambda - 1) R_2] + \xi du - \lambda R_1 (r du + r_1 dv),$$
$$\delta y = d(\lambda R_1) + \eta_1 dv + (\lambda - 1) R_2 (r du + r_1 dv).$$

The necessary and sufficient condition that the line L generate a normal congruence is that there exist a point P on L such that for displacements of M in all directions on the surface the displacements of P are perpendicular to L, *i. e.*,

$$R_2 \delta x + R_1 \delta y + 0 \delta z = 0. \tag{1}$$

This condition reduces to

$$R_2 dR_2 = d[\lambda (R_1^2 + R_2^2)] + (R_1^2 + R_2^2) d\lambda.$$

Upon substitution of the value of λ this becomes

$$R_2 dR_2 = (R_1^2 + R_2^2)^{1/2} dt. \tag{2}$$

Unless R_2 is a constant and consequently t also, it follows from (2) that $\frac{\partial (t, R_2)}{\partial (u, v)} = 0$, *i. e.*, t is a function of R_2, and hence by (2) that R_1 is a function of R_2. Conversely, if in (2) R_1 is assumed to be a function of R_2, t is so defined that the congruence will be normal. When R_2 is a constant, t is also a constant. Hence, if one of the radii of geodesic curvature is a constant, the corresponding center of geodesic curvature generates an orthogonal surface of the associated normal congruence.

The establishment of the existence for any surface of systems of curves of the kind required, and hence of the existence of associated normal congruences, will be deferred until later. Let us now consider an exceptional case, which has not been included in the preceding discussion; namely, when one of the radii is infinite. This happens when one of the families of the orthogonal system is composed of geodesics. Such systems are known to exist on any surface. If $R_2 = \infty$, the direction cosines of the line of the congruence become $1, 0, 0$, and the coordinates of the point P on L, $(t, R_1, 0)$. The condition (1) that P generates an orthogonal surface of the congruence is satisfied by $t = \text{const.}$ The congruence is therefore normal. In particular, the point $(0, R_1, 0)$ generates an orthogonal surface of the congruence which is complementary to the given surface for the geodesic family.

§ 2. *Geodesic Curvature Constant in Both Families. Pseudospherical Surfaces.*

Let S be a surface whose parametric curves form an orthogonal system, and assume that the geodesic curvatures of $u = \text{const.}$ and $v = \text{const.}$ are a and b respectively. Then, if the linear element is of the form $ds^2 = E\,du^2 + G\,dv^2$, it follows that

$$a = \frac{1}{\sqrt{EG}} \frac{\partial \sqrt{G}}{\partial u}, \qquad b = -\frac{1}{\sqrt{EG}} \frac{\partial \sqrt{E}}{\partial v}. \tag{3}$$

Differentiating these with respect to v and u respectively and adding the resulting equations, we obtain

$$\frac{\partial^2}{\partial u\,\partial v} \log \frac{E}{G} = 0.$$

Therefore E/G is equal to the ratio of a function of u and a function of v. The system is thus isothermic and the linear element of the surface can be put in the form $ds^2 = \lambda^2 (du^2 + dv^2)$. Formulæ (3) under these conditions become

$$a = \frac{1}{\lambda^2} \frac{\partial \lambda}{\partial u}, \qquad b = -\frac{1}{\lambda^2} \frac{\partial \lambda}{\partial v}.$$

Integrating these we see that $\lambda = (bv - au)^{-1}$ and that the linear element of the surface is *

$$ds^2 = \frac{du^2 + dv^2}{(bv - au)^2}. \tag{4}$$

The total curvature of this quadratic form is $K = -(a^2 + b^2)$. Hence every surface with this linear element is pseudospherical. Since all pseudospherical surfaces with the same total curvature are applicable on one another, and since the geodesic curvature of any curve on a surface is unchanged by deformation of the surface, there exists on every pseudospherical surface at least one orthogonal system for which, when parametric, the surface has a linear element of the form (4).

Let us inquire whether there is more than one such system on a pseudospherical surface, assuming in the discussion that the surface is referred to some orthogonal system. Then any other orthogonal system is determined by

$$\frac{dv}{du} = \frac{\xi}{\eta_1} \tan \theta, \tag{5}$$

$$\frac{dv}{du} = -\frac{\xi}{\eta_1} \cot \theta, \tag{6}$$

where θ is the angle the curves given by (5) make with $v = \text{const.}$ If the curves

* Bianchi, "Lezioni Di Geometria Differenziale," second ed., Vol. I, p. 211.

whose differential equations are (5) and (6) have geodesic curvatures b and a respectively, then

$$b\,\xi\,\eta_1 = \frac{\partial}{\partial u}(\eta_1 \sin\theta) - \frac{\partial}{dv}(\xi\cos\theta),$$

$$a\,\xi\,\eta_1 = \frac{\partial}{\partial u}(\eta_1\cos\theta) + \frac{\partial}{\partial v}(\xi\sin\theta).$$

Solving these for θ_u and θ_v, we have

$$\theta_u = \frac{\xi_v}{\eta_1} + \xi\,(b\cos\theta - a\sin\theta), \tag{7}$$

$$\theta_v = -\frac{\eta_{1u}}{\xi} + \eta_1\,(a\cos\theta + b\sin\theta). \tag{8}$$

The condition of integrability $\theta_{uv} = \theta_{vu}$ reduces to

$$a^2 + b^2 = \frac{r_{1u} - r_v}{\xi\,\eta_1} = -K.$$

Hence the following theorem:

THEOREM 1. *There exists on a pseudospherical surface of total curvature* $-(a^2+b^2)$ *a single infinity of orthogonal systems whose families have geodesic curvatures a and b.*

If a and b are replaced by other constants c and d, the condition of integrability requires only that $c^2 + d^2 = a^2 + b^2$. Hence we have

THEOREM 2. *There exists on any pseudospherical surface a double infinity of systems of orthogonal curves such that the curves in each family have the same constant geodesic curvature.*

The substitution $t = \tan\theta/2$ will reduce equations (7) and (8) to a Riccati equation, and we have

THEOREM 3. *Given any orthogonal system of curves on a pseudospherical surface, the determination of all orthogonal systems, the two families of which have given constant geodesic curvatures, requires the solution of a Riccati equation.*

§ 3. *Transformations of Bianchi.*

Let us return to the study of a pseudospherical surface with the linear element (4) and of the associated normal congruence formed by joining corresponding centers of geodesic curvature. If M denotes a point on the surface, L the corresponding line of the congruence, and P_1 the foot of the perpendicular from M to L, we wish to prove that P_1 generates an orthogonal surface of the congruence, that this surface is pseudospherical and of the same total curvature

as the original surface, and finally that the lines of curvature on both surfaces correspond. The distances from M to the two centers of geodesic curvature of the parametric curves are constant; hence the distance from either of these centers to P_1 is constant. But in § 1 it was shown that a point at a constant distance on L from the center of constant geodesic curvature generates an orthogonal surface. Hence P_1 generates such a surface.

In order to obtain the total curvature of this surface, we will find the product of the focal distances, measured from P_1, of the congruence of lines L. If a and b are the geodesic curvatures of $u = \text{const.}$ and $v = \text{const.}$ respectively, the coordinates of any point P on L will be

$$\left(\frac{-a}{a^2+b^2}+\frac{bt}{\sqrt{a^2+b^2}},\quad \frac{b}{a^2+b^2}+\frac{at}{\sqrt{a^2+b^2}},\quad 0\right),$$

where t is the distance from P_1. The displacements of P are

$$\delta x=\frac{b\,dt}{\sqrt{a^2+b^2}}+\frac{du}{bv-au}-(r\,du+r_1\,dv)\left(\frac{b}{a^2+b^2}+\frac{at}{\sqrt{a^2+b^2}}\right),$$

$$\delta y=\frac{a\,dt}{\sqrt{a^2+b^2}}+\frac{dv}{bv-au}+(r\,du+r_1\,dv)\left(\frac{-a}{a^2+b^2}+\frac{bt}{\sqrt{a^2+b^2}}\right),$$

$$\delta z=(p\,du+p_1\,dv)\left(\frac{b}{a^2+b^2}+\frac{at}{\sqrt{a^2+b^2}}\right)-(q\,du+q_1\,dv)\left(\frac{-a}{a^2+b^2}+\frac{bt}{\sqrt{a^2+b^2}}\right).$$

Since b, a and 0 are proportional to the direction-cosines of the line of the congruence, the conditions that P be displaced in the direction of the line L are

$$\frac{\delta x}{b}=\frac{\delta y}{a}=\frac{\delta z}{0}.$$

These are equivalent to the equations

$$a\,\delta x-b\,\delta y=0,\qquad \delta z=0,$$

which may be reduced to the following, when the expressions for δx, δy and δz are substituted:

$$\frac{dv}{du}=\frac{a-b\sqrt{a^2+b^2}\,t}{b+a\sqrt{a^2+b^2}\,t}, \tag{9}$$

$$\frac{dv}{du}=-\frac{pb+qa+\sqrt{a^2+b^2}\,(pa-pb)\,t}{p_1b+q_1a+\sqrt{a^2+b^2}\,(p_1a-q_1b)\,t}. \tag{10}$$

The elimination of dv/du between these equations gives a quadratic in t whose

roots are the distances from P_1 to the focal points of the congruence. The result of this elimination is the equation

$$(a^2+b^2)\,[p\,(a^2-b^2)-a\,b\,(q+p_1)]\,t^2 + \sqrt{a^2+b^2}\,[(p_1+q)\,(a^2-b^2)+4\,a\,b\,p]\,t - [p\,(a^2-b^2)-a\,b\,(q+p_1)] = 0,$$

for which the product of the roots is $-(a^2+b^2)^{-1}$. Therefore the total curvature of the surface generated by P_1 is $-(a^2+b^2)$, the same as the total curvature of the given surface.

To find the equation of the developable surfaces of the congruence let us eliminate t from (9) and (10). The result of this operation is the equation

$$q_1\,dv^2+(p_1+q)\,du\,dv+p\,du^2=0.$$

As this is the equation of the lines of curvature on the given surface, the developables of the normal congruence, or what amounts to the same thing, the lines of curvature on the orthogonal surfaces of the congruence, correspond to the lines of curvature on the given surface.

Combining these results with the fact that MP_1 is equal to $(a^2+b^2)^{-1/2}$, *i. e.*, equal to $(-K)^{-1/2}$, we have

THEOREM 4. *The point P_1 generates a Bianchi transform of the given surface.*

By virtue of this theorem, any properties which are possessed by Bianchi transforms of the given surface or their congruences of normals are properties of the orthogonal surface generated by P_1 or of the associated normal congruences. One of these properties may be stated thus:

THEOREM 5. *The focal points of this associated normal congruence lie on the tangents to the lines of curvature of the given pseudospherical surface.*

We will now give another proof of this theorem similar to one stated by Darboux.* The tangents to the lines of curvature of an orthogonal surface of a cyclic system pass through the focal points of the associated cyclic congruence. When the circles are in the tangent planes of a pseudospherical surface, with their centers at the points of tangency, their radii are equal to $(-K)^{-1/2}$ and the orthogonal surfaces are Bianchi transforms of the given surface. The associated cyclic congruence is the congruence of normals of the given surface. The relation between a surface and any one of its Bianchi transforms is reciprocal in the sense that either is a Bianchi transform of the other. Therefore the tangents to the lines of curvature on the given surface pass through the focal points of the congruence of normals of the surface generated by P_1.

* Darboux, "Surfaces," Vol. III, p. 430.

§ 4. *Coincidence of Normal Congruences.*

Thus far it has been shown that on any pseudospherical surface of total curvature $-(a^2+b^2)$ there exists a single infinity of orthogonal systems whose families have given constant geodesic curvatures a and b, and that each of these systems determines a congruence of normals of a Bianchi transform of the given surface. It is now our intention to prove that the centers of geodesic curvature of a single infinity of orthogonal systems lie on each of these congruences, or in other words, that each associated normal congruence and each Bianchi transform may be determined by any one of a single infinity of orthogonal systems. If on a surface whose linear element is (4) the geodesic curvatures of an orthogonal system are m and n, then

$$m^2+n^2=a^2+b^2,$$

and if θ denotes the angle which the curves of geodesic curvature n make with $v=$ const., then

$$n=(bv-au)^2\left[\frac{\partial}{\partial u}\frac{\sin\theta}{bv-au}-\frac{\partial}{\partial v}\frac{\cos\theta}{bv-au}\right], \tag{12}$$

$$m=(bv-au)^2\left[\frac{\partial}{\partial u}\frac{\cos\theta}{bv-au}+\frac{\partial}{\partial v}\frac{\sin\theta}{bv-au}\right]. \tag{13}$$

The coordinates of the centers of geodesic curvature of this system of curves, with respect to the moving trihedral, are

$$\left(-\frac{1}{n}\sin\theta,\ \frac{1}{n}\cos\theta,\ 0\right),\quad \left(-\frac{1}{m}\cos\theta,\ -\frac{1}{m}\sin\theta,\ 0\right). \tag{14}$$

If these centers lie on the lines of the congruence determined by the parametric curves, they must satisfy the relation

$$y=ax/b+1/b, \tag{15}$$

which is the equation of the lines with respect to the moving trihedral. Substituting the coordinates of the centers of geodesic curvature (14) in (15), and then solving for $\sin\theta$ and $\cos\theta$, we obtain

$$\cos\theta=\frac{am+bn}{a^2+b^2},\qquad \sin\theta=\frac{an-bm}{a^2+b^2}.$$

These give a constant value for θ which satisfies (12) and (13) and therefore enable us to state the following theorem:

THEOREM 6. *Corresponding curves of any two orthogonal systems whose families have constant geodesic curvature and determines the same normal congruence intersect under a constant angle.*

Since m and n are restricted by the relation (11) only, we have

THEOREM 7. *A single infinity of these systems determines the same associated normal congruence.*

Theorems 6 and 7 show that there exists a double infinity of systems of curves whose families have constant geodesic curvature and intersect under a constant angle, and whose centers of geodesic curvature lie on the lines of the associated normal congruence determined by the parametric curves. If a surface is referred to a system of curves for which g and h are the constant geodesic curvatures of $v = \text{const.}$ and $u = \text{const.}$ respectively, and ω is the constant angle between these parametric curves, Voss* has shown that the linear element of the surface is

$$ds^2 = \frac{du^2 + 2\cos\omega\, du\, dv + dv^2}{\left[\frac{(g - h\cos\omega)\, v + (-h + g\cos\omega)\, u}{\sin\omega}\right]^2}.$$

The total curvature of this quadratic form is

$$K = \frac{g^2 + h^2 - 2gh\cos\omega}{\sin^2\omega},$$

and hence the surface must be pseudospherical. If the x-axis of the moving trihedral is tangent to $v = \text{const.}$, it can be shown by the same method as that employed in the case where the curves were orthogonal that the surface generated by P_1 is a Bianchi transform of the given surface. Therefore the congruence determined by any system of curves of this kind must be a congruence of normals of a Bianchi transform of the given surface.

§ 5. *Existence of Associated Normal Congruences for Any Surface.*

In order to prove the existence of associated normal congruences for any surface, assume that an orthogonal system of curves is parametric on the surface and that the curves whose differential equations are (5) and (6) have geodesic curvatures T and T_1 respectively. These geodesic curvatures are then determined by the expressions

$$T = \frac{1}{\xi\eta_1}\left[\frac{\partial}{\partial u}(\eta_1 \sin\theta) - \frac{\partial}{\partial v}(\xi\cos\theta)\right], \tag{18}$$

$$T_1 = \frac{1}{\zeta\eta_1}\left[\frac{\partial}{\partial u}(\eta_1\cos\theta) + \frac{\partial}{\partial v}(\xi\sin\theta)\right]. \tag{19}$$

Assuming that T_1 is a constant or an analytic function of T, equations (18) and (19) by the elimination of T give a partial differential equation of the

* *München Sitzungsberichte*, 1906, pp. 247–296.

first order. When T_1 is a constant or equals a constant times T, the differential equation is also of the first degree. Hence the existence of associated normal congruences for any surface is established for cases 2 and 3, and we are enabled to state the following theorem:

THEOREM 8. *Given any analytic functional relation between the corresponding radii of geodesic curvature of the curves of an orthogonal system on a surface, or given that one of these radii is a constant, there exist associated normal congruences and the problem of determining them when any orthogonal system is parametric leads to the solution of a partial differential equation of the first order.*

The results of solving equations (18) and (19) for θ_u and θ_v, when T_1 has been replaced by $f(T)$, are

$$\theta_u = \frac{\xi_v}{\eta_1} + \xi\,(T\cos\theta - f(T)\sin\theta),$$

$$\theta_v = -\frac{\eta_{1u}}{\xi} + \eta_1\,(T\sin\theta + f(T)\cos\theta).$$

Denoting by K the total curvature of the surface and by f' the derivative of $f(T)$ with respect to T, the condition of integrability of these is

$$K + T^2 + f^2 = \frac{T_v}{\eta_1}(\cos\theta - f'\sin\theta) - \frac{T_u}{\xi}(\sin\theta + f'\cos\theta). \qquad (20)$$

In order that this equation be independent of θ, so that as in the case of pseudospherical surfaces θ involves a parameter, we must have

$$K + T^2 + f^2 = 0,$$
$$\xi T_v - \eta_1 f' T_u = 0,$$
$$\xi f' T_v + \eta_1 T_u = 0.$$

Since the determinant of the system composed of the last two equations is $\xi\eta_1(f'^2+1)$, the only real solution is $T_u = 0$, $T_v = 0$, or $T = \text{const.}$ Then by the first condition the surface must be pseudospherical and of total curvature $-(f^2+T^2)$. Another result arising from the consideration of (20) is that for any set of values T and $f(T)$ there can not exist more than two angles θ.

§ 6. *Some Characteristic Properties of Associated Normal Congruences.*

We now desire to know whether all normal congruences whose lines lie in the tangent planes of a surface are *associated normal congruences.* To carry out the investigation, we assume parametric an orthogonal system of curves on the surface, obtain the condition that a line in the tangent plane generate

a normal congruence and finally derive the condition that this line pass through corresponding centers of geodesic curvature of some orthogonal system. Let ϕ be the angle that the perpendicular from the point of tangency, M, to the line of the congruence makes with the tangent to $v = \text{const.}$, which is taken as the x-axis of an associated moving trihedral, and let h be the length of this perpendicular MP_1. Then the coordinates of any point P on the line will be $(h\cos\phi - t\sin\phi,\ h\sin\phi + t\cos\phi,\ 0)$, where t is the distance from P_1 to P. In order that P move perpendicular to the line, for all infinitesimal displacements of M we must have

$$h\,d\phi + dt - \xi\sin\phi\,du + \eta_1\cos\phi\,dv + (r\,du + r_1\,dv)\,h = 0.$$

This being true for all displacements is equivalent to

$$h\phi_u + t_u - \xi\sin\phi + rh = 0, \tag{21}$$

$$h\phi_v + t_v + \eta_1\cos\phi + r_1 h = 0. \tag{22}$$

The condition of integrability for t of these differential equations is

$$h_u(\phi_v + r_1) - h_v(\phi_u + r) + \eta_{1u}\cos\phi + \xi_v\sin\phi - \eta_1\sin\phi\,\phi_u + \xi\cos\phi\,\phi_v - h\,\xi\,\eta_1 K = 0. \tag{23}$$

Since this does not involve t, any pair of solutions ϕ and h will make (21) and (22) consistent and will therefore determine a normal congruence whose lines lie in the tangent planes of the surface. An inspection of (23) shows that, with certain restrictions, h or ϕ can be chosen arbitrarily and the other determined by a partial differential equation of the first order and first degree.

Denote the radii of geodesic curvature of the two families of the orthogonal system defined by (5) and (6) by R_1 and R_2 respectively. Then the coordinates of the centers of geodesic curvature are $(-R_1\sin\theta,\ R_1\cos\theta,\ 0)$ and $(-R_2\cos\theta,\ -R_2\sin\theta,\ 0)$ respectively. Since the equation of the line of the normal congruence under consideration is

$$x\cos\phi + y\sin\phi = h,$$

the conditions that these two points lie on the line are

$$-\sin\theta\cos\phi + \cos\theta\sin\phi = h/R_1,$$
$$-\cos\theta\cos\phi - \sin\theta\sin\phi = h/R_2,$$

where ϕ and h are solutions of equation (23), and $1/R_1$ and $1/R_2$ are the geodesic curvatures of the curves defined by (5) and (6) respectively. Sub-

stituting for $1/R_1$ and $1/R_2$ their values in terms of ξ, η_1 and θ, and solving for θ_u and θ_v, we have

$$\theta_u = -r + \frac{\xi}{h}\sin\phi, \tag{24}$$

$$\theta_v = -r_1 - \frac{\eta_1}{h}\cos\phi. \tag{25}$$

The condition of integrability of these equations is

$$\eta_{1u}\cos\phi + \xi_v \sin\phi - \eta_1 \sin\phi\,\phi_u + \xi\cos\phi\,\phi_v - h\,\xi\,\eta_1 K - \frac{1}{h}(\eta_1\cos\phi\,h_u + \xi\sin\phi\,h_v) = 0. \tag{26}$$

Since this does not involve θ, any pair of solutions ϕ and h of (23) and (26) will determine a normal congruence made up of the joins of corresponding centers of geodesic curvature of (5) and (6). Further, any set of values ϕ and h, solutions of (23) and (26), will determine by (24) and (25) an infinity of functions θ, which differ only by additive constants, and therefore will give a single infinity of orthogonal system whose centers of geodesic curvature lie on the lines of the normal congruence determined by ϕ and h. Combining this result with Caronnet's theorem, we have the following, a special case of which is given by Theorems 6 and 7.

THEOREM 9. *If the joins of the centers of geodesic curvature of the curves of an orthogonal system on a surface form a normal congruence, then the centers of geodesic curvature of a single infinity of orthogonal systems of curves lie on the lines of this congruence and corresponding curves of any two of these systems intersect under a constant angle.*

Subtracting (26) from (23) we obtain

$$h_u\left(\phi_v + r_1 + \frac{\eta_1\cos\phi}{h}\right) = h_v\left(\phi_u + r - \frac{\xi\sin\phi}{h}\right). \tag{27}$$

From (24), (25) and (27) and from (21), (22) and (27) it follows that

$$h_u(\phi_v - \theta_v) - h_v(\phi_u - \theta_u) = 0, \tag{28}$$

$$h_u t_v - h_v t_u = 0. \tag{29}$$

THEOREM 10. *The three quantities h, t and $\phi - \theta$ are functions of one another.*

If in (21) and (22) h is assumed to be a function of t, the vanishing of the Jacobian $\dfrac{\partial(h,t)}{\partial(u,v)}$ is equivalent to

$$\begin{vmatrix} \xi\sin\phi - h\,\phi_u - r\ h, & h_u \\ -\eta_1\cos\phi - h\,\phi_v - r_1\,h, & h_v \end{vmatrix} = 0.$$

This equation reduces to (26) and hence, together with Theorem 10, proves the following, which may be used as a second definition of associated normal congruences:

THEOREM 11. *The necessary and sufficient condition that a normal congruence whose lines lie in the tangent planes of a surface be an associated normal congruence is that h be a function of t, or that h or t be a constant, or that both be constants, and that ϕ be a solution of* (26).

§ 7. *Verification and Extension of the Results in the Preceding Section.*

Let us now verify formula (29) by means of the formulæ obtained in the proof of Caronnet's theorem. In that proof t was measured from $(-R_2, 0, 0)$ to any normal surface of the congruence and was given by

$$dt = \frac{R_2\, dR_2}{\sqrt{R_1^2 + R_2^2}}.$$

If the distance is to be measured from P_1, to t must be added the distance from $(-R_2, 0, 0)$ to P_1. The result of this substitution is

$$dt = \frac{R_1 R_2}{(R_1^2 + R_2^2)^{3/2}} (R_2\, dR_1 - R_1\, dR_2).$$

Since h is the length MP_1,

$$\left.\begin{aligned} h &= \frac{R_1 R_2}{\sqrt{R_1^2 + R_2^2}}, \qquad dh = \frac{R_2^3\, dR_1 + R_1^3\, dR_2}{(R_1^2 + R_2^2)^{3/2}}, \\ \frac{\partial(ht)}{\partial(uv)} &= \frac{R_1^2 R_2^2}{(R_1^2 + R_2^2)^2} (R_{1v} R_{2u} - R_{1u} R_{2v}). \end{aligned}\right\} \quad (30)$$

But R_2 is a function of R_1, or either R_1 or R_2 is a constant or both are constants; hence the Jacobian is equal to zero and the condition that t be a function of h or that either or both be constants is satisfied.

In consequence of Theorem 9 and (30) it follows that

$$h = \frac{R_1 R_2}{\sqrt{R_1^2 + R_2^2}} = \frac{R_1' R_2'}{\sqrt{R_1'^2 + R_2'^2}}, \quad (31)$$

where R_1' and R_2' are the radii of geodesic curvature of another orthogonal system of curves whose centers of geodesic curvature lie on the lines of the same associated normal congruence. Since R_2 is a function of R_1 and R_2' is a function of R_1', it is necessarily true from (31) that R_1 is a function of R_1'. But since the curves which have radii of geodesic curvature R_1 and those which have radii of geodesic curvature R_1' intersect under a constant angle, by Theorem 9, we have

THEOREM 12. *There exists on any surface a double infinity of systems of curves whose geodesic radii are functions of one another and the curves of whose families intersect under a constant angle. These systems determine the same associated normal congruences.*

It becomes necessary now to prove that every system of this kind determines a normal congruence and that the congruence so determined is identical with one determined by an orthogonal system. Assuming that $v = \text{const.}$ and $u = \text{const.}$ have geodesic radii R_1 and R_2 respectively and intersect under a constant angle ω, and that the x-axis of an associated moving trihedral is tangent to $v = \text{const.}$, the following relations are obtained easily from the formulæ of the moving trihedral and the formulæ of the geodesic curvature of the parametric curves:

$$E = \xi^2, \quad F = \xi\,\xi_1, \quad G = \frac{\xi_1^2}{\cos^2\omega} = \frac{\eta_1^2}{\sin^2\omega}, \quad \xi_1 \sin\omega = \eta_1 \cos\omega,$$

$$r = \frac{\xi_{1u} - \xi_v}{\eta_1}, \quad r_1 = \frac{\eta_{1u} - \xi_v \sin\omega\cos\omega}{\sin^2\omega},$$

$$R_1 = \xi/r, \quad R_2 = \xi_1/(r_1\cos\omega) = \eta_1/(r_1\sin\omega).$$

The coordinates of any point P on the line of the congruence are

$$\left(\frac{-\sin\omega\, R_2\, t}{\sqrt{R_1^2 + R_2^2 - 2R_1R_2\cos\omega}}, \quad R_1 - \frac{(R_1 - R_2\cos\omega)\, t}{\sqrt{R_1^2 + R_2^2 - 2R_1R_2\cos\omega}}, \quad 0\right),$$

where t is the distance from $(0, R, 0)$ to P. The direction cosines of the line are proportional to

$$R_2\sin\omega, \quad R_1 - R_2\cos\omega, \quad 0.$$

The condition that P generate an orthogonal surface of the congruence reduces to

$$dt\,\sqrt{R_1^2 + R_2^2 - 2R_1R_2\cos\omega} = (R_1 - R_2\cos\omega)\, dR_1.$$

By a course of reasoning the same as that employed in § 1, this shows that R_1 is a function of R_2, R_1 or R_2 is a constant or both are constants.

If in § 6 the equations of the system of curves had been

$$\frac{dv}{du} = \frac{\xi}{\eta_1}\tan\theta, \quad \frac{dv}{du} = \frac{\xi}{\eta_1}\tan(\theta + \omega),$$

and the conditions that their centers of geodesic curvature should lie on the line of the congruence had been imposed, equations (24) and (25) would again have been obtained. Hence the problem of determining these new systems is the same as that of determining the orthogonal systems.

THEOREM 13. *The necessary and sufficient condition that the lines joining corresponding centers of geodesic curvature of a system whose curves intersect under a constant angle shall form an associated normal congruence, is that the radii of geodesic curvature be functions of one another or that one radius be a constant or that both be constant.*

Theorem 13 gives a third definition of associated normal congruences. Together with the previous discussion this shows that the consideration of the orthogonal systems is equivalent to the consideration of these new systems. When a single orthogonal system determining an associated normal congruence is known, all systems of curves whose families intersect under a constant angle and determine the same normal congruence are known.

We will now verify formula (28). Since the slope of MP_1 is $\tan\phi$, the following equations are easily obtained:

$$\tan\phi = -\frac{R_2 - R_1\tan\theta}{R_1 + R_2\tan\theta}, \qquad \tan(\phi-\theta) = -R_2/R_1, \tag{32}$$

$$d(\phi-\theta) = \frac{R_2\,dR_1 - R_1\,dR_2}{R_1^2+R_2^2} = \frac{\sqrt{R_1^2+R_2^2}}{R_1R_2}\,dt. \tag{33}$$

The curves intersect under right angles, and hence h depends only on R_1 and R_2 and is thus the same as in the previous discussion. These facts enable us to write the relation

$$\begin{vmatrix} \phi_u - \theta_u, & h_u \\ \phi_v - \theta_v, & h_v \end{vmatrix} = \frac{(R_1^2+R_2^2)^{1/2}}{R_1R_2}\begin{vmatrix} t_u & t_v \\ h_u & h_v \end{vmatrix} = 0,$$

and hence equation (28) is verified. From (33), when t equals a constant, $\phi-\theta$ equals a constant and conversely. From (32), when R_2 is equal to a constant times R_1, $\phi-\theta$ is equal to a constant and conversely, except when $1/R_1 = 0$. In the latter case $\phi-\theta$ is equal to zero and the curves defined by (5) are geodesics.

THEOREM 14. *A necessary and sufficient condition that P_1 generate an orthogonal surface of the associated normal congruence, is that the radius of geodesic curvature of the curves of one family of the orthogonal system be equal to a constant times the radius of geodesic curvature of the curves of the other family, or that one radius of geodesic curvature be infinite.*

§ 8. *Developables Corresponding to the Lines of Curvature.*

Suppose now that the lines of curvature are parametric and that equations (5) and (6) define an orthogonal system on the surface, such that if T is the geodesic curvature of (5), $f(T)$ is the geodesic curvature of (6). Under

these conditions we wish to find when the lines of curvature on the given surface correspond to the developables of the associated normal congruence. To do this, we will obtain the displacements of a point P on the line of the congruence, impose the conditions that P move tangent to this line, eliminate t to derive the equations of the developable surfaces, and finally equate the coefficients of du^2 and dv^2 to zero. The developables will then be given by $du\,dv = 0$, or in other words, they will correspond to the lines of curvature on the given surface. The system of equations obtained by this process is:

$$q\,(\cos\theta - f'\sin\theta)\,T_u = 0,$$
$$p_1\,(\sin\theta + f'\cos\theta)\,T_v = 0,$$

where f' is the derivative of $f(T)$ with respect to T. This system is equivalent to the following systems:

$$\left.\begin{aligned}\sin\theta + f'\cos\theta &= 0,\\ \cos\theta - f'\sin\theta &= 0,\end{aligned}\right\}\ (34), \qquad \left.\begin{aligned}p_1 &= 0,\\ q &= 0,\end{aligned}\right\}\ (35), \qquad \left.\begin{aligned}T_u &= 0,\\ T_v &= 0,\end{aligned}\right\}\ (36),$$

$$\left.\begin{aligned}T_v &= 0,\\ \cos\theta &= f'\sin\theta,\end{aligned}\right\}\ (37), \qquad \left.\begin{aligned}T_u &= 0,\\ \sin\theta &= -f'\cos\theta,\end{aligned}\right\}\ (38),$$

$$\left.\begin{aligned}p_1 &= 0,\\ \cos\theta &= f'\sin\theta,\end{aligned}\right\}\ (39), \qquad \left.\begin{aligned}q &= 0,\\ \sin\theta &= -f'\cos\theta,\end{aligned}\right\}\ (40),$$

$$\left.\begin{aligned}T_v &= 0,\\ q &= 0,\end{aligned}\right\}\ (41), \qquad \left.\begin{aligned}T_u &= 0,\\ p_1 &= 0,\end{aligned}\right\}\ (42).$$

The determinant of (34) is $1+f'^2$; hence it expresses an impossible condition. Since the lines of curvature are parametric, system (35) defines a plane. System (36) states that the geodesic curvatures of both families of curves are constant. This has been fully discussed in §§ 2–4. Systems (37), (39) and (41) are similar respectively to (38), (40) and (42). Hence it is advisable to consider the first set only. Since, as may be easily shown, system (39) is satisfied only by developable surfaces of revolution, it is not desirable to treat it further. The only surfaces that satisfy condition (41) are the plane and developable surfaces of revolution; but as this statement is by no means evident, it is desirable to prove it. The lines of curvature being parametric, system (41) is equivalent to the two systems

$$T_v = 0,\ q = 0,\ p_1 = 0,\ r_v = r_{1u},\ r = -\frac{\xi_v}{\eta_1},\ r_1 = \frac{\eta_{1u}}{\xi}, \tag{43}$$

$$T_v = 0,\ q = 0,\ r = 0,\ p_{1u} = 0,\ r_{1u} = 0,\ \xi_v = 0,\ r_1 = \frac{\eta_{1u}}{\xi}. \tag{44}$$

Evidently a surface which satisfies equations (43) must be a plane. The linear element of a surface conditioned by equations (44) is reducible to the form $ds^2 = du^2 + (v_1 u + v_2)^2 dv^2$, where v_1 and v_2 are functions of v alone. Since q and the total curvature of this quadratic form are zero, the curves $v = \text{const.}$ must be tangents to the edge of regression. If $v_2 = 0$ or $v_1 = 0$ or $v_1/v_2 = \text{const.}$, the surface is a right circular cone or cylinder, since it is then a developable surface of revolution on which the lines of curvature are parametric. In general, T and $f(T)$, which are functions of u alone, and v_1 and v_2 must be such quantities that

$$T\xi\eta_1 = \frac{\partial}{\partial u}(\eta_1 \sin\theta) - \frac{\partial}{\partial v}(\xi \cos\theta), \tag{45}$$

$$f(T)\xi\eta_1 = \frac{\partial}{\partial u}(\eta_1 \cos\theta) + \frac{\partial}{\partial v}(\xi \sin\theta) \tag{46}$$

possess solutions θ. Substituting for ξ and η_1 their values given by the linear element, and solving for θ_u and θ_v, we obtain

$$\theta_u = R\cos\theta - f\sin\theta,$$
$$\theta_v = -v_1 + (v_1 u + v_2)(R\sin\theta + f\cos\theta).$$

The condition of integrability of these equations is

$$T^2 + f^2 + T_u \sin\theta + f_u \cos\theta = 0.$$

By virtue of the identity $\sin^2\theta + \cos^2\theta = 1$, this equation shows that θ must be a function of u alone. Further, since θ is a function of u alone, $\dfrac{v_1}{v_1 u + v_2}$ is a function of u alone. Therefore $v_2/v_1 = \text{const.}$, and consequently the surface is a developable surface of revolution.

System (37), in which R, f and θ must be functions of u alone, will now be investigated at length. Since θ is a function of u alone, equations (45) and (46) are equivalent to

$$\frac{\eta_{1u}}{\xi\eta_1} = (T\sin\theta + f\cos\theta), \tag{47}$$

$$\eta_1\theta_u = \xi_v + \xi\eta_1(T\cos\theta - f\sin\theta). \tag{48}$$

To these must be added the identity

$$\sin^2\theta + \cos^2\theta = 1, \tag{49}$$

and from system (37) the equation

$$f_u \sin\theta = T_u \cos\theta. \tag{50}$$

From (48) and (49) we have

$$\sin\theta = \pm \frac{T_u}{\sqrt{T_u^2 + f_u^2}}, \tag{51}$$

$$\cos\theta = \pm \frac{f_u}{\sqrt{T_u^2 + f_u^2}}, \tag{52}$$

where the upper signs are to be taken together. According to equation (47) the geodesic curvature of $u = \text{const.}$ is a function of u alone. From (51) and (52) by differentiation we have

$$\theta_u = \frac{f_u T_{uu} - T_u f_{uu}}{T_u^2 + f_u^2}, \tag{53}$$

and from (53) and (48),

$$r = \frac{T_u f_{uu} - f_u T_{uu}}{T_u^2 + f_u^2} \pm \xi \frac{T f_u - f T_u}{\sqrt{T_u^2 + f_u^2}}. \tag{54}$$

By (47) the coordinates, with respect to the moving trihedral, of the centers of geodesic curvature of $u = \text{const.}$ are

$$\left(\mp \frac{\sqrt{T_u^2 + f_u^2}}{T T_u + f f_u},\ 0,\ 0\right), \tag{55}$$

and by use of (51) and (52) these are identical with the coordinates of the point of intersection of the line of the congruence and the x-axis. Therefore the center of geodesic curvature of $u = \text{const.}$ lies on the line of the congruence. The displacements of the point (55) are:

$$\delta x = \mp d\left(\frac{\sqrt{T_u^2 + f_u^2}}{T T_u + f f_u}\right) + \xi\, du,$$

$$\delta y = \mp \frac{1}{T T_u + f f_u}\left[\frac{T_u f_{uu} - f_u T_{uu}}{\sqrt{T_u^2 + f_u^2}} + \xi\,(T f_u - f T_u)\right] du,$$

$$\delta z = \pm q \frac{\sqrt{T_u^2 + f_u^2}}{T T_u + f f_u}\, du.$$

Since none of these displacements involve dv, the point (55) must generate a curve as M moves over the surface. It follows also that ξ and q must be functions of u alone. Hence without loss of generality ξ can be taken equal to unity. This then proves with (47) that η_1 is a function of u alone. Therefore since the linear element of the surface is of the form $ds^2 = du^2 + \eta_1^2\, dv^2$, and the lines of curvature are parametric, the surface must be a surface of revolution.

Since ξ is unity and r is zero, from (54) it follows that

$$\pm \frac{T f_u - f T_u}{\sqrt{T_u^2 + f_u^2}} = \frac{f_u T_{uu} - T_u f_{uu}}{T_u^2 + f_u^2}, \tag{56}$$

and from (46) that

$$\frac{\eta_{1u}}{\eta_1} = \pm \frac{T T_u + f f_u}{\sqrt{T_u^2 + f_u^2}}. \tag{57}$$

In equations (56) and (57) there are three functions of u. If it is possible to choose $\eta_1 (u)$ arbitrarily and determine functions T and f that satisfy (56) and (57), the preceding discussion will hold true for *any* surface of revolution. Let then η_1 be a known analytic function of u and denote by U the function $\frac{\eta_{1u}}{\eta_1}$. In order to further simplify the notation, use only upper signs and let dT/du be μ and df/du be λ. Then the equations to be considered are:

$$U \sqrt{\lambda^2 + \mu^2} = T\mu + f\lambda, \tag{58}$$

$$(T\mu + f\lambda)(T\lambda - f\mu) = U\left(\lambda \frac{d\mu}{du} - \mu \frac{d\lambda}{du}\right). \tag{59}$$

Solving (58) for μ we have

$$\mu = \lambda \frac{-Tf \pm U\sqrt{T^2 + f^2 - U^2}}{T^2 - U^2}. \tag{60}$$

Both of these values of μ satisfy (58). For convenience choose one of them and let μ equal $\lambda\phi(T, f, u)$. By substituting the values of μ and $d\mu/du$ in equation (59), it is found that the coefficient of $d\lambda/du$ is identically zero and that (59) reduces to

$$\lambda^2\left[\lambda U\left(\phi \frac{\partial \phi}{\partial T} + \frac{\partial \phi}{\partial f}\right) + U\frac{\partial \phi}{\partial u} - (T - f\phi)(T\phi + f)\right] = 0. \tag{61}$$

If λ equals zero, $f(T)$ degenerates into a constant. By formula (52) this means that θ equals $\pi/2$ or $3\pi/2$, and hence that the curves whose equation is (5) are the parallels of the surface of revolution, and those whose equation is (6) are the meridians. The constant f must then be zero and T must be U, which is equal to $\frac{\eta_{1u}}{\eta_1}$. These results satisfy equations (58) and (59).

Let us now consider the other value of λ. Since neither of the coefficients $U\left(\phi \frac{\partial \phi}{\partial T} + \frac{\partial \phi}{\partial f}\right)$ and $U\frac{\partial \phi}{\partial u} - (T - f\phi)(T\phi + f)$ is identically zero, equation (61) will in general give a definite value of λ which is not zero or infinite.

If we denote these values of λ and μ by the following analytic functions of T, f and u,

$$\mu = \Phi(T, f, u), \qquad \lambda = \Psi(T, f, u),$$

equations (58) and (59) will become equivalent to

$$\frac{dT}{du} = \Phi(T, f, u), \qquad \frac{df}{du} = \Psi(T, f, u).$$

Hence we can say that for any given $\eta_1(u)$ there exist functions T and f, which satisfy equations (58) and (59). When T and f are determined, the angle θ can be found by means of equation (51) and (52). In this discussion we have used the upper signs. The same results would follow if we had used the lower signs. As no difficulty arises in the determination of p, q, r and r_1, we have as the result of the investigation in this section the theorem:*

THEOREM 15. *The necessary and sufficient condition that there exist associated normal congruences whose developables correspond to the lines of curvature on the given surface is that the surface be a surface of revolution or a pseudospherical surface.*

If in the preceding discussion of surfaces of revolution $f(T)$ is equal to a constant times T, the point (55), which is the center of geodesic curvature of the parallels of the surface of revolution, is the point P_1. This point generates an orthogonal surface of the congruence and the congruence so obtained is the same as that determined by the parallels and meridians. Conversely, if it is assumed that the point (55) generates an orthogonal surface of the congruence, by means of (54) we find that f must equal a constant times T.

For any parallel the totality of lines of the congruence will envelope a right circular cone. Hence the orthogonal surfaces will be surfaces of revolution. When f is equal to a constant times T, the lines corresponding to any parallel will be perpendicular to the axis and also to a tangent to the corresponding meridian. The orthogonal surfaces of the congruence will, in this case, be right circular cylinders.

§ 9. *The Geodesic Curvature of One Family only of an Orthogonal System on a Surface is a Constant. Surfaces on Which One Family of Lines of Curvature Have Constant Geodesic Curvature.*

It has already been established that the center of constant geodesic curvature generates an orthogonal surface of the associated normal congruence,

* The total curvature of the surface of revolution is

$$K = -(T^2 + f^2) \mp (T_u^2 + f_u^2)^{1/2}.$$

that the point P_1 can not lie on an orthogonal surface of this congruence, and that, unless the orthogonal system is made up of the lines of curvature of a surface of revolution, the developables of the congruence can not correspond to the lines of curvature on the given surface. For convenience of notation let c equal $1/a$, where a is the geodesic curvature of the curves $u = \text{const.}$ of the orthogonal parametric system. Then with respect to the moving trihedral, chosen as usnal, the coordinates of the center of constant geodesic curvature are $(-c, 0, 0)$. The displacements of this point are

$$\delta x = \xi\, du, \quad \delta y = -c\, r\, du, \quad \delta z = (q\, du + q_1\, dv)\, c,$$

and consequently the linear element of the orthogonal surface generated by it is

$$ds^2 = (\xi^2 + c^2 r^2 + c^2 q^2)\, du^2 + 2\, c^2 q\, q_1\, du\, dv + c^2 q_1^2\, dv^2. \tag{62}$$

If the parametric system is to be orthogonal on this surface, q must equal zero. This means that the curves $v = \text{const.}$ on the given surface are asymptotic lines. These asymptotic lines can not be straight lines, as then both geodesic curvatures would be constant and the surface would be pseudospherical.

Since

$$\eta = \xi_1 = 0, \qquad p_1 \eta - q_1 \xi = p \eta_1 - q \xi_1,$$

it follows that if q_1 equals zero, p must equal zero also. This condition will soon be discussed.

If the curves $u = \text{const.}$ are to be lines of curvature, the conditions imposed on the surface are

$$\eta_{1u} = a \xi \eta_1, \quad r = -\xi_v/\eta_1, \quad r_1 = \eta_{1u}/\xi, \quad p_{1u} = -q r_1, \quad q_v = r p_1, \quad r_v - r_{1u} = -p_1 q.$$

These are equivalent to

$$r_1 = a \eta_1, \tag{63}$$
$$\eta_{1u} = a \xi \eta_1, \tag{64}$$
$$p_{1u} = -a q \eta_1, \tag{65}$$
$$q_v = r p_1, \tag{66}$$
$$r_v = a^2 \xi \eta_1 - p_1 q, \tag{67}$$
$$\xi_v = -r \eta_1. \tag{68}$$

Multiplying (66) by q and (67) by r, and adding, gives

$$q q_v + r r_v + a^2 \xi \xi_v = 0,$$

or after integration,

$$q^2 + r^2 + a^2 \xi^2 = U^2,$$

where U is a function of u alone. By the transformation $du_1 = U\, du$, this

becomes $r'^2 + q'^2 + a^2\xi'^2 = 1$; hence without loss of generality U may be taken as unity. Let then U be unity and let ξ, q and r be represented by $\xi = c\sin\theta$, $q = \sin\phi\cos\theta$, $r = \cos\phi\cos\theta$.

From the conditions above we also have

$$\eta_1 = -\frac{c\,\theta_v}{\cos\theta}, \qquad r_1 = -\frac{\theta_v}{\cos\phi}, \qquad p_1 = \phi_v - \tan\phi\tan\theta\theta_v.$$

The fundamental quantities are now expressed in terms of the new variables ϕ and θ, and equations (64) and (65) are left to determine these. Substituting in (65) and (64) respectively gives

$$\begin{aligned}\phi_{uv} &- \tan\phi\tan\theta\theta_{uv} - \tan\phi\sec^2\theta\theta_u\theta_v \\ &- \sec^2\phi\tan\theta\phi_u\theta_v - \tan\phi\cos\theta\theta_v = 0, \qquad (69)\end{aligned}$$

$$\theta_{uv} + \tan\phi\phi_u\theta_v - \sin\theta\theta_v = 0. \qquad (70)$$

Eliminating θ_{uv} from (69) by means of (70) we have

$$\phi_{uv} - \tan\theta\theta_v\phi_u - \tan\phi\sec^2\theta\theta_u\theta_v - \tan\phi\sec\theta\theta_v = 0. \qquad (71)$$

Any pair of solutions ϕ and θ of equations (70) and (71) except those for which θ_v is zero will determine ξ, η_1, r, r_1, p_1 and q of a surface on which the lines of curvature u=const. have constant geodesic curvature a.

It may now be shown that the curves $u = \text{const.}$ are spherical lines of curvature. The normal curvature of $u = \text{const.}$, denoted by $1/\rho_2$, is equal to p_1/η_1. Substituting the values of p_1 and η_1 just found, we have

$$-\rho_2 = \frac{a\,\theta_v}{\cos\phi\phi_v - \sin\phi\tan\theta\theta_v}.$$

The curves u=const. are spherical if

$$-\rho_2 = U_1 + \frac{U_2}{\sqrt{\mathcal{E}\mathcal{G}}}\frac{\partial\sqrt{\mathcal{G}}}{du},$$

where U_1 and U_2 are functions of u alone. Since in our notation $\mathcal{E} = q^2$ and $\mathcal{G} = p_1^2$, it follows that

$$\frac{1}{\sqrt{\mathcal{E}\mathcal{G}}}\frac{\partial\sqrt{\mathcal{G}}}{\partial u} = \frac{\phi_{uv} - \sec^2\phi\tan\theta\phi_u\theta_v - \tan\phi\sec^2\theta\theta_u\theta_v - \tan\phi\tan\theta\theta_{uv}}{\sin\phi\cos\theta\phi_v - \sin\phi\tan\phi\sin\theta\theta_v}.$$

By using (69) this becomes

$$= \frac{\tan\phi \cos\theta\, \theta_v}{\sin\phi \cos\theta\, \phi_v - \sin\phi \tan\phi \sin\theta\, \theta_v}$$

$$= \frac{\theta_v}{\cos\phi\, \phi_v - \sin\phi \tan\theta\, \theta_v}$$

$$= \frac{-\rho_2}{a}.$$

This proves that the curves $u = \text{const.}$ are spherical.

Since the lines of curvature are parametric, q_1 equals zero and the linear element (62) becomes $ds^2 = c^2\, du^2$. Hence the orthogonal surface of the congruence generated by the center of constant geodesic curvature is a curve and therefore a focal surface. It may now be geometrically shown that the curves $u = \text{const.}$ are spherical. Denoting the centers of geodesic curvature of $u = \text{const.}$ and $v = \text{const.}$ by A and B respectively, and the line of the congruence by AB, we see that for any curve $u = \text{const.}$, A remains fixed as v varies and M, the point of tangency of the tangent planes, must move so that it remains at a constant distance c from A. Since the tangent to $v = \text{const.}$ passes through A, the sphere on which $u = \text{const.}$ lies cuts the surface at right angles. Any surface orthogonal to the congruence AB is generated by a point P at a constant distance from A on AB. When $u = \text{const.}$, as v varies, P_1 generates a circle since it lies at a constant distance from A on AB, which is perpendicular to the curve generated by A. Hence any orthogonal surface of the associated normal congruence is the locus of spheres of constant radius whose centers are on the curve generated by A. These circles are lines of curvature.

The lines joining A and the corresponding center N of normal curvature of $v = \text{const.}$ form a normal congruence.* We desire to show that the congruence generated by AB and AN are the same. Any point P on AN has the coordinates

$$\left(-c + \frac{c\,t}{\sqrt{c^2 + \rho_1^2}},\ 0,\ \frac{\rho_1\, t}{\sqrt{c^2 + \rho_1^2}}\right),$$

where t is the distance from a to P and ρ_1 is the radius of normal curvature of $v = \text{const.}$ The direction cosines of AN are proportional to c, 0 and ρ_1. If the condition that P move perpendicular to AN for all displacements on

* *Comptes Rendus*, Vol. CXV, pp. 589–592.

the surface in the neighborhood of M is expressed, it is found that the resulting equation is satisfied by $t = \text{const.}$ Hence the curve generated by A is orthogonal to both congruences. All the lines AB and AN corresponding to any curve $u = \text{const.}$ thus lie in the same plane. These two congruences, which are differently defined, may be considered as the same, since any line of either is a line of the other. The common perpendicular to any two lines AB and AN is the tangent at A of the curve A generates. If ω is the angle between these two lines, it is easily shown that

$$\cos \omega = \frac{c^2}{\sqrt{(c^2 + R^2)\,(c^2 + \rho_1^2)}}.$$

Let us now obtain the equations of a surface on which $u = \text{const.}$ are lines of curvature and have constant geodesic curvature a. In order to do this let $x_0(u)$, $y_0(u)$ and $z_0(u)$ define the locus of centers of spheres of radius c, and $\alpha(u, v)$, $\beta(u, v)$ and $\gamma(u, v)$ be the direction cosines of the radii of the spherical curves $u = \text{const.}$ Then the equations of the desired surface are

$$x = x_0 + c\alpha, \quad y = y_0 + c\beta, \quad z = z_0 + c\gamma. \tag{72}$$

Since these radii are tangent to $v = \text{const.}$, we have

$$x_u = x_{0u} + c\alpha_u = r\alpha, \tag{73}$$

$$y_u = y_{0u} + c\beta_u = r\beta, \tag{74}$$

$$z_u = z_{0u} + c\gamma_u = r\gamma, \tag{75}$$

where r is a factor of proportionality. To these must be added

$$\alpha^2 + \beta^2 + \gamma^2 = 1. \tag{76}$$

Multiplying equations (73), (74) and (75) by α, β and γ respectively, and adding, we have $r = \Sigma x_{0u}\alpha$. Hence, if u measures the arc of the curve of centers and ω is the angle between a tangent to $v = \text{const.}$ and the corresponding tangent to this curve, it follows that $r = \cos\omega$. It is necessary to prove that the equations (73), (74), (75) and (76) are consistent. To do so eliminate γ from (73), (74) and (75) by means of (76), and then, in the equation resulting from the last elimination, substitute the values of α_u and β_u from the equations obtained by the first two eliminations. The equality so obtained is an identity. This shows that the three equations in α and β are not independent and that these quantities may be determined by the following:

$$c\alpha_u = x_{0u}(\alpha^2 - 1) + y_{0u}\alpha\beta + z_{0u}\alpha\sqrt{1 - \alpha^2 - \beta^2},$$

$$c\beta_u = x_{0u}\alpha\beta + y_{0u}(\beta^2 - 1) + z_{0u}\beta\sqrt{1 - \alpha^2 - \beta^2}.$$

Therefore for any given set of initial coordinates α_0, β_0, u_0 and v_0 for which $\alpha_0^2 + \beta_0^2 \leqq 1$ and x_0, y_0 and z_0 are analytic, there exist solutions $\alpha(u, v)$ and $\beta(u, v)$ which, for $u = u_0$, reduce to functions $\phi_1(v)$ and $\phi_2(v)$ respectively, and for $u = u_0$ and $v = v_0$ reduce to α_0 and β_0. After α and β have been determined, γ is known and hence also the equation of the surface.

APPENDIX.

Conformal Representation on the Plane of Systems of Curves of Constant Geodesic Curvature that lie on a Pseudospherical Surface.

If the linear element of a pseudospherical surface is

$$ds^2 = c^2 (du^2 + \varepsilon^{2u} dv^2),$$

Darboux has shown, by means of the transformation*

$$v = x, \qquad \varepsilon^{-u} = y,$$

that curves on the surface of geodesic curvatures a and b may be represented conformally on the upper half plane by the circles

$$(x - h)^2 + (y - l)^2 = \frac{l^2 c^2}{a^2}, \tag{I}$$

$$(x - m)^2 + (y - n)^2 = \frac{n^2 c^2}{b^2}. \tag{II}$$

Since systems of curves whose two families have geodesic curvatures a and b and intersect under a constant angle ω determine associated normal congruences, we wish to find the corresponding systems of circles in the plane that represent them conformally. If we suppose that the circles (I) and (II) intersect under an angle ω, it follows that

$$(h - m)^2 + (l - n)^2 = c^2 \left(\frac{l^2}{a^2} + \frac{n^2}{b^2} - 2\frac{l n}{a b}\cos\omega\right). \tag{III}$$

Previously it has been shown that

$$c^2 = \frac{a^2 + b^2 - 2ab\cos\omega}{\sin^2\omega}. \tag{IV}$$

This relation enables us to write equation (III) thus:

$$\left[\sin\omega\,(h - m) + l\left(\cos\omega - \frac{b}{a}\right) + n\left(\cos\omega - \frac{a}{b}\right)\right]\left[\sin\omega\,(h - m) - l\left(\cos\omega - \frac{b}{a}\right) - n\left(\cos\omega - \frac{a}{b}\right)\right] = 0. \tag{V}$$

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* Darboux, "Leçons," Vol. III, Ch. XI.

There is a single infinity of circles in each family of the system, and hence we will suppose that the pairs h and l and m and n are functions respectively of the parametrics s and t. Considering the first factor only of (V), differentiation with respect to s and t gives the two equations

$$\sin\omega\, h_s + \left(\cos\omega - \frac{b}{a}\right) l_s = 0,$$

$$\sin\omega\, m_t - \left(\cos\omega - \frac{a}{b}\right) n_t = 0.$$

Since $\frac{l_s}{h_s}$ and $\frac{n_t}{m_t}$ are the tangents of the angles the directions of motion of the centers (h, l) and (m, n) make with the x-axis, and are constants, the curves of centers must necessarily be straight lines. From the first factor of (V) we obtain, then, that the equations of the lines of centers must be of the form

$$h\sin\omega + l\left(\cos\omega - \frac{b}{a}\right) + K = 0, \qquad \text{(VI)}$$

$$m\sin\omega - n\left(\cos\omega - \frac{a}{b}\right) + K = 0, \qquad \text{(VII)}$$

where K is the constant that determines the particular system for given a and b. These lines intersect at the point $\left(-\frac{K}{\sin\omega}, 0\right)$ under an angle ω.

As corresponding curves of any two orthogonal systems, of the kind under consideration, that determine the same associated normal congruence intersect under a constant angle, we desire to prove that the same is true for corresponding circles of two orthogonal systems of circles whose lines of centers pass through the same point on the x-axis. Thus, for the present, assume that ω is a right angle and that we select two sets of values a_1 and b_1 and a_2 and b_2 for a and b. If we take the same value of K, say K_1, for both systems, the lines of centers of the circles will intersect on the axis of x at the same point, and the lines of centers of the circles corresponding to the curves of geodesic curvature a_1 and a_2 will intersect under some angle ω'. If we can show that a_1, a_2 and ω' satisfy a relation of the form (III), where

$$a_i^2 + b_i^2 = c^2, \qquad i = 1, 2,$$

the system of circles so obtained will correspond conformally to a system of curves on the surface which intersect under a constant ω', and hence will them-

selves intersect under the same angle. From the equations of the lines of centers it follows that

$$\cos\omega' = \frac{a_1 a_2 + b_1 b_2}{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}},$$

$$\therefore c^2 = \frac{a_1^2 + a_2^2 - 2 a_1 a_2 \cos\omega'}{\sin^2\omega'}.$$

Therefore the desired result is proved.

An examination of the pairs of equations (I) and (VI) and (II) and (VII) yields that all the circles of each family are tangent to each other at the point $\left(-\frac{K}{\sin\omega}, 0\right)$, that there are only two circles of infinite radius in the system and that through any point in the upper half plane there pass one and only one pair of circles of the system.

THEOREM. *That part of all systems of circles, lying in the upper half plane, which satisfies conditions* (I), (II), (V), (VI) *and* (VII), *where a particular value for* $\frac{K}{\sin\omega}$ *has been chosen, represents conformally all systems of curves of geodesic curvatures a and b on a pseudospherical surface, which determine the same associated normal congruence.*

UNIVERSITY OF PENNSYLVANIA, *October*, 1911.

A Theory of Geometrical Relations.—*Continued.**

By Arthur Richard Schweitzer.

Chapter VI.

On the Foundations of Grassmann's Extensive Algebra.

(I) Preliminary Remarks on the Genesis of Euclidean Geometry.

In Chapter V we have shown that by the introduction of suitable definitions the system ${}^3K_3$ (and therefore the system ${}^3R_3$) implies a "geometry of position" which is defined by Hilbert's axioms I and II of his "Foundations of Geometry." This geometry may also be defined by Pasch's axioms in §§ 1–2 of his "Vorlesungen über neuere Geometrie" or Peano's axioms I–XVI in his "I principii di Geometria."†

The system ${}^3K_3$ may be made *complete*‡ for euclidean geometry in a manner that is purely descriptive.§ The procedure necessary for such completion in the case of the "geometries of position" due to Pasch and Peano has been admirably sketched by Burali-Forti‖ and is easily adapted to the system ${}^3K_3$. Burali-Forti says:

"Le système de postulats de M. Pasch, ou celui de M. Peano, donnent toute la Géométrie de position..... Pour obtenir la théorie des parallèles, il faut joindre aux postulats précédentes celui de la continuité due à M. Dedekind et qu'on peut exprimer¶ en disant que 'Un ensemble convexe de points, contenu sur un segment, est lui-même un segment.'

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* This article is a continuation of one under the same title in Vol. XXXI (1909), pp. 365–410; reference to the latter article will be made by pages under the notation "T. G. R."

† Compare also, G. Peano, "Sui fondamenti della Geometria," *Rivista di Matematica*, Vol. IV (1894), p. 51; A. B. Kempe, *Proceedings of the London Math. Soc.*, Vol. XXI (1890), pp. 176–179; F. Enriques, *Encyklopaedie der Math. Wiss.*, Band III, Heft 1, pp. 21–27, 70–72.

‡ Cf. F. Riesz, "Die Genesis des Raumbegriffs," *Math. und Naturw. Berichte aus Ungarn* (1905–1906), p. 309.

§ Cf. A. B. Kempe, *l. c.*; F. Enriques, *l. c.*, p. 34; A. R. Schweitzer, *Bull. of the Am. Math. Soc.*, Vol. XV (1909), pp. 314–315.

‖ "Verhandlungen des Ersten Internationalen Mathematiker-Kongresses," Leipzig, 1898, pp. 248–249.

¶ G. Peano, "I principii," pp. 9, 10, 28, 39.

"La definition de demi-droites parallèles donnée* on demontre que deux demi-droites parallèles sont complanaires; qu'un point externe à la demi-droite a est l'origine d'une seule demi-droite parallèle à a; que la relation exprimée par le mot parallèle est réflexive, symétrique et transitive..... Il en résulte aussi que: par un point extérieur à une droite on peut mener une ou deux droites parallèles à la droite même, et qu'un de ces deux cas doit toujours se vérifier. Pour obtenir la Géométrie d'Euclide il faut poser le postulat suivant: 'Quelle que soit la droite a il existe un point b extérieur à la droite a par lequel passe une seule parallèle à la droite a'; car on déduit aisément qu'une seule parallèle à la droite a passe aussi par un point quelconque."

For further details of the development corresponding to the preceding position we refer to Burali-Forti, *l. c.*; E. H. Moore, *Science*, New Series, Vol. XVII, p. 403,† and finally the "Programm" of F. Klein, Erlangen, 1872, reprinted in the *Mathematische Annalen*, Vol. XLIII (1893).‡

To complete, therefore, the system ${}^3K_3$ for euclidean geometry, we must first add to this system an axiom of Dedekind continuity. The latter may be expressed in a variety of ways;§ a form attractive for our purposes is an axiom with the (geometric) content of the so-called Heine-Borel theorem,‖ stated for three dimensions. Finally, a parallel axiom corresponding to that given by Burali-Forti is easily stated. In connection with this axiom it is desirable to give the following definitions:¶

1. *The line* $\delta\xi'$ *does not intersect the* (*coplanar*) *line* $a\beta$ means, $a\beta\delta\xi' K_2$ and the existence of ξ, ε such that $\delta\xi'\xi\Delta K_1$ and $a\beta\delta\varepsilon K$ implies $a\beta\delta\varepsilon K a\beta\xi\varepsilon$.

2. *The line* $\delta\xi'$ *does not intersect the plane* $a\beta\gamma$ means, $a\beta\gamma\delta K$ and the existence of ξ such that $\delta\xi'\xi\Delta K_1$ implies $a\beta\gamma\delta K a\beta\gamma\xi$.

3. *The plane* $\delta\xi'\xi''$ *does not intersect the plane* $a\beta\gamma$ means, $a\beta\gamma\delta K$, $\delta\xi'\xi'' a K$ and the existence of ξ such that $\delta\xi'\xi''\xi K_2$ implies $a\beta\gamma\delta K a\beta\gamma\xi$.

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* G. Peano, "I principii," pp. 38, 39.

† Cf. O. Veblen, *Trans. Am. Math. Soc.*, Vol. V (1904), pp. 343–384.

‡ Cf. Fano, *Encyklopaedie der Math. Wiss.*, Band III, Heft 2, p. 295, etc., and the remarks of the editors of Grassmann's *Gesammelte Werke*, Band I, 1, p. 406, and Band I, 2, pp. 434, 436, 438.

§ See, for instance, O. Veblen, *l. c.*, pp. 346, 348; F. Riesz, *l. c.*, pp. 318–319; "Atti del IV Congresso Internazionale dei Matematici," Vol. II (1909), pp. 18–24; R. L. Moore, *Trans. Am. Math. Soc.*, Vol. IX (1908), pp. 488–489.

‖ Cf. E. Borel, *Journal de Mathématiques Pures et Appliquées*, 1903, p. 329; *Comptes Rendus*, 1905, p. 298; O. Veblen, *Bull. of the Am. Math. Soc.*, 1904, p. 436.

¶ For the definition of the symbols used, see "T. G. R." pp. 375–376, 398.

(II) METRICAL COMPLETION* OF THE SYSTEM ${}^3K_3$ FOR EUCLIDEAN GEOMETRY. EXPLICIT INTRODUCTION OF THE NUMBER SYSTEM. NUMERICAL DERIVATION OF POINTS.†

Although the above observations are important in showing that the outer product of Grassmann may serve as a generating relation for euclidean geometry even when divested very largely of its metrical connotations, yet in our opinion this treatment can not be regarded as conforming to the basal geometric position of Grassmann in his "Ausdehnungslehre" of 1844. Namely, we assume that this position finds its most suitable expression through the use of the outer product (as an indefinable) in connection with the early, fundamental introduction of the number system and the deducing of properties by means of the "numerical derivation" of geometric elements.‡

In the following sections we attempt, therefore, a more properly grassmannian development of euclidean geometry.

§ 1. *Quotient Relations: The Classes* $\mathfrak{K}$, $\breve{\mathfrak{K}}$, $\bar{\mathfrak{K}}$.

Associated with the system ${}^3K_3$§ is the system ${}^3\breve{K}_3$ or *conjugate*‖ of ${}^3K_3$. The latter system is phrased exclusively in terms of the conjugate relation $\breve{K}$ defined as follows: $\alpha\beta\gamma\delta\breve{K}\xi\eta\zeta\tau$ if, and only if, $\alpha\beta\gamma\delta K\eta\xi\zeta\tau$. Thus the validity of the system ${}^3K_3$ implies the validity of ${}^3\breve{K}_3$ and conversely.¶

More generally, on the basis of the system ${}^3K_3$, we define two tetrads $\alpha\beta\gamma\delta$, $\xi\eta\zeta\tau$ to be in the *quotient relation*, written $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$, if, and only if, $\alpha\beta\gamma\delta K\xi\eta\zeta\tau$ or $\alpha\beta\gamma\delta\breve{K}\xi\eta\zeta\tau$ or ($\alpha\beta\gamma\delta\bar{K}$ and $\xi\eta\zeta\tau K$); corresponding to the terms of the latter disjunction, the relation $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is said to be in the class $\mathfrak{K}$ or $\breve{\mathfrak{K}}$ or $\bar{\mathfrak{K}}$. The tetradic terms of the preceding quotient relation give rise, through our subsequent developments, to *symmetric*

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* As is easily recognized, this completion differs from that under (I) in *relativity*; the latter completion may be said to be *less absolute* than the former.

† In this part we work out, in more exact detail, a position we have briefly indicated in the *Bull. of the Am. Math. Soc.*, Vol. XV (1908), p. 81.

‡ Cf. Grassmann, *Werke*, Vol. I, 1, p. 10, paragraph 1; p. 138, note 1; F. Caspary, *Bull. des Sciences Math.*, Vol. XIII (1889), pp. 203, 240; G. Peano, *Atti di Torino*, Vol. XXXI (1895–1896), pp. 952–975; Vol. XXXIII (1897–1898), p. 534, concluding remark.

§ Cf. "T. G. R.," pp. 395–396.

‖ With this term compare the "obverse element" of A. B. Kempe, *l. c.*, p. 155; J. Royce, *Trans. Am. Math. Soc.*, Vol. VI (1905), pp. 357, 365; and the "semi-space" of Laguerre, *Oeuvres*, Tome II (1905), pp. 592–604 and 620, note. See also, "T. G. R.", p. 376, note.

¶ Use is here made of a logical postulate analogous to that of Russell, "Principles of Mathematics," p. 25, paragraph 2, and p. 97, § 95, (2). Cf. also "T. G. R.," p. 375.

products, as distinguished from *alternating* or *outer* products, which we define below.* Grassmann† has used a relation $\frac{A_1}{A}$ analogous to our quotient relation in the statement

$$A_1 = \frac{A_1}{A} \cdot A,$$

where A and A_1 are elementary (spatial) magnitudes which are "gleichartig" ($A \neq 0$). From our viewpoint, Grassmann's relation is associated primarily with the system ${}^3R_3$ rather than ${}^3K_3$.‡

Axioms 1–16 of the system ${}^3K_3$, as phrased in terms of quotient relations, we shall denote by the notations G_1–G_{16} respectively. By G_{17} we denote correspondingly an axiom which ensures that the system ${}^3K_3$ possesses, on the basis of G_1–G_{16}, the property of Dedekind continuity; suitable forms for this axiom are easily derived from those cited under (I).

§ 2. *Identification of the Classes* $\mathfrak{R}$, $\breve{\mathfrak{R}}$, $\bar{\mathfrak{R}}$ *with Systems of Real Numbers.*

The classes $\mathfrak{R}$, $\breve{\mathfrak{R}}$, $\bar{\mathfrak{R}}$ are respectively identified with systems of real numbers by means of axioms which we denote by G_{18_1}, G_{18_2}, G_{18_3}. Throughout these axioms we shall denote by "$\theta_1\theta_2\theta_3\theta_4/\theta_1'\theta_2'\theta_3'\theta_4' = q$" that the real number q *corresponds uniquely* § to the quotient relation $\theta_1\theta_2\theta_3\theta_4/\theta_1'\theta_2'\theta_3'\theta_4'$.

Axiom G_{18_1}.

1. If the quotient relations ‖ $\theta_1\theta_2\theta_3\theta_4/\alpha_0\beta_0\gamma_0\delta_0$, $\theta_2\theta_3\theta_1\theta_4/\alpha_0\beta_0\gamma_0\delta_0$, $\theta_3\theta_4\theta_1\theta_2/\alpha_0\beta_0\gamma_0\delta_0$ are in the class $\mathfrak{R}$, then $\theta_1\theta_2\theta_3\theta_4/\alpha_0\beta_0\gamma_0\delta_0 = k^2$, $\theta_2\theta_3\theta_1\theta_4/\alpha_0\beta_0\gamma_0\delta_0 = k^2$, $\theta_3\theta_4\theta_1\theta_2/\alpha_0\beta_0\gamma_0\delta_0 = k^2$.

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* Cf. "T. G. R.," pp. 376, 394.

† *Gesammelte Werke*, Vol. I, 1, pp. 41 (§ 8), 123, 127–129, 163 (§ 97), 303.

‡ Developments similar to those given in this and the subsequent sections might be based directly on the system ${}^3R_3$; cf. "T. G. R.," p. 377, note.

§ For the purpose of this correspondence the quotient relation is particularized in so far as it is conceived to hold between specific tetrads; to the effective relation in its generality, *i. e.*, to the class of particularized relations, corresponds a class consisting of some or all real numbers; compare B. Russell, "The Principles of Mathematics," pp. 161, 432–433; Grassmann, *Werke*, Vol. I, 1, p. 123, note; p. 138, note 1; pp. 343–345; G. Peano, *Formulaire Mathématique* (1902–03), p. 338: "$u/\psi\varepsilon q$"; p. 260: "$u \times v \varepsilon q$." Grassmann's position (with which Peano seems in harmony) is presumably that the quotients of cospatial magnitudes of the same grade (*i. e.*, "gleichartige Grössen") obey the laws that govern any system of some or all real numbers (cf. *Werke*, Vol. I, 1, p. 134, note), and in accordance with this position it would seem desirable for us to subject our quotient relations postulationally to such laws.

‖ By the notation $\alpha_0\beta_0\gamma_0\delta_0$ we designate in axioms G_{18_1}, G_{18_2}, G_{18_3} the same fixed tetrad (in the K relation to itself); this tetrad may conveniently be identified with the tetrad of axiom 2 of the system ${}^3K_3$, Chapter V, p. 395. The symbol k^2 denotes a positive number.

2. If $\theta_1\theta_2\theta_3\theta_4/\alpha_0\beta_0\gamma_0\delta_0$ and $\alpha_0\beta_0\gamma_0\delta_0/\theta_1\theta_2\theta_3\theta_4$ are in the class $\mathfrak{R}$, then $\theta_1\theta_2\theta_3\theta_4/\alpha_0\beta_0\gamma_0\delta_0 = k^2$ implies $\alpha_0\beta_0\gamma_0\delta_0/\theta_1\theta_2\theta_3\theta_4 = 1/k^2$.

3. If $\theta_1\theta_2\theta_3\theta_4/\alpha_0\beta_0\gamma_0\delta_0$ and $\alpha_0\beta_0\gamma_0\delta_0/\theta_1'\theta_2'\theta_3'\theta_4'$ and $\theta_1\theta_2\theta_3\theta_4/\theta_1'\theta_2'\theta_3'\theta_4'$ are in the class $\mathfrak{R}$, then $\theta_1\theta_2\theta_3\theta_4/\alpha_0\beta_0\gamma_0\delta_0 = k^2$ and $\alpha_0\beta_0\gamma_0\delta_0/\theta_1'\theta_2'\theta_3'\theta_4' = 1/k'^2$ imply $\theta_1\theta_2\theta_3\theta_4/\theta_1'\theta_2'\theta_3'\theta_4' = k^2/k'^2$.

Axiom G_{18_2}.

1. If $\beta\alpha\gamma\delta/\xi\eta\zeta\tau$ is in the class $\mathfrak{R}$, then $\beta\alpha\gamma\delta/\xi\eta\zeta\tau = k^2$ implies $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$.

2. If $\alpha\beta\gamma\delta/\eta\xi\zeta\tau$ is in the class $\mathfrak{R}$, then $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$ implies $\alpha\beta\gamma\delta/\eta\xi\zeta\tau = k^2$.

Axiom G_{18_3}.

1. If $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ is in the class $\bar{\mathfrak{R}}$, then $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = 0$.

2. If $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ is in the class $\bar{\mathfrak{R}}$ and $\alpha_0\beta_0\gamma_0\delta_0/\xi\eta\zeta\tau$ is in the class $\mathfrak{R}$ or $\breve{\mathfrak{R}}$, then $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = 0$ and $\alpha_0\beta_0\gamma_0\delta_0/\xi\eta\zeta\tau = k^2$ or $-k^2$ imply $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = 0$.

The classes $\mathfrak{R}$, $\breve{\mathfrak{R}}$, $\bar{\mathfrak{R}}$ as satisfying axioms G_{18_1}–G_{18_3} we denote by $\mathfrak{R}_+$, $\mathfrak{R}_-$, $\mathfrak{R}_0$ respectively. We proceed to prove the following theorems:

Theorem 1. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in the class $\mathfrak{R}$, then $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$ implies $\beta\alpha\gamma\delta/\eta\xi\zeta\tau = k^2$.

For by G_{18_2}, 1, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$ implies $\beta\alpha\gamma\delta/\xi\eta\zeta\tau = -k^2$; therefore by G_{18_2}, 2, $\beta\alpha\gamma\delta/\eta\xi\zeta\tau = k^2$.

Theorem 2. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in the class $\mathfrak{R}$, then $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$ implies $\alpha\beta\gamma\delta/\eta\xi\zeta\tau = -k^2$.

By theorem 1, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$ implies $\beta\alpha\gamma\delta/\eta\xi\zeta\tau = k^2$; therefore by G_{18_2}, 1, $\alpha\beta\gamma\delta/\eta\xi\zeta\tau = -k^2$.

Theorem 3. If $\beta\alpha\gamma\delta/\xi\eta\zeta\tau$ is in the class $\mathfrak{R}$, then $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$ implies $\beta\alpha\gamma\delta/\xi\eta\zeta\tau = k^2$.

By G_{18_2}, 2, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$ implies $\alpha\beta\gamma\delta/\eta\xi\zeta\tau = k^2$; hence by theorem 1, $\beta\alpha\gamma\delta/\xi\eta\zeta\tau = k^2$.

Theorem 4. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in the class $\mathfrak{R}$, then $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$.

By hypothesis, $\alpha\beta\gamma\delta/\alpha\beta\gamma\delta$ and $\xi\eta\zeta\tau/\xi\eta\zeta\tau$ are both in the class $\mathfrak{R}$. Therefore, by axiom G_{16}, ($\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ or $\beta\alpha\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$) and ($\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ or $\eta\xi\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$) are in $\mathfrak{R}$. Four cases arise.

CASE I. $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ and $\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ are in $\mathfrak{R}$.

Hence by axiom G_{18_1}, 1, we may put $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = k_1^2$ and $\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0 = k_2^2$. But if $\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ is in $\mathfrak{R}$, so also is $\alpha_0\beta_0\gamma_0\delta_0/\xi\eta\zeta\tau$,

and by axiom G_{18_1}, 2, $\alpha_0\beta_0\gamma_0\delta_0/\xi\eta\zeta\tau = 1/k_2^2$. Hence by G_{18_1}, 3, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k_1^2/k_2^2$, which verifies the theorem.

CASE II. $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ is in $\mathfrak{R}$ and $\eta\xi\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ is in $\mathfrak{R}$.

Then $\alpha\beta\gamma\delta/\eta\xi\zeta\tau$ is in $\mathfrak{R}$ and this contradicts the hypothesis.

CASE III. $\beta\alpha\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ and $\eta\xi\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ are in $\mathfrak{R}$.

Then, as under Case I, we show that $\beta\alpha\gamma\delta/\eta\xi\zeta\tau = k^2$. Therefore, by theorem 1, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$.

CASE IV. $\beta\alpha\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ is in $\mathfrak{R}$ and $\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ is in $\mathfrak{R}$.

Therefore $\beta\alpha\gamma\delta/\xi\eta\zeta\tau$ is in $\mathfrak{R}$, which contradicts the hypothesis.

Theorem 5. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in the class $\breve{\mathfrak{R}}$, then $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$.

Since $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in $\breve{\mathfrak{R}}$, $\beta\alpha\gamma\delta/\xi\eta\zeta\tau$ is in $\mathfrak{R}$ (cf. § 1). Hence by theorem 4, $\beta\alpha\gamma\delta/\xi\eta\zeta\tau = k^2$. Therefore by axiom G_{18_2}, 1, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$.

Theorem 6. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in the class $\bar{\mathfrak{R}}$, then $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = 0$.

Since $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in the class $\bar{\mathfrak{R}}$, $\alpha\beta\gamma\delta/\alpha\beta\gamma\delta$ is not in the class $\mathfrak{R}$, but $\xi\eta\zeta\tau/\xi\eta\zeta\tau$ is in the class $\mathfrak{R}$. Hence, by axiom G_{16}, $\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ or $\eta\xi\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ is in the class $\mathfrak{R}$; that is, $\alpha_0\beta_0\gamma_0\delta_0/\xi\eta\zeta\tau$ or $\alpha_0\beta_0\gamma_0\delta_0/\eta\xi\zeta\tau$ is in the class $\mathfrak{R}$. If $\alpha_0\beta_0\gamma_0\delta_0/\xi\eta\zeta\tau$ is in $\mathfrak{R}$, then, by theorem 4, $\alpha_0\beta_0\gamma_0\delta_0/\xi\eta\zeta\tau = k^2$. If $\alpha_0\beta_0\gamma_0\delta_0/\eta\xi\zeta\tau$ is in $\mathfrak{R}$, then $\alpha_0\beta_0\gamma_0\delta_0/\xi\eta\zeta\tau$ is in $\breve{\mathfrak{R}}$ and, by theorem 5, $\alpha_0\beta_0\gamma_0\delta_0/\xi\eta\zeta\tau = -k^2$. Finally, by axiom G_{18_3}, 1, $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = 0$. Thus the hypotheses of axiom G_{18_3}, 2, are fulfilled and we have, therefore, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = 0$.

Theorem 7. The quotient relation $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in the class $\mathfrak{R}$, or $\breve{\mathfrak{R}}$, or $\bar{\mathfrak{R}}$ according as $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$, or $-k^2$, or 0.

Suppose that $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$. Then, by definition, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in one of the classes $\mathfrak{R}$, $\breve{\mathfrak{R}}$, $\bar{\mathfrak{R}}$. Now $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ can not be in the class $\breve{\mathfrak{R}}$; for if so, by theorem 5, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$. Similarly, by theorem 6, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ can not be in the class $\bar{\mathfrak{R}}$. Hence $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in the class $\mathfrak{R}$.

The remaining cases are proved in an analogous manner.

Theorem 8. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$, then $\xi\eta\zeta\tau/\alpha\beta\gamma\delta = 1/k^2$.

As in theorem 4 we find two essential cases.

CASE I. $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = k_1^2$, $\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0 = k_2^2$.

Therefore by G_{18_1}, 2, $\alpha_0\beta_0\gamma_0\delta_0/\alpha\beta\gamma\delta = 1/k_1^2$, $\alpha_0\beta_0\gamma_0\delta_0/\xi\eta\zeta\tau = 1/k_2^2$. Hence by G_{18_1}, 3, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k_1^2/k_2^2$, $\xi\eta\zeta\tau/\alpha\beta\gamma\delta = k_2^2/k_1^2$. Thus, since we may take $k^2 = k_1^2/k_2^2$, the theorem is verified.

CASE II. $\beta\alpha\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = k_1^2$, $\eta\xi\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0 = k_2^2$.

Then as under Case I we show that $\beta\alpha\gamma\delta/\eta\xi\zeta\tau = k_1^2/k_2^2$, $\eta\xi\zeta\tau/\beta\alpha\gamma\delta = k_2^2/k_1^2$. That is, by theorem 1, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k_1^2/k_2^2$, $\xi\eta\zeta\tau/\alpha\beta\gamma\delta = k_2^2/k_1^2$, and thus the theorem is verified.

Theorem 9. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$ and $\xi\eta\zeta\tau/\alpha'\beta'\gamma'\delta' = 1/k'^2$, then $\alpha\beta\gamma\delta/\alpha'\beta'\gamma'\delta' = k^2/k'^2$.

We have either ($\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ and $\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$) or ($\beta\alpha\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ and $\eta\xi\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$) in $\mathfrak{R}$. Similarly, ($\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ and $\alpha'\beta'\gamma'\delta'/\alpha_0\beta_0\gamma_0\delta_0$) or ($\eta\xi\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ and $\beta'\alpha'\gamma'\delta'/\alpha_0\beta_0\gamma_0\delta_0$) are in $\mathfrak{R}$. Hence ($\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ and $\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ and $\alpha'\beta'\gamma'\delta'/\alpha_0\beta_0\gamma_0\delta_0$) or ($\beta\alpha\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ and $\eta\xi\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ and $\beta'\alpha'\gamma'\delta'/\alpha_0\beta_0\gamma_0\delta_0$) are in $\mathfrak{R}$.

In the former case we may put, by G_{18_1}, 1, $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = l^2$, $\alpha'\beta'\gamma'\delta'/\alpha_0\beta_0\gamma_0\delta_0 = l'^2$, $\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0 = m^2$. Consequently, by G_{18_1}, 2, 3, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = l^2/m^2$, $\xi\eta\zeta\tau/\alpha'\beta'\gamma'\delta' = m^2/l'^2$, $\alpha\beta\gamma\delta/\alpha'\beta'\gamma'\delta' = l^2/l'^2$. Since we may take $l^2/m^2 = k^2$, $m^2/l'^2 = 1/k'^2$ and therefore $l^2/l'^2 = k^2/k'^2$, the theorem is verified.

The remaining case is proved, through theorem 1, in a similar manner.

Theorem 10. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$, then $\alpha\beta\gamma\delta/\alpha\beta\gamma\delta = 1$.*

By theorem 8, if $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$, then $\xi\eta\zeta\tau/\alpha\beta\gamma\delta = 1/k^2$. Hence by theorem 9, $\alpha\beta\gamma\delta/\alpha\beta\gamma\delta = 1$.

On the preceding proof we refer to Peano, *Atti di Torino*, Vol. XXXIII, p. 513, axioms 1–3; *Formulaire Mathématique* (1903), p. 3, § 1, and p. 315; Russell, "The Principles of Mathematics," p. 219, § 209, and p. 163.

Theorem 11. If $\alpha\beta\gamma\delta/\alpha\beta\gamma\delta = 1$, then $\beta\gamma\alpha\delta/\alpha\beta\gamma\delta = 1$ and $\gamma\delta\alpha\beta/\alpha\beta\gamma\delta = 1$.

By axioms G_{16} and G_{18_1}, 1, we have $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ or $\beta\alpha\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = k^2$. In the former case, by G_{18_1}, 1, $\beta\gamma\alpha\delta/\alpha_0\beta_0\gamma_0\delta_0 = k^2$ and $\gamma\delta\alpha\beta/\alpha_0\beta_0\gamma_0\delta_0 = k^2$. Hence by G_{18_1}, 2, 3, $\beta\gamma\alpha\delta/\alpha\beta\gamma\delta = 1$ and $\gamma\delta\alpha\beta/\alpha\beta\gamma\delta = 1$. In the latter case we have by G_{18_1}, 1, $\gamma\beta\alpha\delta/\alpha_0\beta_0\gamma_0\delta_0 = k^2$ and $\delta\gamma\alpha\beta/\alpha_0\beta_0\gamma_0\delta_0 = k^2$, and therefore, by G_{18_1}, 2, 3, $\gamma\beta\alpha\delta/\beta\alpha\gamma\delta = 1$ and $\delta\gamma\alpha\beta/\beta\alpha\gamma\delta = 1$; that is, by theorem 1, $\beta\gamma\alpha\delta/\alpha\beta\gamma\delta = 1$ and $\gamma\delta\alpha\beta/\alpha\beta\gamma\delta = 1$.

Theorem 12. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$, then $\beta\gamma\alpha\delta/\xi\eta\zeta\tau = k^2$ and $\gamma\delta\alpha\beta/\xi\eta\zeta\tau = k^2$.

By theorem 4, since $\beta\gamma\alpha\delta/\xi\eta\zeta\tau$ and $\gamma\delta\alpha\beta/\xi\eta\zeta\tau$ are in the class $\mathfrak{R}$, $\beta\gamma\alpha\delta/\xi\eta\zeta\tau = l^2$ and $\gamma\delta\alpha\beta/\xi\eta\zeta\tau = m^2$. Hence by theorem 9, $\beta\gamma\alpha\delta/\alpha\beta\gamma\delta = l^2/k^2$ and $\gamma\delta\alpha\beta/\alpha\beta\gamma\delta = m^2/k^2$. But by theorem 10, $\alpha\beta\gamma\delta/\alpha\beta\gamma\delta = 1$, and by theorem 11, $\beta\gamma\alpha\delta/\alpha\beta\gamma\delta = 1$ and $\gamma\delta\alpha\beta/\alpha\beta\gamma\delta = 1$. Therefore, since the correspondence is unique, we have $l^2 = k^2$ and $m^2 = k^2$, and thus the theorem is verified.

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* If a tetrad is in the quotient relation to itself, then this quotient relation is necessarily in the class $\mathfrak{R}$. For by definition, $\alpha\beta\gamma\delta/\alpha\beta\gamma\delta$ is in $\mathfrak{R}$ or $\breve{\mathfrak{R}}$ or $\overline{\mathfrak{R}}$. If $\alpha\beta\gamma\delta/\alpha\beta\gamma\delta$ is in $\overline{\mathfrak{R}}$, then $\alpha\beta\gamma\delta\overline{K}$ and $\alpha\beta\gamma\delta K$. If $\alpha\beta\gamma\delta/\alpha\beta\gamma\delta$ is in $\breve{\mathfrak{R}}$, then $\alpha\beta\gamma\delta K\beta\alpha\gamma\delta$; that is, by G_{11}, $\alpha\beta\gamma\delta K\alpha\beta\gamma\delta$, which is impossible by G_4.

Theorem 13. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$, then $\xi\eta\zeta\tau/\alpha\beta\gamma\delta = -1/k^2$.

Since $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$, by theorem 3, $\beta\alpha\gamma\delta/\xi\eta\zeta\tau = k^2$; hence by theorem 8, $\xi\eta\zeta\tau/\beta\alpha\gamma\delta = 1/k^2$; that is, by theorem 2, $\xi\eta\zeta\tau/\alpha\beta\gamma\delta = -1/k^2$.

Theorem 14. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$ and $\xi\eta\zeta\tau/\alpha'\beta'\gamma'\delta' = -1/k'^2$, then $\alpha\beta\gamma\delta/\beta'\alpha'\gamma'\delta' = -k^2/k'^2$.

By theorem 3, $\beta\alpha\gamma\delta/\xi\eta\zeta\tau = k^2$ and by G_{18_2}, 2, $\xi\eta\zeta\tau/\beta'\alpha'\gamma'\delta' = 1/k'^2$. Therefore, by theorem 9, $\beta\alpha\gamma\delta/\beta'\alpha'\gamma'\delta' = k^2/k'^2$; that is, by G_{18_2}, 1, $\alpha\beta\gamma\delta/\beta'\alpha'\gamma'\delta' = -k^2/k'^2$.

Theorem 14'. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$, then $\beta\alpha\gamma\delta/\eta\xi\zeta\tau = -k^2$.

By axiom G_{18_2}, 2, $\alpha\beta\gamma\delta/\eta\xi\zeta\tau = k^2$; hence by theorem 1, $\beta\alpha\gamma\delta/\xi\eta\zeta\tau = k^2$. Therefore, by theorem 2, $\beta\alpha\gamma\delta/\eta\xi\zeta\tau = -k^2$.

Theorem 15. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$, then $\alpha\beta\gamma\delta/\beta\alpha\gamma\delta = -1$.*

By theorem 13, $\xi\eta\zeta\tau/\alpha\beta\gamma\delta = -1/k^2$. Hence by theorem 14, $\alpha\beta\gamma\delta/\beta\alpha\gamma\delta = -1$.

Theorem 16. If $\alpha\beta\gamma\delta/\beta\alpha\gamma\delta = -1$, then $\beta\gamma\alpha\delta/\beta\alpha\gamma\delta = -1$ and $\gamma\delta\alpha\beta/\beta\alpha\gamma\delta = -1$.

Since $\alpha\beta\gamma\delta/\beta\alpha\gamma\delta = -1$, we have by axiom G_{18_2}, 2, $\alpha\beta\gamma\delta/\alpha\beta\gamma\delta = 1$, and hence by theorem 11, $\beta\gamma\alpha\delta/\alpha\beta\gamma\delta = 1$ and $\gamma\delta\alpha\beta/\alpha\beta\gamma\delta = 1$. Therefore, by theorem 2, $\beta\gamma\alpha\delta/\beta\alpha\gamma\delta = -1$ and $\gamma\delta\alpha\beta/\beta\alpha\gamma\delta = -1$.

Theorem 17. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$, then $\beta\gamma\alpha\delta/\xi\eta\zeta\tau = -k^2$ and $\gamma\delta\alpha\beta/\xi\eta\zeta\tau = -k^2$.

Compare the proof of theorem 16 and use theorem 12 instead of theorem 11.

If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in the class $\bar{\mathfrak{R}}$, then, as we have seen, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = 0$, and conversely. By the statement $\alpha\beta\gamma\delta = 0$ we mean, there exists $\xi'\eta'\zeta'\tau'$ such that $\alpha\beta\gamma\delta/\xi'\eta'\zeta'\tau' = 0$. Thus $\alpha\beta\gamma\delta\overline{K}$ and $\alpha\beta\gamma\delta = 0$ are equivalent. Therefore, $\alpha\beta\gamma\delta K$ and $\alpha\beta\gamma\delta \neq 0$ are equivalent. Further, $\alpha\beta\gamma \neq 0$ means that there exists a point δ such that $\alpha\beta\gamma\delta \neq 0$; $\alpha\beta\gamma = 0$ means that there exists no such δ; $\alpha\beta \neq 0$ means $\alpha \neq \beta$.†

If $\alpha\beta\gamma\delta \neq 0$, then $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = k^2$ or $\beta\alpha\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = k^2$. We define: $\alpha\beta\gamma\delta > 0$ means, $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = k^2$. $\alpha\beta\gamma\delta < 0$ means, $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = -k^2$. Thus, if $\alpha\beta\gamma\delta \neq 0$, then $\alpha\beta\gamma\delta > 0$ or $\alpha\beta\gamma\delta < 0$. In particular, $\alpha_0\beta_0\gamma_0\delta_0 > 0$. By a non-vanishing *alternating product* (Peano)

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* Compare theorem 10 and the corresponding note; we may show analogously that if the statement $\alpha\beta\gamma\delta/\beta\alpha\gamma\delta$ is valid, then $\alpha\beta\gamma\delta/\beta\alpha\gamma\delta$ is necessarily in the class $\bar{\mathfrak{R}}$.

† Cf. Grassmann, *Werke*, Vol. I, 1, § 119.

or *outer product* (Grassmann) we mean a tetrad which is greater [less] than zero. When $\alpha\beta\gamma\delta \neq 0$ we term the tetrad a non-vanishing *symmetric product.* Since for any two tetrads $\alpha'\beta'\gamma'\delta'$ and $\alpha''\beta''\gamma''\delta''$ we have the relations $\alpha'\beta'\gamma'\delta'/\alpha_0\beta_0\gamma_0\delta_0 = q'$ and $\alpha''\beta''\gamma''\delta''/\alpha_0\beta_0\gamma_0\delta_0 = q''$, $q', q'' \gtreqless 0$, and we may define $\alpha'\beta'\gamma'\delta' \gtreqless \alpha''\beta''\gamma''\delta''$ according as $q' \gtreqless q''$, symmetric products have the *character of magnitudes** with reference to a fixed tetrad.

It may, perhaps, be remarked here that Grassmann, in the "Ausdehnungslehre" of 1844, §§ 137–139 (cf. §§ 3–10), assumes that the outer product is *associative.*† The latter property is easily expressed in terms of our notations of Chapter III‡ by representing an outer product, say $\alpha_1\alpha_2\alpha_3\alpha_4$, by reflexive relations, $\alpha_1\alpha_2\alpha_3\alpha_4 R^{1111}_{1234}$, $\alpha_1\alpha_2(\alpha_3\alpha_4) R^{112}_{123}$, etc., and assuming that these relations are identical. The associativity of the outer product does not seem to enter very essentially into the discussion of the present chapter.

§ 3. *Numerical Derivation of Points.*

By the class $\mathfrak{Q}$ we mean the logical product of the classes $\mathfrak{K}_+$, $\mathfrak{K}_-$, $\mathfrak{K}_0$ (that is, the classes $\mathfrak{K}$, $\breve{\mathfrak{K}}$, $\bar{\mathfrak{K}}$ as conditioned by the axioms G_1–G_{17}, G_{18_1}, G_{18_2}, G_{18_3}). The class $\mathfrak{Q}$ consists, then, of *quotient relations* to each of which corresponds *uniquely* a real number which may be either positive, negative, or zero. As heretofore, the real numbers corresponding to elements of the class $\mathfrak{Q}$ will be designated by small letters of the Latin alphabet.

We proceed to deduce an arbitrary point of a descriptive m-space ($m = 1, 2, 3$) as *numerically derived*§ from $m+1$ independent‖ points of this space. For this purpose we further condition the class $\mathfrak{Q}$ by certain axioms, after setting up the following

Definition 1.

$\alpha\beta\gamma\delta = k\,\alpha'\beta'\gamma'\delta'$ means, there exists $\xi\eta\zeta\tau$ such that $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = m$, $\alpha'\beta'\gamma'\delta'/\xi\eta\zeta\tau = m'$, $m = k\,m'$ and k corresponds to some element of the class $\mathfrak{Q}$.

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* Cf. Grassmann, *Werke,* Vol. I, 1, pp. 161 (§ 96), 303, note; Veronese, "Grundzüge der Geometrie," p. 18, note 2; Russell, "The Principles of Mathematics," pp. 161, 163.

† Cf. Peano, "Formulaire Mathématique," (1903), p. 339.

‡ See "T. G. R.," p. 376; our Chapters II, III should be compared with Grassmann, *Werke,* Vol. I, 1, pp. 22–45.

§ Grassmann, *Werke,* Vol. I, 2, § 1.

‖ Grassmann, *Werke,* Vol. I, 1, § 107.

Thus if* $\alpha\beta\gamma\delta/\alpha'\beta'\gamma'\delta' = q$, then $\alpha\beta\gamma\delta = q\ \alpha'\beta'\gamma'\delta'$; if $\alpha'\beta'\gamma'\delta' \neq 0$, then conversely, $\alpha\beta\gamma\delta = q\ \alpha'\beta'\gamma'\delta'$ implies $\alpha\beta\gamma\delta/\alpha'\beta'\gamma'\delta' = q$. Again, if $\alpha\beta\gamma\delta = q\ \alpha'\beta'\gamma'\delta'$ and $\alpha'\beta'\gamma'\delta' = 0$, then $\alpha\beta\gamma\delta = 0$; but if $\alpha\beta\gamma\delta = 0$, then not necessarily $\alpha'\beta'\gamma'\delta' = 0$ unless $q \neq 0$. Also, if $\alpha\beta\gamma\delta = q\ \alpha'\beta'\gamma'\delta'$ and $q = 0$, then $\alpha\beta\gamma\delta = 0$. We notice that any tetrad $\alpha_1\beta_1\gamma_1\delta_1$ has the properties $\alpha_1\beta_1\gamma_1\delta_1 = 1\ \alpha_1\beta_1\gamma_1\delta_1$ and $\alpha_1\beta_1\gamma_1\delta_1 = -1\ \beta_1\alpha_1\gamma_1\delta_1$.

Hereafter we shall write instead of $\alpha\beta\gamma\delta = -1\ \alpha'\beta'\gamma'\delta'$ and $\alpha\beta\gamma\delta = 1\ \alpha'\beta'\gamma'\delta'$ the expressions, $\alpha\beta\gamma\delta = -\alpha'\beta'\gamma'\delta'$ and $\alpha\beta\gamma\delta = \alpha'\beta'\gamma'\delta'$, respectively.

As final axioms of this chapter we give

Axioms G_{19}–G_{23}.

G_{19}. If $\alpha\beta\xi = 0$, $\alpha \neq \beta \neq \xi \neq \alpha$, and there exist two points δ_0', δ_0'' such that $\alpha\beta\delta_0'\delta_0'' \neq 0$ and $\xi\beta\delta_0'\delta_0'' = k_0\,\alpha\xi\delta_0'\delta_0''$, then the existence of δ', δ'' such that $\alpha\beta\delta'\delta'' \neq 0$ and $\xi\beta\delta'\delta'' = k\,\alpha\xi\delta'\delta''$ implies $k = k_0$.

G_{20}.* If $\alpha'\beta'\gamma'\delta' = q'\,\alpha_0\beta_0\gamma_0\delta_0$ and $\alpha''\beta''\gamma''\delta'' = q''\,\alpha_0\beta_0\gamma_0\delta_0$ and q', q'', $q' + q'' \neq 0$, then there exists $\xi\eta\zeta\tau$ such that $\xi\eta\zeta\tau = (q' + q'')\,\alpha_0\beta_0\gamma_0\delta_0$.

G_{21}. If $\alpha\beta\xi = 0$, $\alpha \neq \beta \neq \xi \neq \alpha$, then the existence of δ', δ'' such that $\alpha\beta\delta'\delta'' \neq 0$, $\alpha\xi\delta'\delta'' = k_1\,\alpha\beta\delta'\delta''$, $\xi\beta\delta'\delta'' = k_2\,\alpha\beta\delta'\delta''$ (where k_1 and k_2 are constants† with reference to $\delta'\delta''$) and the existence of $\xi\eta\zeta\tau$ such that $\xi\eta\zeta\tau = (k_1 + k_2)\,\alpha\beta\delta'\delta''$ imply $k_1 + k_2 = 1$.

G_{22}. If $q \neq 0$ corresponds to an element of the class $\mathfrak{Q}$, then $\alpha\beta\gamma\delta \neq 0$ implies the existence of $\xi\eta\zeta\tau$ such that $\xi\eta\zeta\tau = q\,\alpha\beta\gamma\delta$.

G_{23}. If the coplanar lines $\alpha\beta$ and $\xi\xi'$ do not intersect,‡ then there exists some point $\xi_0 \neq \xi, \xi'$ such that: $\xi_0\xi\xi' = 0$ and the existence of δ such that $\xi_0\xi\alpha\delta \neq 0$, $\xi_0\xi'\alpha\delta \neq 0$ implies $\xi_0\xi\beta\delta = \xi_0\xi\alpha\delta$ and $\xi_0\xi'\beta\delta = \xi_0\xi'\alpha\delta$.

In connection with the above definition of the relation $\alpha\beta\gamma\delta = k\,\alpha'\beta'\gamma'\delta'$ we give the following

Definition 2.

$\sum_{i=1}^{m} p_i\alpha_i\beta_i\gamma_i\delta_i = \sum_{j=1}^{n} p^{(j)}\alpha^{(j)}\beta^{(j)}\gamma^{(j)}\delta^{(j)}$ means, there exist for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ the tetrads $\xi_i\eta_i\zeta_i\tau_i$ and $\xi^{(j)}\eta^{(j)}\zeta^{(j)}\tau^{(j)}$ such that $\xi_i\eta_i\zeta_i\tau_i = p_i\alpha_i\beta_i\gamma_i\delta_i$, $\xi^{(j)}\eta^{(j)}\zeta^{(j)}\tau^{(j)} = p^{(j)}\alpha^{(j)}\beta^{(j)}\gamma^{(j)}\delta^{(j)}$ and $\xi_i\eta_i\zeta_i\tau_i = q_i\alpha_0\beta_0\gamma_0\delta_0$, $\xi^{(j)}\eta^{(j)}\zeta^{(j)}\tau^{(j)} = q^{(j)}\alpha_0\beta_0\gamma_0\delta_0$ and $\sum_{i=1}^{m} q_i = \sum_{j=1}^{n} q^{(j)}$.

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* See axioms G_{18_1}, G_{18_3} for the definition of the tetrad $\alpha\,\beta_0\gamma_0\delta_0$ used in G_{20}.

† Cf. axiom G_{19}.

‡ For definition see 1 under (I) of the present chapter.

By means of our axioms it is easily shown that the relation "=" between the two sums in the above definition is linearly transitive and symmetrical; that terms on one side of the relation may be transposed to the other side after change of sign, etc.* Our definitions 1 and 2 should be carefully compared with Grassmann's "Erklärung 1" and "Erklärung 2," *Werke*, Vol. I, 1, pp. 343–345.

Theorems.

Theorem 18. If $\alpha\beta\gamma\delta = k\,\alpha'\beta'\gamma'\delta'$, then for any tetrad $\theta_1\theta_2\theta_3\theta_4$ such that $\alpha\beta\gamma\delta = p\cdot\theta_1\theta_2\theta_3\theta_4$ and $\alpha'\beta'\gamma'\delta' = p'\,\theta_1\theta_2\theta_3\theta_4$ and $\theta_1\theta_2\theta_3\theta_4 \neq 0$, $p = k\cdot p'$.

If $k = 0$, then (see under definition 1) $\alpha\beta\gamma\delta = 0$ and therefore $p = 0$; hence $p = k\,p'$. If $k \neq 0$ and $\alpha'\beta'\gamma'\delta' = 0$, then $\alpha\beta\gamma\delta = 0$, and therefore $p = 0$, $p' = 0$; hence $p = k\,p'$. Let now $k \neq 0$, $\alpha'\beta'\gamma'\delta' \neq 0$. Then $\alpha\beta\gamma\delta/\theta_1\theta_2\theta_3\theta_4 = p$, $\alpha'\beta'\gamma'\delta'/\theta_1\theta_2\theta_3\theta_4 = p'$. By hypothesis and definition 1 there exists $\xi\eta\zeta\tau$ such that $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = m$, $\alpha'\beta'\gamma'\delta'/\xi\eta\zeta\tau = m'$ and $m = k\,m'$. Hence by the theorems of § 2, $\xi\eta\zeta\tau/\theta_1\theta_2\theta_3\theta_4 = p/m$, $\xi\eta\zeta\tau/\theta_1\theta_2\theta_3\theta_4 = p'/m'$. Therefore, $p/m = p'/m'$. But $m = k\,m'$. Hence $p = k\,p'$.

Theorem 18′. If $\alpha\beta\gamma\delta = k\,\alpha'\beta'\gamma'\delta'$ and $k \neq 0$, then $\alpha'\beta'\gamma'\delta' = 1/k\;\alpha\beta\gamma\delta$.

This is easily shown by the theorems of § 2 if $\alpha'\beta'\gamma'\delta' \neq 0$. Suppose that $\alpha'\beta'\gamma'\delta' = 0$; then $\alpha\beta\gamma\delta = 0$, and we must show that $1/k$ corresponds to an element of the class $\mathfrak{Q}$. Since $k \neq 0$ and $\alpha_0\beta_0\gamma_0\delta_0 \neq 0$ (cf. note on axiom G_{20}), there exists by axiom G_{22} the tetrad $\xi\eta\zeta\tau$ such that $\xi\eta\zeta\tau = k\,\alpha_0\beta_0\gamma_0\delta_0$; that is, by the theorems of § 2, $\alpha_0\beta_0\gamma_0\delta_0 = 1/k\;\xi\eta\zeta\tau$. Hence $1/k$ corresponds to an element of the class $\mathfrak{Q}$.

Theorem 18″. If $\alpha\beta\gamma\delta = k\,\alpha'\beta'\gamma'\delta'$ and $\alpha'\beta'\gamma'\delta' = k'\,\alpha''\beta''\gamma''\delta''$, then $\alpha\beta\gamma\delta = k\,k'\,\alpha''\beta''\gamma''\delta''$.

If $\alpha'\beta'\gamma'\delta' = 0$, then the theorem is easily verified when $k = 0$, or when $k \neq 0$ and $k' = 0$. Let now $\alpha'\beta'\gamma'\delta' = 0$ and $k, k' \neq 0$; then $\alpha''\beta''\gamma''\delta'' = 0$, $\alpha\beta\gamma\delta = 0$, and we must show that $k\,k'$ corresponds to an element of the class $\mathfrak{Q}$. Since $\alpha_0\beta_0\gamma_0\delta_0 \neq 0$ there exist by axiom G_{22} the tetrads $\xi_1\eta_1\zeta_1\tau_1$ and $\xi_2\eta_2\zeta_2\tau_2$ such that $\xi_1\eta_1\zeta_1\tau_1 = k\,\alpha_0\beta_0\gamma_0\delta_0$ and $\xi_2\eta_2\zeta_2\tau_2 = k'\,\xi_1\eta_1\zeta_1\tau_1$. Hence, by the theorems of § 2, $\xi_2\eta_2\zeta_2\tau_2 = k\,k'\,\alpha_0\beta_0\gamma_0\delta_0$, and therefore $k\,k'$ corresponds to an element of the class $\mathfrak{Q}$.

In the case $\alpha'\beta'\gamma'\delta' \neq 0$, $k, k' \neq 0$ the theorem is verified by referring to the theorems of § 2. Finally, when $\alpha'\beta'\gamma'\delta' \neq 0$ and $k = 0$, the theorem is obviously valid.

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* In connection with definition 2, compare axiom G_{20}.

Theorem 19. If $\alpha'\beta'\gamma'\delta' = q'\,\xi\eta\zeta\tau$ and $\alpha''\beta''\gamma''\delta'' = q''\,\xi\eta\zeta\tau$, where $\xi\eta\zeta\tau \neq 0$, then $\alpha'\beta'\gamma'\delta' + \alpha''\beta''\gamma''\delta'' = (q'+q'')\,\xi\eta\zeta\tau$.

We must show that there exists $\xi_1\eta_1\zeta_1\tau_1$ such that $\xi_1\eta_1\zeta_1\tau_1 = (q'+q'')\,\xi\eta\zeta\delta$ and that if $\alpha'\beta'\gamma'\delta' = p'\,\alpha_0\beta_0\gamma_0\delta_0$, $\alpha''\beta''\gamma''\delta'' = p''\,\alpha_0\beta_0\gamma_0\delta_0$ and $\xi_1\eta_1\zeta_1\tau_1 = p\,\alpha_0\beta_0\gamma_0\delta_0$, then $p'+p'' = p$.

Let $\xi\eta\zeta\tau = p_1\,\alpha_0\beta_0\gamma_0\delta_0$. Then, since by hypothesis $\alpha'\beta'\gamma'\delta' = q'\,\xi\eta\zeta\tau$ and $\alpha''\beta''\gamma''\delta'' = q''\,\xi\eta\zeta\tau$, we have $\alpha'\beta'\gamma'\delta' = p_1 q'\,\alpha_0\beta_0\gamma_0\delta_0$ and $\alpha''\beta''\gamma''\delta'' = p_1 q''\,\alpha_0\beta_0\gamma_0\delta_0$. Hence by G_{20} there exists a tetrad, say $\xi_1\eta_1\zeta_1\tau_1$, such that $\xi_1\eta_1\zeta_1\tau_1 = (p_1 q' + p_1 q'')\,\alpha_0\beta_0\gamma_0\delta_0$. But $\alpha_0\beta_0\gamma_0\delta_0 = \dfrac{1}{p_1}\,\xi\eta\zeta\tau$ since $\xi\eta\zeta\tau \neq 0$ and $\alpha_0\beta_0\gamma_0\delta_0 \neq 0$. Hence $\xi_1\eta_1\zeta_1\tau_1 = (q'+q'')\,\xi\eta\zeta\tau$.

From the preceding we evidently have $p' = p_1 q'$, $p'' = p_1 q''$, $p = p_1 q' + p_1 q''$, and hence $p' + p'' = p$.

Theorem 20. If $\alpha\beta\gamma\delta = k\,\xi\eta\zeta\tau$ and $\xi\eta\zeta\tau = \alpha'\beta'\gamma'\delta' + \alpha''\beta''\gamma''\delta''$, then $\alpha\beta\gamma\delta = k\,\alpha'\beta'\gamma'\delta' + k\,\alpha''\beta''\gamma''\delta''$.

If $k = 0$, then $\alpha\beta\gamma\delta = 0$ and the theorem is obviously valid. Therefore we suppose $k \neq 0$. Likewise, we may assume that $\alpha'\beta'\gamma'\delta' \neq 0$ and $\alpha''\beta''\gamma''\delta'' \neq 0$.

Since k corresponds to an element of the class $\mathfrak{Q}$ (see definition 1), by axiom G_{22} there exist, say, $\lambda'\mu'\nu'\omega'$ and $\lambda''\mu''\nu''\omega''$, such that $\lambda'\mu'\nu'\omega' = k\,\alpha'\beta'\gamma'\delta'$ and $\lambda''\mu''\nu''\omega'' = k\,\alpha''\beta''\gamma''\delta''$. Let $\alpha'\beta'\gamma'\delta' = q'\,\alpha_0\beta_0\gamma_0\delta_0$ and $\alpha''\beta''\gamma''\delta'' = q''\,\alpha_0\beta_0\gamma_0\delta_0$. Then $\lambda'\mu'\nu'\omega' = k q'\,\alpha_0\beta_0\gamma_0\delta_0$ and $\lambda''\mu''\nu''\omega'' = k q''\,\alpha_0\beta_0\gamma_0\delta_0$, and we must show that $\alpha\beta\gamma\delta = (k q' + k q'')\,\alpha_0\beta_0\gamma_0\delta_0$. Now by hypothesis $\xi\eta\zeta\tau = \alpha'\beta'\gamma'\delta' + \alpha''\beta''\gamma''\delta''$, and therefore, by definition 2, $\xi\eta\zeta\tau = (q'+q'')\,\alpha_0\beta_a\gamma_0\delta_0$. If $\xi\eta\zeta\tau = 0$, $q'+q'' = 0$, $\alpha\beta\gamma\delta = 0$ and therefore $\alpha\beta\gamma\delta = (k q' + k q'')\,\alpha_0\beta_0\gamma_0\delta_0$. If $\xi\eta\zeta\tau \neq 0$, then from $\alpha\beta\gamma\delta = k\,\xi\eta\zeta\tau$ and $\xi\eta\zeta\tau = (q+q')\,\alpha_0\beta_0\gamma_0\delta_0$ we obtain $\alpha\beta\gamma\delta = (k q + k q')\,\alpha_0\beta_0\gamma_0\delta_0$.

Theorem 21. If $\xi\eta\zeta\tau = (k+k')\,\alpha\beta\gamma\delta$, where k and k' correspond to elements of the class $\mathfrak{Q}$, then $\xi\eta\zeta\tau = k\,\alpha\beta\gamma\delta + k'\,\alpha\beta\gamma\delta$.

We assume $\alpha\beta\gamma\delta \neq 0$. By axiom G_{22} there exist $\lambda\mu\nu\omega$ and $\lambda'\mu'\nu'\omega'$ such that $\lambda\mu\nu\omega = k\,\alpha\beta\gamma\delta$ and $\lambda'\mu'\nu'\omega' = k'\,\alpha\beta\gamma\delta$. Let $\lambda\mu\nu\omega = q\,\alpha_0\beta_0\gamma_0\delta_0$ and $\lambda'\mu'\nu'\omega' = q'\,\alpha_0\beta_0\gamma_0\delta_0$. Then we must show that $\xi\eta\zeta\tau = (q+q')\,\alpha_0\beta_0\gamma_0\delta_0$.

We suppose that $k \neq 0$; then $\alpha\beta\gamma\delta = 1/k\,\lambda\mu\nu\omega$ and since $\lambda\mu\nu\omega = q\,\alpha_0\beta_0\gamma_0\delta_0$ we have, $\alpha\beta\gamma\delta = q/k\,\alpha_0\beta_0\gamma_0\delta_0$. But by hypothesis, $\xi\eta\zeta\tau = (k+k')\,\alpha\beta\gamma\delta$. Hence $\xi\eta\zeta\tau = (q + q k'/k)\,\alpha_0\beta_0\gamma_0\delta_0$. Since $\lambda'\mu'\nu'\omega' = k'\,\alpha\beta\gamma\delta$ and $\alpha\beta\gamma\delta = q/k\cdot\alpha_0\beta_0\gamma_0\delta_0$, we have $\lambda'\mu'\nu'\omega' = q k'/k\,\alpha_0\beta_0\gamma_0\delta_0$. But $\lambda'\mu'\nu'\omega' = q'\,\alpha_0\beta_0\gamma_0\delta_0$. Therefore $q' = q k'/k$; that is, $\xi\eta\zeta\tau = (q+q')\,\alpha_0\beta_0\gamma_0\delta_0$.

Theorem 22. If $p\,\alpha\beta\gamma\delta = q\,\xi\eta\zeta\tau$ and $p'\,\alpha'\beta'\gamma'\delta' = q'\,\xi'\eta'\zeta'\tau'$, then $p\,\alpha\beta\gamma\delta + p'\,\alpha'\beta'\gamma'\delta' = q\,\xi\eta\zeta\tau + q'\,\xi'\eta'\zeta'\tau'$.

By hypothesis and definition there exist tetrads, say, $\lambda'\mu'\nu'\omega'$, $\lambda''\mu''\nu''\omega''$, $\lambda_1\mu_1\nu_1\omega_1$, $\lambda_2\mu_2\nu_2\omega_2$, such that $\lambda'\mu'\nu'\omega' = p\,\alpha\beta\gamma\delta$, $\lambda''\mu''\nu''\omega'' = p'\,\alpha'\beta'\gamma'\delta'$, $\lambda_1\mu_1\nu_1\omega_1 = q\,\xi\eta\zeta\tau$, $\lambda_2\mu_2\nu_2\omega_2 = q'\,\xi'\eta'\zeta'\tau'$. Let $\lambda'\mu'\nu'\omega' = r'\,\alpha_0\beta_0\gamma_0\delta_0$, $\lambda''\mu''\nu''\omega'' = r''\,\alpha_0\beta_0\gamma_0\delta_0$, $\lambda_1\mu_1\nu_1\omega_1 = r_1\,\alpha_0\beta_0\gamma_0\delta_0$, $\lambda_2\mu_2\nu_2\omega_2 = r_2\,\alpha_0\beta_0\gamma_0\delta_0$. Then by theorem 19, $\lambda'\mu'\nu'\omega' + \lambda''\mu''\nu''\omega'' = (r' + r'')\,\alpha_0\beta_0\gamma_0\delta_0$ and $\lambda_1\mu_1\nu_1\omega_1 + \lambda_2\mu_2\nu_2\omega_2 = (r_1 + r_2)\,\alpha_0\beta_0\gamma_0\delta_0$. But by hypothesis, $r' = r_1$ and $r'' = r_2$; hence $r' + r'' = r_1 + r_2$, which verifies the theorem.

Theorems 19–22 have obvious extensions to an arbitrary number of terms. These theorems are essentially involved in the proof of the remaining theorems of this section.

In the following we shall denote (cf. Chapter V) by the notations Δ', Δ'', Δ_1, Δ_2, $\Delta^{(1)}$, $\Delta^{(2)}$, etc., *arbitrary points.*

Theorem 23.* If $\alpha\beta\xi = 0$, $\alpha \neq \beta \neq \xi \neq \alpha$, then $\xi\beta\Delta_1\Delta_2 = k\,\alpha\xi\Delta_1\Delta_2$, where k is constant with reference to Δ_1, Δ_2.

Let δ_1, δ_2 be two given points. If $\alpha\beta\delta_1\delta_2 \neq 0$, then by axioms G_{18_1}–G_{18_3}, G_{19} and the theorems of Chapter V we readily show that $\xi\beta\delta_1\delta_2 = k\,\alpha\xi\delta_1\delta_2$.

Suppose now $\alpha\beta\delta_1\delta_2 = 0$. Then we show that $\xi\beta\delta_1\delta_2 = 0$ and $\alpha\xi\delta_1\delta_2 = 0$. If $\alpha\beta\delta_1 \neq 0$, then since $\alpha\beta\xi = 0$, we have by theorem 7, Chapter V, $\xi\beta\delta_1\delta_2 = 0$, $\alpha\xi\delta_1\delta_2 = 0$. If $\alpha\beta\delta_1 = 0$, then since $\alpha\beta\xi = 0$, supposing $\xi \neq \delta_1$, we have by theorem 11, Chapter V, $\xi\beta\delta_1\delta_2 = 0$, $\alpha\xi\delta_1\delta_2 = 0$.

Theorem 24. If $\alpha\beta\xi = 0$, $\alpha \neq \beta \neq \xi \neq \alpha$, then $\alpha\beta\Delta_1\Delta_2 = \alpha\xi\Delta_1\Delta_2 + \xi\beta\Delta_1\Delta_2$.

If $\alpha\beta\xi = 0$ and α, β, ξ are distinct, then by axioms 1 and 2 of the system ${}^3K_3$ and theorems 5, 8 of Chapter V there exist two points δ_0', δ_0'' such that $\alpha\beta\delta_0'\delta_0'' \neq 0$, $\xi\beta\delta_0'\delta_0'' \neq 0$, $\alpha\xi\delta_0'\delta_0'' \neq 0$. Also $\alpha\beta\xi = 0$ implies $\xi\beta\alpha = 0$ and $\alpha\xi\beta = 0$. Thus from axiom G_{19} follow two theorems which are obtained from this axiom by interchanging ξ and α, ξ and β respectively. Hence for any two points δ', δ'' such that $\xi\beta\delta'\delta'' \neq 0$ and $\alpha\xi\delta'\delta'' \neq 0$ we have, $\alpha\beta\delta'\delta'' = k_1\,\xi\alpha\delta'\delta''$ and $\beta\xi\delta'\delta'' = k_2\,\alpha\beta\delta'\delta''$, where k_1 and k_2 are constants with reference to $\delta'\,\delta''$. Then by axiom G_{18_2}, 1, 2, and theorems 2, 8, 13 of § 2, we have $\alpha\xi\delta'\delta'' = -\frac{1}{k_1}\,\alpha\beta\delta'\delta''$ and $\xi\beta\delta'\delta'' = -k_2\,\alpha\beta\delta'\delta''$. Let now $\alpha\beta\delta'\delta'' =$

* In this, and the following theorems, arbitrary points occurring homogeneously in relations may be omitted by introducing a suitable convention.

$l \cdot \alpha_0 \beta_0 \gamma_0 \delta_0$; then by theorems of § 2 we have, $\alpha \xi \delta' \delta'' = -l/k_1 \, \alpha_0 \beta_0 \gamma_0 \delta_0$ and $\xi \beta \delta' \delta'' = -l k_2 \alpha_0 \beta_0 \gamma_0 \delta_0$. Hence by axiom G_{20} and definition 2 we have,

$$\alpha \xi \delta' \delta'' + \xi \beta \delta' \delta'' = -(l/k_1 + l k_2) \, \alpha_0 \beta_0 \gamma_0 \delta_0;$$

that is, since $\alpha_0 \beta_0 \gamma_0 \delta_0 = 1/l \, \alpha \beta \delta' \delta''$,

$$\alpha \xi \delta' \delta'' + \xi \beta \delta' \delta'' = -(1/k_1 + k_2) \, \alpha \beta \delta' \delta'',$$

where k_1 and k_2 are constants with reference to δ', δ''. Accordingly, by axiom G_{21} we have $-(1/k_1 + k_2) = 1$, which establishes the theorem when $\alpha \beta \delta' \delta'' \neq 0$.

If $\alpha \beta \delta' \delta'' = 0$, then, as under the preceding theorem, we show that $\alpha \xi \delta' \delta'' = 0$, $\xi \beta \delta' \delta'' = 0$.

Theorem 25. If the line $\xi \xi'$ does not intersect the coplanar line $\alpha \beta$, then $\alpha \xi \xi' \Delta = \beta \xi \xi' \Delta$ and $\alpha \beta \xi \Delta = \alpha \beta \xi' \Delta$.

We show first that if $\xi \xi'$ does not intersect $\alpha \beta$, then the existence of δ such that $\alpha \xi \xi' \delta \neq 0$ implies $\alpha \xi \xi' \delta = \beta \xi \xi' \delta$. By axiom G_{23} there exists the point $\xi_0 \neq \xi, \xi'$ such that: $\xi_0 \xi \xi' = 0$ and the existence of δ such that $\xi_0 \xi \alpha \delta \neq 0$ and $\xi_0 \xi' \alpha \delta \neq 0$ implies $\xi_0 \xi \alpha \delta = \xi_0 \xi \beta \delta$ and $\xi_0 \xi' \alpha \delta = \xi_0 \xi' \beta \delta$. By definition 1 under (I) of the present chapter, since $\xi \xi'$ does not intersect $\alpha \beta$, $\alpha \xi \xi' \neq 0$. Let the point δ_1 be such that $\alpha \xi \xi' \delta_1 \neq 0$; then since $\xi_0 \xi \xi' = 0$, we have by theorem 8, Chapter V, $\xi_0 \xi \alpha \delta_1 \neq 0$, $\xi_0 \xi' \alpha \delta_1 \neq 0$. Hence by axiom G_{23}, $\xi_0 \xi \alpha \delta_1 = \xi_0 \xi \beta \delta_1$ and $\xi_0 \xi' \alpha \delta_1 = \xi_0 \xi' \beta \delta_1$. Therefore, by theorem 22,

$$\xi \xi_0 \alpha \delta_1 + \xi_0 \xi' \alpha \delta_1 = \xi \xi_0 \beta \delta_1 + \xi_0 \xi' \beta \delta_1.$$

But by theorem 24, since $\xi_0 \xi \xi' = 0$,

$$\xi \xi_0 \alpha \delta_1 + \xi_0 \xi' \alpha \delta_1 = \xi \xi' \alpha \delta_1,$$
$$\xi \xi_0 \beta \delta_1 + \xi_0 \xi' \beta \delta_1 = \xi \xi' \beta \delta_1.$$

Therefore, by definition 2, $\xi \xi' \alpha \delta_1 = \xi \xi' \beta \delta_1$.

Let δ_2 be such that $\alpha \xi \xi' \delta_2 = 0$; then, since $\alpha \xi \xi' \beta = 0$ and $\alpha \xi \xi' \neq 0$, we have by theorem 7 of Chapter V, $\beta \xi \xi' \delta_2 = 0$.

We have thus verified the first part of the preceding theorem; and since the relation of non-intersection is symmetrical, it is clear that the remaining part of the theorem is proved in a similar manner.

Theorem 26. If $\alpha \beta \gamma \neq 0$ and $\alpha \beta \gamma \xi = 0$, then $\xi \beta \gamma \Delta + \alpha \xi \gamma \Delta + \alpha \beta \xi \Delta = \alpha \beta \gamma \Delta$.

If $\xi = \alpha$ or β or γ, then the theorem obviously is verified.

Let now $\xi \neq \alpha, \beta, \gamma$. Then by the theorems of Chapter V there exists a point η such that ($\beta\eta\xi = 0$ and $\alpha\gamma\eta = 0$) or ($\alpha\eta\xi = 0$ and $\beta\gamma\eta = 0$) or ($\gamma\eta\xi = 0$ and $\alpha\beta\eta = 0$). If $\eta = \alpha$ or β or γ, the theorem is easily verified by theorem 24 and definition 2. Assuming $\eta \neq \alpha, \beta, \gamma$, we have by theorem 24, if $\beta\eta\xi = 0$ and $\alpha\gamma\eta = 0$,

$$\alpha\eta\Delta'\Delta + \eta\gamma\Delta'\Delta = \alpha\gamma\Delta'\Delta, \tag{1}$$

$$\beta\eta\Delta''\Delta + \eta\xi\Delta''\Delta = \beta\xi\Delta''\Delta. \tag{2}$$

Hence, putting in (1) $\Delta' = \xi, \beta$ and in (2) $\Delta'' = \alpha, \gamma$, we obtain respectively,

$$\left.\begin{aligned} \alpha\eta\xi\Delta + \eta\gamma\xi\Delta &= \alpha\gamma\xi\Delta, \\ \alpha\eta\beta\Delta + \eta\gamma\beta\Delta &= \alpha\gamma\beta\Delta, \\ \beta\eta\alpha\Delta + \eta\xi\alpha\Delta &= \beta\xi\alpha\Delta, \\ \beta\eta\gamma\Delta + \eta\xi\gamma\Delta &= \beta\xi\gamma\Delta. \end{aligned}\right\} \tag{3}$$

From the right-hand side of (3) we obtain expressions for $\xi\beta\gamma\Delta$, $\alpha\xi\gamma\Delta$, $\alpha\beta\xi\Delta$, $\alpha\beta\gamma\Delta$; then the theorem is seen to be valid by finding the sum (cf. theorem 22) $\xi\beta\gamma\Delta + \alpha\xi\gamma\Delta + \alpha\beta\xi\Delta$ and comparing with the expression for $\alpha\beta\gamma\Delta$.

Corollary. If $\alpha\beta\gamma = 0$, then for any ξ, $\xi\beta\gamma\Delta + \alpha\xi\gamma\Delta + \alpha\beta\xi\Delta = \alpha\beta\gamma\Delta$.

For if $\xi\beta\gamma$, $\alpha\xi\gamma$, $\alpha\beta\xi$ are all equal to zero, then by axiom 7 of system ${}^3K_3$, Chapter V, $\alpha\beta\gamma = 0$; thus the theorem is verified. Let now $\xi\beta\gamma$ or $\alpha\xi\gamma$ or $\alpha\beta\xi$ be not equal to zero, say $\xi\beta\gamma \neq 0$; then by theorem 26 for the point α we have, since $\xi\beta\gamma\alpha = 0$,

$$\alpha\beta\gamma\Delta + \xi\alpha\gamma\Delta + \xi\beta\alpha\Delta = \xi\beta\gamma\Delta;$$

that is,

$$\xi\beta\gamma\Delta + \alpha\xi\gamma\Delta + \alpha\beta\xi\Delta = \alpha\beta\gamma\Delta.$$

Theorem 27. If $\alpha\beta\gamma\delta \neq 0$, then for any ξ, $\xi\beta\gamma\delta + \alpha\xi\gamma\delta + \alpha\beta\xi\delta + \alpha\beta\gamma\xi = \alpha\beta\gamma\delta$.

If $\xi = \alpha$ or β or γ or δ, then the theorem obviously is verified.

Let $\xi \neq \alpha, \beta, \gamma, \delta$. Then by the theorems of Chapter V there exists a point η such that ($\alpha\eta\xi = 0$ and $\beta\gamma\delta\eta = 0$) or ($\beta\eta\xi = 0$ and $\alpha\gamma\delta\eta = 0$) or ($\gamma\eta\xi = 0$ and $\alpha\beta\delta\eta = 0$) or ($\delta\eta\xi = 0$ and $\alpha\beta\gamma\eta = 0$). If η is collinear with an edge of $\alpha\beta\gamma\delta$, then the theorem is easily verified by theorem 26 and

definition 2. Assuming $\alpha\beta\eta \neq 0$, $\beta\gamma\eta \neq 0$, etc., we have by theorem 24, if $\alpha\eta\xi = 0$ and $\beta\gamma\delta\eta = 0$,

$$\alpha\eta\Delta\Delta' + \eta\xi\Delta\Delta' = \alpha\xi\Delta\Delta',$$

and therefore,

$$\left.\begin{aligned} \alpha\eta\beta\gamma + \eta\xi\beta\gamma &= \alpha\xi\beta\gamma, \\ \alpha\eta\gamma\delta + \eta\xi\gamma\delta &= \alpha\xi\gamma\delta, \\ \alpha\eta\beta\delta + \eta\xi\beta\delta &= \alpha\xi\beta\delta. \end{aligned}\right\} \quad (1)$$

Also by theorem 26 we have for $\eta\gamma\delta$ and the point β, since $\beta\gamma\delta\eta = 0$,

$$\beta\gamma\delta\Delta'' + \eta\beta\delta\Delta'' + \eta\gamma\beta\Delta'' = \eta\gamma\delta\Delta'',$$

and consequently for $\Delta'' = \alpha, \xi$ we obtain respectively,

$$\left.\begin{aligned} \beta\gamma\delta\alpha + \eta\beta\delta\alpha + \eta\gamma\beta\alpha &= \eta\gamma\delta\alpha, \\ \beta\gamma\delta\xi + \eta\beta\delta\xi + \eta\gamma\beta\xi &= \eta\gamma\delta\xi. \end{aligned}\right\} \quad (2)$$

From (1) and (2) we obtain expressions for $\xi\beta\gamma\delta$, $\alpha\xi\gamma\delta$, $\alpha\beta\xi\delta$, $\alpha\beta\gamma\xi$; whence by addition and comparing the sum obtained with the expression for $\alpha\beta\gamma\delta$ in (2), we easily verify the theorem.

Corollary. If $\alpha\beta\gamma\delta = 0$, then for any ξ, $\xi\beta\gamma\delta + \alpha\xi\gamma\delta + \alpha\beta\xi\delta + \alpha\beta\gamma\xi = \alpha\beta\gamma\delta$.

For if $\xi\beta\gamma\delta$, $\alpha\xi\gamma\delta$, $\alpha\beta\xi\delta$, $\alpha\beta\gamma\xi$ are all equal to zero, then by axiom 7 of the system ${}^3K_3$, Chapter V, $\alpha\beta\gamma\delta = 0$ and the theorem is verified. Let now $\xi\beta\gamma\delta$, $\alpha\xi\gamma\delta$, $\alpha\beta\xi\delta$, $\alpha\beta\gamma\xi$ be not all zero, say $\xi\beta\gamma\delta \neq 0$. Then by theorem 27, for the point α we have

$$\alpha\beta\gamma\delta + \xi\alpha\gamma\delta + \xi\beta\alpha\delta + \xi\beta\gamma\alpha = \xi\beta\gamma\delta;$$

that is,

$$\xi\beta\gamma\delta + \alpha\xi\gamma\delta + \alpha\beta\xi\delta + \alpha\beta\gamma\xi = \alpha\beta\gamma\delta.$$

Theorem 28.* If $\alpha\beta\xi = 0$ and $\alpha \neq \beta \neq \xi \neq \alpha$,† then

$$\xi\Delta_1\Delta_2\Delta_3 = a'\alpha\Delta_1\Delta_2\Delta_3 + b'\beta\Delta_1\Delta_2\Delta_3,$$

where $a' + b' = 1$.

Since $\alpha\beta\xi = 0$ and α, β, ξ are distinct, we have by theorem 23,

$$\xi\beta\Delta'\Delta'' = k'\xi\alpha\Delta'\Delta''. \quad (1)$$

Let $\delta_1, \delta_2, \delta_3$ be three given points and suppose first that $\delta_1\delta_2\delta_3 \neq 0$. Then we distinguish three cases.

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* Compare Peano, "Calcolo Geometrico," pp. 33–35.

† If $\xi = \alpha$ we have, by definition 2, $\xi\Delta_1\Delta_2\Delta_3 = 1\,\alpha\Delta_1\Delta_2\Delta_3 + 0\,\beta\Delta_1\Delta_2\Delta_3$, and similarly if $\xi = \beta$.

CASE I. The plane $\delta_1\delta_2\delta_3$ meets the line $\alpha\beta$ in the unique point $\xi^{(0)}$.* Then by theorem 24 we have,

$$\left.\begin{aligned}\xi\beta\Delta'\Delta'' &= \xi\xi^{(0)}\Delta'\Delta'' + \xi^{(0)}\beta\Delta'\Delta'',\\ \xi\alpha\Delta'\Delta'' &= \xi\xi^{(0)}\Delta'\Delta'' + \xi^{(0)}\alpha\Delta'\Delta''.\end{aligned}\right\} \tag{2}$$

Hence,

$$\left.\begin{aligned}\xi\beta\delta_1\delta_2 &= \xi\xi^{(0)}\delta_1\delta_2 + \xi^{(0)}\beta\delta_1\delta_2,\\ \xi\alpha\delta_1\delta_2 &= \xi\xi^{(0)}\delta_1\delta_2 + \xi^{(0)}\alpha\delta_1\delta_2.\end{aligned}\right\} \tag{2'}$$

But from (1),

$$\xi\beta\delta_1\delta_2 = k'\xi\alpha\delta_1\delta_2. \tag{1'}$$

Hence from (1′) and the second relation of (2′), by theorem 20,

$$\xi\beta\delta_1\delta_2 = k'\xi\xi^{(0)}\delta_1\delta_2 + k'\xi^{(0)}\alpha\delta_1\delta_2. \tag{3}$$

Therefore from the first relation of (2′) and from (3),

$$\xi\xi^{(0)}\delta_1\delta_2 + \xi^{(0)}\beta\delta_1\delta_2 = k'\xi\xi^{(0)}\delta_1\delta_2 + k'\xi^{(0)}\alpha\delta_1\delta_2.$$

That is,

$$\xi\xi^{(0)}\delta_1\delta_2 - k'\xi\xi^{(0)}\delta_1\delta_2 = -k'\alpha\xi^{(0)}\delta_1\delta_2 + \beta\xi^{(0)}\delta_1\delta_2.$$

Now by axiom G_{22} there exist $\xi'\eta'\zeta'\tau'$ and $\xi''\eta''\zeta''\tau''$ such that $\xi'\eta'\zeta'\tau' = \xi\xi^{(0)}\delta_1\delta_2$ and $\xi''\eta''\zeta''\tau'' = -k'\xi\xi^{(0)}\delta_1\delta_2$; hence by theorem 19 there exists the sum, say $\xi\eta\zeta\tau$, such that

$$\xi'\eta'\zeta'\tau' + \xi''\eta''\zeta''\tau'' = (1-k')\,\xi\xi^{(0)}\delta_1\delta_2,$$

and therefore,

$$(1-k')\,\xi\xi^{(0)}\delta_1\delta_2 = -k'\alpha\xi^{(0)}\delta_1\delta_2 + \beta\xi^{(0)}\delta_1\delta_2.$$

Since $k' \neq 1$,† we have $\xi\xi^{(0)}\delta_1\delta_2 = 1/(1-k')\,\xi\eta\zeta\tau$, and since $\xi\eta\zeta\tau = \xi_1\eta_1\zeta_1\tau_1 + \beta\xi^{(0)}\delta_1\delta_2$, where $\xi_1\eta_1\zeta_1\tau_1 = -k'\alpha\xi^{(0)}\delta_1\delta_2$, we have by theorem 20,

$$\xi\xi^{(0)}\delta_1\delta_2 = 1/(1-k')\,\xi_1\eta_1\zeta_1\tau_1 + 1/(1-k')\,\beta\xi^{(0)}\delta_1\delta_2.$$

That is,

$$\xi\xi^{(0)}\delta_1\delta_2 - 1/(1-k')\,\beta\xi^{(0)}\delta_1\delta_2 = 1/(1-k')\,\xi_1\eta_1\zeta_1\tau_1.$$

But $\xi_1\eta_1\zeta_1\tau_1 = -k'\alpha\xi^{(0)}\delta_1\delta_2$ and consequently by the theorems of § 2 and definitions 1 and 2 of the present section we have,

$$\xi\xi^{(0)}\delta_1\delta_2 - 1/(1-k')\,\beta\xi^{(0)}\delta_1\delta_2 = -k'/(1-k')\,\alpha\xi^{(0)}\delta_1\delta_2,$$

or if we put

$$a' = -k'/(1-k'), \quad b' = 1/(1-k'), \quad a'+b' = 1,$$

we obtain

$$\xi\xi^{(0)}\delta_1\delta_2 = a'\alpha\xi^{(0)}\delta_1\delta_2 + b'\beta\xi^{(0)}\delta_1\delta_2. \tag{4}$$

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* We treat here essentially $\xi^{(0)} \neq a, \beta, \xi$ and $\xi\xi^{(0)}\delta_1\delta_2 \neq 0$, $\alpha\xi^{(0)}\delta_1\delta_2 \neq 0$, $\beta\xi^{(0)}\delta_1\delta_2 \neq 0$.

† If $k'=1$, then by (1), $\xi\beta\Delta'\Delta'' = \xi\alpha\Delta'\Delta''$. But by theorem 24, $\alpha\xi\Delta'\Delta'' + \xi\beta\Delta'\Delta'' = \alpha\beta\Delta'\Delta''$. Hence $\alpha\xi\Delta'\Delta'' + \xi\alpha\Delta'\Delta'' = \alpha\beta\Delta'\Delta''$; that is, $\alpha\beta\Delta'\Delta'' = 0$ or $\alpha = \beta$ by axioms 1, 2 and theorem 5 of Chapter V.

In a similar manner we derive the relations

$$\xi\,\xi^{(0)}\,\delta_2\,\delta_3 = a'\,\alpha\,\xi^{(0)}\,\delta_2\,\delta_3 + b'\,\beta\,\xi^{(0)}\,\delta_2\,\delta_3, \tag{5}$$

$$\xi\,\xi^{(0)}\,\delta_1\,\delta_3 = a'\,\alpha\,\xi^{(0)}\,\delta_1\,\delta_3 + b'\,\beta\,\xi^{(0)}\,\delta_1\,\delta_3. \tag{6}$$

Now by theorem 27, since $\xi_0\,\delta_1\,\delta_2\,\delta_3 = 0$ we have,

$$\xi\,\xi^{(0)}\,\delta_2\,\delta_3 + \xi\,\delta_1\,\xi^{(0)}\,\delta_3 + \xi\,\delta_1\,\delta_2\,\xi^{(0)} = \xi\,\delta_1\,\delta_2\,\delta_3,$$

$$\alpha\,\xi^{(0)}\,\delta_2\,\delta_3 + \alpha\,\delta_1\,\xi^{(0)}\,\delta_3 + \alpha\,\delta_1\,\delta_2\,\xi^{(0)} = \alpha\,\delta_1\,\delta_2\,\delta_3,$$

$$\beta\,\xi^{(0)}\,\delta_2\,\delta_3 + \beta\,\delta_1\,\xi^{(0)}\,\delta_3 + \beta\,\delta_1\,\delta_2\,\xi^{(0)} = \beta\,\delta_1\,\delta_2\,\delta_3,$$

and therefore, by using axiom G_{22} and an obvious extension of theorem 20,

$$a'\,\alpha\,\xi^{(0)}\,\delta_2\,\delta_3 + a'\,\alpha\,\delta_1\,\xi^{(0)}\,\delta_3 + a'\,\alpha\,\delta_1\,\delta_2\,\xi^{(0)} = a'\,\alpha\,\delta_1\,\delta_2\,\delta_3,$$

$$b'\,\beta\,\xi^{(0)}\,\delta_2\,\delta_3 + b'\,\beta\,\delta_1\,\xi^{(0)}\,\delta_3 + b'\,\beta\,\delta_1\,\delta_2\,\xi^{(0)} = b'\,\beta\,\delta_1\,\delta_2\,\delta_3.$$

Hence by addition (cf. theorem 22) from (4), (5), (6),

$$\xi\,\delta_1\,\delta_2\,\delta_3 = a'\,\alpha\,\delta_1\,\delta_2\,\delta_3 + b'\,\beta\,\delta_1\,\delta_2\,\delta_3. \tag{7}$$

CASE II. The plane $\delta_1\,\delta_2\,\delta_3$ does not intersect* the line $\alpha\,\beta$.
Then we have†

$$\xi\,\delta_1\,\delta_2\,\delta_3 = \alpha\,\delta_1\,\delta_2\,\delta_3 = \beta\,\delta_1\,\delta_2\,\delta_3.$$

Now by Case I for three points $\delta_1', \delta_2', \delta_3'$ such that the plane $\delta_1'\,\delta_2'\,\delta_3'$ meets the line $\alpha\,\beta$ in a uniqe point,

$$\xi\,\delta_1'\,\delta_2'\,\delta_3' = a'\,\alpha\,\delta_1'\,\delta_2'\,\delta_3' + b'\,\beta\,\delta_1'\,\delta_2'\,\delta_3',$$

where $a' + b' = 1$ and a', b' correspond to elements of the class $\mathfrak{Q}$. Since $a' + b' = 1$ we may write

$$\xi\,\delta_1\,\delta_2\,\delta_3 = (a' + b')\,\alpha\,\delta_1\,\delta_2\,\delta_3,$$

and therefore, since a', b' correspond to elements of the class $\mathfrak{Q}$, by theorem 21,

$$\xi\,\delta_1\,\delta_2\,\delta_3 = a'\,\alpha\,\delta_1\,\delta_2\,\delta_3 + b'\,\alpha\,\delta_1\,\delta_2\,\delta_3.$$

But from the preceding, $\alpha\,\delta_1\,\delta_2\,\delta_3 = \beta\,\delta_1\,\delta_2\,\delta_3$, and hence by definition 2,

$$\xi\,\delta_1\,\delta_2\,\delta_3 = a'\,\alpha\,\delta_1\,\delta_2\,\delta_3 + b'\,\beta\,\delta_1\,\delta_2\,\delta_3.$$

CASE III. The plane $\delta_1\,\delta_2\,\delta_3$ passes through the line $\alpha\,\beta$.
Then

$$\xi\,\delta_1\,\delta_2\,\delta_3 = \alpha\,\delta_1\,\delta_2\,\delta_3 = \beta\,\delta_1\,\delta_2\,\delta_3 = 0.$$

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* See the definitions under (I) of the present chapter.

† By the theorems of Chapter V and axiom G_{23} there exists a point θ such that, say, $\delta_1\,\delta_3\,\theta = 0$ and $\theta\,\delta_2\,\alpha\,\Delta = \theta\,\delta_2\,\beta\,\Delta$; putting $\Delta = \delta_3, \delta_1$ and using theorems 22, 24, we obtain by addition, $\alpha\,\delta_1\,\delta_2\,\delta_3 = \beta\,\delta_1\,\delta_2\,\delta_3$.

Therefore, by definition 2, for the two numbers a', b' such that $a' + b' = 1$ and a', b' correspond to elements of the class $\mathfrak{Q}$ (cf. Case I), we have

$$\xi\,\delta_1\,\delta_2\,\delta_3 = a'\,\alpha\,\delta_1\,\delta_2\,\delta_3 + b'\,\beta\,\delta_1\,\delta_2\,\delta_3.$$

Suppose, secondly, that $\delta_1\,\delta_2\,\delta_3 = 0$; then, as in Case III,

$$\xi\,\delta_1\,\delta_2\,\delta_3 = \alpha\,\delta_1\,\delta_2\,\delta_3 = \beta\,\delta_1\,\delta_2\,\delta_3,$$

and we show similarly,

$$\xi\,\delta_1\,\delta_2\,\delta_3 = a'\,\alpha\,\delta_1\,\delta_2\,\delta_3 + b'\,\beta\,\delta_1\,\delta_2\,\delta_3.$$

Therefore the theorem is verified.

Theorem 29.* If $\alpha\beta\gamma \neq 0$ and $\alpha\beta\gamma\xi = 0$, then $\xi\,\Delta_1\,\Delta_2\,\Delta_3 = a'\,\alpha\,\Delta_1\,\Delta_2\,\Delta_3 + b'\,\beta\,\Delta_1\,\Delta_2\,\Delta_3 + c'\,\gamma\,\Delta_1\,\Delta_2\,\Delta_3$, where $a' + b' + c' = 1$.

If $\xi = \alpha$ or β or γ, then by definition 2 the theorem is easily verified.

Let $\xi \neq \alpha, \beta, \gamma$. Suppose that ξ is collinear with $\alpha\beta$ or $\beta\gamma$ or $\gamma\alpha$, say, $\alpha\beta\xi = 0$. Then by theorem 28 and definition 2 we have

$$\xi\,\Delta_1\,\Delta_2\,\Delta_3 = a'\,\alpha\,\Delta_1\,\Delta_2\,\Delta_3 + b'\,\beta\,\Delta_1\,\Delta_2\,\Delta_3 + 0\cdot\gamma\,\Delta_1\,\Delta_2\,\Delta_3,$$

and similarly if $\beta\gamma\xi = 0$ or $\gamma\alpha\xi = 0$. Let ξ be non-collinear with $\alpha\beta$ or $\beta\gamma$ or $\gamma\alpha$. Then there exists (see proof of theorem 26) a point η such that, say, $\beta\xi\eta = 0$ and $\alpha\gamma\eta = 0$. Hence, by theorem 28,

$$\xi\,\Delta_1\,\Delta_2\,\Delta_3 = b\,\beta\,\Delta_1\,\Delta_2\,\Delta_3 + y\,\eta\,\Delta_1\,\Delta_2\,\Delta_3, \qquad b + y = 1,$$
$$\eta\,\Delta_1\,\Delta_2\,\Delta_3 = a\,\alpha\,\Delta_1\,\Delta_2\,\Delta_3 + c\,\gamma\,\Delta_1\,\Delta_2\,\Delta_3, \qquad \alpha + c = 1.$$

By axiom G_{22}, definition 2, theorem 20 and the theorems of § 2, we find from the latter relations,

$$\xi\,\Delta_1\,\Delta_2\,\Delta_3 = y\,a\,\alpha\,\Delta_1\,\Delta_2\,\Delta_3 + b\,\beta\,\Delta_1\,\Delta_2\,\Delta_3 + c\,y\,\gamma\,\Delta_1\,\Delta_2\,\Delta_3,$$

or if we put

$$y\,a = a', \quad b = b', \quad c\,y = c',$$

then

$$a' + b' + c' = 1$$

and

$$\xi\,\Delta_1\,\Delta_2\,\Delta_3 = a'\,\alpha\,\Delta_1\,\Delta_2\,\Delta_3 + b'\,\beta\,\Delta_1\,\Delta_2\,\Delta_3 + c'\,\gamma\,\Delta_1\,\Delta_2\,\Delta_3,$$

which verifies the theorem.

Theorem 30. If $\alpha\beta\gamma\delta \neq 0$, then for any ξ, $\xi\,\Delta_1\,\Delta_2\,\Delta_3 = a'\,\alpha\,\Delta_1\,\Delta_2\,\Delta_3 + b'\,\beta\,\Delta_1\,\Delta_2\,\Delta_3 + c'\,\gamma\,\Delta_1\,\Delta_2\,\Delta_3 + d'\,\delta\,\Delta_1\,\Delta_2\,\Delta_3$, where $a' + b' + c' + d' = 1$.

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* Compare Grassmann, *Werke*, Vol. I, 1, § 110.

If $\xi = \alpha$ or β or γ or δ, or if ξ is collinear with an edge of $\alpha\beta\gamma\delta$, or if ξ is coplanar with a face of $\alpha\beta\gamma\delta$, then by theorems 28 and 29 and definition 2, the theorem is easily verified.

Let now ξ be non-coplanar with each of the faces $\alpha\beta\gamma$, $\delta\beta\gamma$, $\alpha\delta\gamma$, $\alpha\beta\delta$ of $\alpha\beta\gamma\delta$. Then there exists (see proof of theorem 27) a point η such that, say, $\alpha\eta\xi = 0$ and $\beta\gamma\delta\eta = 0$, and therefore by theorem 28 and 29 we have

$$\xi\Delta_1\Delta_2\Delta_3 = a\alpha\Delta_1\Delta_2\Delta_3 + y\eta\Delta_1\Delta_2\Delta_3,$$
$$\eta\Delta_1\Delta_2\Delta_3 = b\beta\Delta_1\Delta_2\Delta_3 + c\gamma\Delta_1\Delta_2\Delta_3 + d\delta\Delta_1\Delta_2\Delta_3.$$

Hence by axiom G_{22}, definition 2, theorem 20 and the theorems of § 2, we find from the latter relations,

$$\xi\Delta_1\Delta_2\Delta_3 = a'\alpha\Delta_1\Delta_2\Delta_3 + b'\beta\Delta_1\Delta_2\Delta_3 + c'\gamma\Delta_1\Delta_2\Delta_3 + d'\delta\Delta_1\Delta_2\Delta_3,$$

where

$$a = a', \quad yb = b', \quad yc = c', \quad yd = d'$$

and

$$a' + b' + c' + d' = 1.$$

The Double Tangents of a Binodal Quartic.

By H. Bateman.

1. The number of conditions required to determine an algebraic curve C_n of the nth degree is usually said to be $\frac{1}{2}n(n+3)$. This statement, however, is not complete, for a number of questions must be settled before we can be sure that the theorem is true in a particular case. It will be sufficient at present to consider the following questions:

1) Are the conditions consistent with the properties of a C_n?

2) Are the conditions independent of one another?

3) Is there a non-degenerate C_n satisfying the given conditions?

Of course it would be useless to try to determine a nodal cubic with three inflexions not on a line,* and in this case the reason is quite clear, but there are other cases in which it is not easy to see that the conditions are inconsistent or not independent.

For instance nine double points should determine a C_6, but Cayley† and Halphen‡ have shown that if the points are chosen arbitrarily, the C_6 consists of a double cubic through the nine points; on the other hand if the points are not chosen arbitrarily and there is one non-degenerate C_6 having the given double points, there is an infinite number of C_6's forming a pencil, which satisfy the conditions.

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*A cubic satisfying the conditions would necessarily consist of three straight lines. It should be noticed that to be given an inflexion on a cubic counts as two conditions, but to be given three inflexions on a line is equivalent to only five. For five conditions are sufficient to make two of the points inflexions on a cubic and to make the third point lie on the cubic; this third point is then necessarily an inflexion.

It can easily be verified in fact that there are ∞^1 cubics with a given double point and three given inflexions on a line. For the cubic

$$xy(y-mx)+ax^2+2hxy+by^2=0$$

has inflexions at infinity in the directions $x=0$, $y=0$, $y-mx=0$ if the *two* conditions

$$2h+\frac{a}{m}=0, \qquad 2h+bm=0$$

are satisfied. The equation thus contains the arbitrary parameter h.

†*Proc. London Math. Soc.*, Vol. III (1869–71), pp. 198–202. A corresponding theorem for quartic surfaces had been given previously, *ibid.*, p. 19.

‡*Bull. de la Société Mathématique de France* (1882).

Other theorems of a similar type have been given for curves of higher degree by Halphen and de Jonquières.* In some cases a C_n can be found to satisfy more than the proper number of conditions and then we must conclude that the conditions are not independent.

For instance if we are given six tangents to a conic, two quartic curves can be found with these lines as double tangents and a given point O as double point.† Again, if $A_1, A_2, \ldots, A_6$ are six points on a conic, we can describe a quartic curve to have a double point at a given point O and to touch OA_1, $OA_2, \ldots, OA_6$ at the points $A_1, A_2, \ldots, A_6$.‡

In each case the quartic can apparently be made to satisfy fifteen conditions instead of fourteen. Now this occurs very frequently in the case of the quartic curve, and I have found that when double points and bitangents are given, the method of counting constants must be used with caution; for instance, we have the following theorems:

If we are given four double tangents and one double point of a binodal quartic, the other double point can be chosen arbitrarily when the double tangents are such that a conic passes through the points of contact of three of them § *and not through the points of contact of the fourth.*

If the points of contact of the double tangents lie on a conic, the second double point can not be chosen arbitrarily; it must lie on one of three conics. When the double point has been chosen in a suitable way, there are ∞^1 *quartic curves with the given double points and double tangents.*

If the double tangents are such that no set of three are syzygetic,|| *the second double point can not be chosen arbitrarily, and when the double points have been suitably chosen, there are* ∞^1 *quartics which satisfy the conditions.*

The properties of the quartic on which these theorems depend are not altogether new, but I have not seen a statement of them in the above form. I have discussed the theorems from several points of view and have shown that the second theorem may be derived from Cayley's theorem by a (2, 1) transformation. The second theorem is also discussed with the aid of a particular (2, 1) transformation. The properties of this transformation seem so interesting that some of them have been developed for their own sake.

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* *Comptes Rendus*, Vol. CV, pp. 971–978.

† E. Duporcq: *Nouvelles Annales*, Ser. 4, t. 2, p. 161.

‡ E. Bertini: *Acc. R. de Lincei* (3), I (1877), pp. 92–97.

§ When this is the case the three double tangents are said to form a syzygetic set. Frobenius, *Crelle's Journal*, Bd. XCIX. See also a paper by Miss Long, *Proc. London Math. Soc.*, Ser. 2, Vol. IX (1911), p. 205.

|| They are said to form an asyzygetic set.

Two of the three theorems can be proved geometrically with the aid of the theorem that the eight bitangents of a bicircular quartic touch a rational curve of the third class. This proof did not occur to me until the paper was almost completed.

2. The properties of the double tangents of a binodal quartic may be studied by taking the case of the bicircular quartic and considering this as the stereographic projection of a cyclic from a point O on the sphere S containing the cyclic.*

Let H_1, H_2, H_3, H_4 be the four quadric cones through the cyclic, V_1, V_2, V_3, V_4 their vertices and F_1, F_2, F_3, F_4 the projections of these vertices on the plane π, which is, of course, parallel to the tangent plane at O to the sphere S. We shall call the line of intersection of a plane through O with π the *trace* of this plane.

The traces of the tangent planes from O to the four cones H_1, H_2, H_3, H_4 are double tangents of the bicircular quartic, and so the double tangents meet in pairs in the points F_1, F_2, F_3, F_4 which are the centers of the four circles which invert the bicircular quartic into itself.†

Let h_1, h_2, h_3, h_4 be the four conics which are the reciprocals of the cones H_1, H_2, H_3, H_4 with regard to the sphere S, and let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ be their projections on the plane π. If h_1 meet T, the tangent plane at O to S in the points L_1, M_1, the points L_1, M_1 will be the poles with regard to S of the tangent planes from O to the cone H_1. It follows then that the lines OL_1, OM_1 are perpendicular to the traces of these planes on the plane π, but OL_1, OM_1 are parallel to the asymptotes of the conic Γ_1; hence the two double tangents through F_1 are perpendicular to the asymptotes of the conic Γ_1. The conics $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ are known to be confocal. This may be proved as follows:

Since the cones H and the sphere S belong to a linear system of two terms the tangential equations of the conics h_r are of the form

$$\Sigma + \lambda_r U = 0 \qquad (r = 1, 2, 3, 4),$$

where Σ is the tangential equation of the sphere S.

The poles of a plane through O with regard to these conics lie on a line which meets T in the pole of the plane with regard to S. It follows then that if B is the trace of the plane through O and b the projection of the line, B and b

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*See for instance Darboux's book, "Une classe remarquable de courbes et de surfaces algébriques."

†This theorem and some of the following ones are due to E. Czuber, *Zeitschr. für Math. und Physik*, Vol. XXXII (1887), p. 267. I have amplified his method of treatment and applied it to the theorems given by Richter.

are perpendicular and b is the locus of the poles of B with regard to the conics Γ. Now when a system of conics are such that the poles of a line B lie on a perpendicular line b for all positions of B, the conics are necessarily confocal; therefore Γ_1, Γ_2, Γ_3, Γ_4 are confocal conics.*

If F_1X, F_1X' are the two double tangents through F_1, F_2X, F_2X' the two double tangents through F_2, their four points of intersection lie on a circle C_{12}. For we may regard the two pairs of lines as two conics whose axes are parallel to the axes of the conics Γ, and when the axes of two conics are parallel their four points of intersection lie on a circle.

Richter† has shown that the six circles of type C_{12} are concentric with the conics Γ. To prove this by stereographic projection we start with the equations of the four cones H_r in the form

$$S + \lambda_r Q = 0 \qquad (r = 1, 2, 3, 4).$$

The equations of the pairs of tangent planes from O to the cones are then given by

$$\lambda_r q(S + \lambda_r Q) = (T + \lambda_r P)^2,$$

where q is the constant obtained by substituting the coordinates of O in the equation of Q, and $T = 0$, $P = 0$ are the polar planes of O with regard to $S=0$, $Q = 0$ respectively.

The lines of intersection of the two pairs of planes

$$\lambda_1 q(S + \lambda_1 Q) = (T + \lambda_1 P)^2,$$
$$\lambda_2 q(S + \lambda_2 Q) = (T + \lambda_2 P)^2$$

lie on the cone

$$\lambda_1\lambda_2(qS - 2PT) = (\lambda_1 + \lambda_2)T^2,$$

whose trace on the plane π ($T = \text{const.}$) is the circle C_{12}. The center of this circle does not depend upon the quantities λ_1, λ_2, and so the six circles of type C_{12} are concentric. If we had drawn tangent cones from O to any two quadrics through the cyclic, their traces on the plane π would be conics touching the bicircular quartic in four points, and the foregoing analysis indicates that the four points of intersection of these conics‡ lie on a circle concentric with the circles C_{12}. To prove that the circles C_{12} are concentric with the conics Γ, we

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*The relation between the conics Γ and the points F is such that $F_2F_3F_4$ is a self-polar triangle for Γ_1, $F_3F_1F_4$ a self-polar triangle for Γ_2 and so on. The foci of the conics Γ are pairs of isogonal conjugate points with regard to the pedal triangle of $F_1F_2F_3$.

†*Zeitschr. für Mathematik und Phys.*, Vol. XXXV, Supplement (1890), p. 1.

‡The axes of the conics are parallel.

consider the point t which is the pole of T with regard to the conic h_1; this point will project into the center of Γ_1.

Since h_1 is the reciprocal of H_1 with regard to S, it follows that the polar plane of t with regard to S is the same as the polar plane of O with regard to H_1. Consequently

$$S'p = s'P,$$

where S' is the polar plane of t with regard to S, and s', p are the values of S' and P at O. Now the equation of the polar plane of t with regard to the cone

$$\lambda_1\lambda_2(qS - 2PT) = (\lambda_1 + \lambda_2)T^2$$

is

$$\lambda_1\lambda_2(qS' - pT - tP) = (\lambda_1 + \lambda_2)tT,$$

and since $p \equiv q$, $t \equiv s'$ this reduces simply to $T = 0$. It follows, then, that the trace of this cone is a circle whose center is the projection of the point t; hence the circles C and the conics Γ are concentric. Richter has also shown that if U_1, U_1' are the feet of the perpendiculars from the center of the circles C on the two double tangents through F_1, then the four lines U_1U_1', U_2U_2', U_3U_3', U_4U_4' meet in a point.

3. When the bicircular quartic is projected into a binodal quartic, the theorems of Richter and Czuber take the following form:

The double tangents of a binodal quartic can be arranged in four pairs, P_1, P_1'; P_2, P_2'; P_3, P_3'; P_4, P_4', meeting respectively in four points F_1, F_2, F_3, F_4. The points of intersection of any two pairs such as P_1, P_1'; P_2, P_2' lie on a conic C_{12} which passes through the double points $\Omega\Omega'$.

The points F_1, F_2, F_3, F_4 are the poles of $\Omega\Omega'$ with regard to the four conics which pass through $\Omega\Omega'$ and the intersections of the tangents from Ω and Ω' to the quartic.

The six conics of type C_{12} have double contact at the points $\Omega\Omega'$.

It follows from the first theorem that if we are given the four double tangents $P_1P_1'P_2P_2'$ and one double point Ω, then the other double point Ω' can not be chosen arbitrarily, for it must lie on a conic through Ω and four of the intersections of the lines $P_1P_1'P_2P_2'$. There are really three conics which correspond to different sets of two out of three pairs of opposite corners of the quadrilateral formed by the double tangents.

We shall now show that this theorem may be deduced from Halphen's theorem on the double points of a sextic. Let seven arbitrary points be chosen as the base points of a cubic $(1,2)$ transformation between two planes X' and X.

This transformation * is such that a point in X' corresponds to two points in X, while a point in X corresponds to only one point in X'. A line in X' corresponds to a cubic curve through the seven base points $A_1, A_2, \ldots, A_7$, while a line in X corresponds to a rational cubic touching the synoptic curve J' at six points.

It has been shown by Clebsch, Noether and de Paolis that the synoptic curve J' is a general quartic,† and the corresponding curve J (the Jacobian) is a sextic with double points at $A_1, \ldots, A_7$. Now let us apply this transformation to Halphen's theorem that if eight double points of a sextic are given, the ninth can not be chosen arbitrarily, its locus being in fact a C_9 with triple points at the eight given points.

Let $A_1, A_2, \ldots, A_7, A_8$ be the eight given double points; then if we assume that the sextic is not described by pairs of complementary points in the plane X, the corresponding curve in the plane X' will be a binodal quartic touching the synoptic curve J' at eight points which correspond to those in which the sextic meets J.

The corresponding theorem is that for a certain set of binodal quartics touching J' at eight points the two nodes can not be chosen arbitrarily, and that when the nodes are properly chosen, there are ∞^1 quartics of the set having the given nodes.‡

If A_8' is the point corresponding to an eighth given double point on Halphen's sextics, the tangents from A_8' to J' will correspond to the nodal cubics through the eight points, and the points of contact of these tangents to the twelve critical centers for cubics through the eight points. The curve of the ninth degree, which is the locus of a ninth double point on the sextic, meets J at the twelve critical centers and corresponds to a curve of degree

$$3 \times 9 - 7 \times 3 = 6$$

touching J' at the twelve points of contact of the tangents from A_8'. Hence, if

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*The properties of this transformation are discussed very fully by de Paolis. *Memorie dell' Accademia dei Lincei*, Ser. 3, Bd. I–II (3 papers). Miss Scott's papers, *Quarterly Journal*, Vol. XXIX, p. 329, Vol. XXXII, p. 209, *Bull. Amer. Math. Soc.*, Vol. VII, p. 24 (1901), deal with the general theory of the rational transformation between two planes and will provide the reader with practically all the information that is required here. We may also refer to R. Sturm, *Die Lehre von den geometrischen Verwandtschaften*, Vol. IV, pp. 232–268.

†This may be seen at once by mapping the plane X on to a cubic surface by means of cubics through $A_1, A_2, \ldots, A_6$. The synoptic curve in X' can then be regarded as the section of the tangent cone to the cubic from the point corresponding to A_7.

‡I notice that on p. 278, Vol. IV, of his "Geometrische Verwandschaften," Sturm obtains the theorem on the cubic surface which corresponds to Halphen's theorem; as the above theorem may be deduced from Sturm's theorem by projection, it is probable that the theorem was known to Sturm.

one node of the binodal quartic be given, the locus of the other is a sextic curve.

To investigate the question analytically, let the equation of the binodal quartic be

$$Q(x', y', z') = 0.$$

When x', y', z' are expressed in terms of (x, y, z), this equation must take the form

$$Q \equiv SS',$$

where S is the equation of the sextic and S' the equation of the complementary sextic. Now S and S' each have double points at the seven base points and meet on J, hence S, S' and J belong to the same pencil and we may put

$$S \equiv H + \lambda J,$$
$$S' \equiv H - \mu J,$$

where λ and μ are constants.

Since the synoptic curve J' corresponds to J, we have

$$J'(x', y', z') \equiv [J(x, y, z)]^2,$$

and x, y, z can be expressed rationally in terms of x', y', z' and $\sqrt{J'}$. When these rational values are substituted in the expressions for S and S' and J is put equal to $+\sqrt{J'}$ in one case and equal to $-\sqrt{J'}$ in the other, the two resulting expressions must be the same, for both S and S' correspond to the same curve Q. This, however, is only possible if $\lambda = \mu$ and H is a rational function of (x', y', z'). The equation for Q now takes the form

$$Q = H^2 - \lambda^2 J^2 = H^2 - \lambda^2 J' = 0,$$

where $H(x', y', z')$ is of the second degree in (x', y', z'). The eight points of contact of the binodal quartic with J' thus lie on a conic $H = 0$.

Let us now change the notation and use (x, y, z) to denote homogeneous coordinates in the plane X', also let $F(x, y, z) = 0$ be the equation of J', so that $J' \equiv F$. If (x, y, z), (x', y', z') are two double points of the curve Q, we have:

$$\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial y} = \frac{\partial Q}{\partial z} = 0,$$

$$\frac{\partial Q'}{\partial x'} = \frac{\partial Q'}{\partial y'} = \frac{\partial Q'}{\partial z'} = 0.$$

These equations give

$$2H\frac{\partial H}{\partial x} = \lambda^2\frac{\partial F}{\partial x}, \text{ etc.};$$

therefore

$$2H\left(x'\frac{\partial H}{\partial x} + y'\frac{\partial H}{\partial y} + z'\frac{\partial H}{\partial z}\right) = \lambda^2\left(x'\frac{\partial F}{\partial x} + y'\frac{\partial F}{\partial y} + z'\frac{\partial F}{\partial z}\right),$$

and similarly

$$2H'\left(x\frac{\partial H'}{\partial x'} + y\frac{\partial H'}{\partial y'} + z\frac{\partial H'}{\partial z'}\right) = \lambda^2\left(x\frac{\partial F'}{\partial x'} + y\frac{\partial F'}{\partial y'} + z\frac{\partial F'}{\partial z'}\right).$$

Now since H is of the second degree,

$$x'\frac{\partial H}{\partial x} + y'\frac{\partial H}{\partial y} + z'\frac{\partial H}{\partial z} = x\frac{\partial H'}{\partial x'} + y\frac{\partial H'}{\partial y'} + z\frac{\partial H'}{\partial z'};$$

hence

$$H\left(x\frac{\partial F'}{\partial x'} + y\frac{\partial F'}{\partial y'} + z\frac{\partial F'}{\partial z'}\right) = H'\left(x'\frac{\partial F}{\partial x} + y'\frac{\partial F}{\partial y} + z'\frac{\partial F}{\partial z}\right),$$

and consequently

$$F\left(x\frac{\partial F'}{\partial x'} + y\frac{\partial F'}{\partial y'} + z\frac{\partial F'}{\partial z'}\right)^2 = F'\left(x'\frac{\partial F}{\partial x} + y'\frac{\partial F}{\partial y} + z'\frac{\partial F}{\partial z}\right)^2.$$

This relation shows that the two double points can not be chosen arbitrarily; when one double point A_8' is given, the locus of the other is a sextic curve which touches J' at the points where it is met by the first polar of J', *i. e.*, at the points of contact of the tangents from A_8'. The sextic has a triple point at A_8' and double points at the three points of intersection of the first polar and polar line of P' with regard to J'.

The case in which J' consists of four straight lines is of special interest, for then the lines form a set of double tangents of the binodal quartic whose points of contact lie on a conic, *i. e.*, a syzygetic set according to the nomenclature of Frobenius.

Putting $F = xyzt$, the equation of the sextic becomes

$$x'y'z't'[t(x'yz + y'zx + z'xy) + xyzt']^2 = xyzt[t'(xy'z' + yz'x' + zx'y') + x'y'z't]^2,$$

or

$$(t'x'yz - txy'z')(t'y'zx - tyz'x')(t'z'xy - tzx'y') = 0.$$

The sextic thus consists of three conics, each of which passes through the two double points and two pairs of opposite corners of the quadrilateral formed by the double tangents. This corresponds exactly with Richter's theorem.

If then we are given four double tangents which form a syzygetic set and one double point of a binodal quartic, the other double point can not be chosen arbitrarily; its locus consists of three conics. When the two double points have been suitably chosen, there are ∞^1 binodal quartics with the given double points and double tangents.*

If we are given the two double points and three double tangents whose points of contact lie on a conic, the fourth double tangent of the syzygetic set must pass through one of three fixed points on the line joining the nodes.

When Hesse's notation for the double tangents is used, the eight double tangents of a binodal quartic can be represented by the symbols †

$$\begin{matrix} 12 & 27 & 28 & 35 \\ 78 & 18 & 17 & 46 \end{matrix}$$

A pair such as 12, 78 meet in a point F_1; any two pairs $\begin{matrix}27\\18\end{matrix}$ and $\begin{matrix}28\\17\end{matrix}$ form a syzygetic set. There are clearly six syzygetic sets. The tangents from the nodes each count as two double tangents and have two symbols associated with them,‡ thus

$$\binom{48}{68} \quad \binom{14}{26} \quad \binom{24}{26} \quad \binom{67}{47}$$

are the tangents from one node, and

$$\binom{51}{31} \quad \binom{52}{32} \quad \binom{57}{37} \quad \binom{58}{38}$$

are the tangents from the other. The line joining the nodes is represented by the four symbols

$$34, \quad 36, \quad 45, \quad 46.$$

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*In the case of the bicircular quartic the double tangents F_1X, F_1X', F_2X, F_2X' determine the circle C_{12} and its center K. The asymptotes of the conics Γ_1 and Γ_2 can then be determined, but the linear dimensions of one of these conics can be chosen arbitrarily. When Γ_1 is known, the line F_3F_4, which is the polar of F_2, can be found; Γ_2 is then determined by the fact that F_1 is the pole of F_3F_4 with regard to Γ_2. Finally F_3, F_4 are determined by the fact that they are conjugate with regard to Γ_1 and Γ_2.

†Equations of the 28 double tangents of a quartic and the corresponding symbols are given by Cayley (*Crelle's Journal* (1883), p. 93; *Collected Papers*, Vol. XI, p. 221, Vol. XII, p. 74), and by de Paolis, *Rend. Lincei*, II, 2 (1877–1878).

‡This is well known. It may be deduced, for instance, from the results given by G. Fonténé. Sur les dégénérescences des 63 systemes de coniques quadruplement tangentes a une quartique. *Bull. de la Société Mathématique de France*, t. 27 (1899), p. 229.

4. We must now investigate the question whether the double points can be chosen arbitrarily when we are given four double tangents which do not form a syzygetic set. There are two cases to be considered.

(1) We may be given four double tangents which are such that the points of contact of no set of three lie on a conic.

(2) We may be given four double tangents which are such that a conic passes through the points of contact of three of them, but does not pass through the points of contact of the fourth.

In the first case the double tangents will belong to one of the sixteen sets:

12	18	17	35	78	27	28	35
12	18	17	46	78	27	28	46
12	27	28	35	78	18	28	35
12	27	28	46	78	18	28	46
12	27	18	35	78	18	17	35
12	27	18	46	78	18	17	46
12	27	17	35	78	27	17	35
12	27	17	46	78	27	17	46

formed by taking one double tangent from each pair. This case may be discussed by means of the particular (2, 1) cubic transformation in which the synoptic curve consists of four straight lines.

Let the net of cubics in the plane X pass through the fixed point O and touch three fixed lines OA, OB, OC at the fixed points A, B, C respectively. Taking ABC as triangle of reference and choosing the coordinates so that O is the point (1, 1, 1), the net of cubics is represented by the equation

$$\xi x^2(y-z)+\eta y^2(z-x)+\zeta z^2(x-y)=0.$$

This corresponds to the line

$$\xi x'+\eta y'+\zeta z'=0$$

if

$$\frac{x'}{x^2(y-z)}=\frac{y'}{y^2(z-x)}=\frac{z'}{z^2(x-y)}.$$

These are the equations of the transformation; the Jacobian consists of the six lines

$$xyz(y-z)(z-x)(x-y)=0,$$

and the synoptic curve of the four lines

$$x'=0,\ y'=0,\ z'=0,\ x'+y'+z'=0.$$

A point (x', y', z') in the plane X' corresponds to two points (x, y, z), (x_1, y_1, z_1), where

$$x_1 : y_1 : z_1 = \frac{x}{zx+xy-yz} : \frac{y}{xy+yz-zx} : \frac{z}{yz+zx-xy}.$$

Let us now consider the curve which corresponds to a quartic having a double point at O and touching OA, OB, OC at A, B, C respectively. The equation of such a quartic is

$$lx^2(y-z)(2x-y-z)+my^2(z-x)(2y-z-x)$$
$$+nz^2(x-y)(2z-x-y)-\lambda x^2(y-z)^2-\mu y^2(z-x)^2-\nu z^2(x-y)^2=0,$$

and the equation of the corresponding curve is found as follows: Put

$$\frac{x'}{x^2(y-z)}=\frac{y'}{y^2(z-x)}=\frac{z'}{z^2(x-y)},$$

$$\theta=mz'-ny'-\lambda x', \quad \phi=nx'-lz'-\mu y', \quad \psi=ly'-mx'-\nu z';$$

then the previous equation may be written in the form:

$$\frac{\theta x'}{x^2}+\frac{\phi y'}{y^2}+\frac{\psi z'}{z^2}=0.$$

Now

$$\frac{x'}{x^2}+\frac{y'}{y^2}+\frac{z'}{z^2}=0,$$

$$\frac{x'}{x}+\frac{y'}{y}+\frac{z'}{z}=0;$$

therefore on eliminating x, y, z we obtain the equation

$$\sqrt{x'(\phi-\psi)}+\sqrt{y'(\psi-\theta)}+\sqrt{z'(\theta-\phi)}=0. \qquad (1)$$

This represents a quartic curve having the point $\theta=\phi=\psi$ as a node and the lines $x'=0$, $y'=0$, $z'=0$, $x'+y'+z'=0$ as double tangents. In order to make the point (x', y', z') a node, we have then to satisfy only two equations, viz.:

$$\theta=\phi=\psi.$$

Now the original quartic curve will have a node at the point (x, y, z) if three conditions

$$\frac{\partial f}{\partial x}=0, \quad \frac{\partial f}{\partial y}=0, \quad \frac{\partial f}{\partial z}=0$$

are satisfied. These, together with the two conditions $\theta=\phi=\psi$, give five linear conditions for the determination of the ratios of $(l, m, n, \lambda, \mu, \nu)$. When

these are determined, the quartic curve (1) will have a double point at (x', y', z') and a second double point corresponding to (x, y, z), both points being arbitrary. To show that the quartic is not degenerate, we have to prove that a proper quartic can be drawn to have a double point at O, to touch OA, OB, OC at A, B, C, to have a second double point at $P(x, y, z)$, and to pass through two assigned complementary points corresponding to (x', y', z'). It is certainly possible to determine a quartic from these conditions, for we have only

$$3 + 6 + 3 + 2 = 14$$

conditions to satisfy; but there is a cubic curve touching OA, OB, OC at A, B, C respectively and passing through O, P and the two complementary points, and it is this cubic curve together with the line OP which is determined by the given conditions. If P is the double point of a proper quartic curve which satisfies the conditions, there will be a pencil of quartics of the same kind.

It follows, then, that for a non-degenerate curve of type

$$\sqrt{x'(\phi - \psi)} + \sqrt{y'(\psi - \theta)} + \sqrt{z'(\theta - \phi)} = 0$$

the two double points can not be chosen arbitrarily, and when the double points have been suitably chosen there are ∞^1 curves with the given double points and double tangents whose equations have the above form.

To find if the points of contact of the double tangents lie on a conic or not, we put $x' = 0$, $y' = 0$, $z' = 0$, $x' + y' + z' = 0$ in succession. The points of contact of these four double tangents are then found to be given by the equations

$$\begin{aligned}
&y'^2(l + n) - z'^2(m + l) - y'z'(\nu + m - n + \mu) = 0, \qquad x' = 0;\\
&z'^2(l + m) - x'^2(m + n) - z'x'(\lambda + \nu + n - l) = 0, \qquad y' = 0;\\
&x'^2(m + n) - y'^2(l + n) - x'y'(\mu + \lambda + l - m) = 0, \qquad z' = 0;\\
&y'^2(\lambda + \mu) + z'^2(\lambda + \nu) + 2\lambda y'z' = 0, \qquad x' + y' + z' = 0.
\end{aligned}$$

It is clear from the form of the coefficients of x'^2, y'^2, z'^2 in the first three equations that a conic does not pass through the points of contact of the first three double tangents, and it is easy to verify that the points of contact of any other set of three do not lie on a conic. For instance, the general equation of a conic through the points of contact of the second and third is

$$\begin{aligned}
y'^2(l + n) + z'^2(l + m) - x'^2(m + n) + x'y'(\mu + \lambda + l - m)&\\
- z'x'(\lambda + \nu + n - l) + ky'z' = 0;&
\end{aligned}$$

this meets the last double tangent $x' + y' + z' = 0$ in points given by

$$z'^2(\lambda + \nu) - y'^2(\lambda + \mu) + (k + \nu - \mu - m - n - 2l)y'z' = 0,$$

and under no circumstances are these the same as the points of contact which are given by the equation

$$y'^2(\lambda+\mu)+z'^2(\lambda+\nu)+2\lambda y'z'=0.$$

Hence we have the following theorem:

If we are given an asyzygetic set of double tangents of a binodal quartic, the two double points can not be chosen arbitrarily, and when the double points have been suitably chosen there are ∞^1 quartics with the given double points and double tangents.

Let (X', Y', Z') be the coordinates of the double point given by the equations $\theta=\phi=\psi$. Then we have

$$\begin{aligned} X'(m+n)-Y'(\mu+l)+Z'(\nu-l)&=0,\\ X'(\lambda-m)+Y'(n+l)-Z'(\nu+m)&=0,\\ -X'(\lambda+n)+Y'(\mu-n)+Z'(l+m)&=0. \end{aligned}$$

Also, if (x', y', z') are the coordinates of a second double point, we find on differentiating the equation

$$\sqrt{x'(\phi-\psi)}+\sqrt{y'(\psi-\theta)}+\sqrt{z'(\theta-\phi)}=0$$

with regard to x', y', z', that

$$\frac{1}{2\sqrt{x'}}\sqrt{(\phi-\psi)}+(m+n)\sqrt{\frac{x'}{\phi-\psi}}+(\lambda-m)\sqrt{\frac{y'}{\psi-\theta}}-(\lambda+n)\sqrt{\frac{z'}{\theta-\phi}}=0,$$

$$\frac{1}{2\sqrt{y'}}\sqrt{(\psi-\theta)}-(\mu+l)\sqrt{\frac{x'}{\phi-\psi}}+(n+l)\sqrt{\frac{y'}{\psi-\theta}}+(\mu-n)\sqrt{\frac{z'}{\theta-\phi}}=0,$$

$$\frac{1}{2\sqrt{z'}}\sqrt{(\theta-\phi)}+(\nu-l)\sqrt{\frac{x'}{\phi-\psi}}-(\nu+m)\sqrt{\frac{y'}{\psi-\theta}}+(l+m)\sqrt{\frac{z'}{\theta-\phi}}=0.$$

Multiplying these equations by X', Y', Z' respectively and adding, we get

$$\frac{X'}{\sqrt{x'}}\sqrt{(\phi-\psi)}+\frac{Y'}{\sqrt{y'}}\sqrt{(\psi-\theta)}+\frac{Z'}{\sqrt{z'}}\sqrt{(\theta-\phi)}=0.$$

Combining this with the equation

$$\sqrt{x'(\phi-\psi)}+\sqrt{y'(\psi-\theta)}+\sqrt{z'(\theta-\phi)}=0,$$

we get

$$\frac{\sqrt{(\phi-\psi)}}{(Y'z'-Z'y')\sqrt{x'}}=\frac{\sqrt{(\psi-\theta)}}{(Z'x'-X'z')\sqrt{y'}}=\frac{\sqrt{(\theta-\phi)}}{(X'y'-Y'x')\sqrt{z'}}.$$

Therefore

$$x'(Y'z'-Z'y')^2+y'(Z'x'-X'z')^2+z'(X'y'-Y'x')^2=0. \qquad (2)$$

If (X', Y', Z') be given, the locus of (x', y', z') is a rational cubic having a double point at (X', Y', Z') and passing through the six corners of the quadrilateral formed by the four lines

$$x'=0, \quad y'=0, \quad z'=0, \quad x'+y'+z'=0.$$

If, on the other hand, the point (x', y', z') be given, the locus of (X', Y', Z') is a conic which consists of a pair of straight lines through the point (x', y', z'). It is in fact part of the Jacobian of the net of cubics passing through the six corners of the quadrilateral and the point (x', y', z').

The two double points of the binodal quartic thus bear different relations to the four double tangents of an asyzygetic set; this is suggested by the nomenclature, for an asyzygetic set

$$12, \quad 18, \quad 17, \quad 35$$

appears to be associated with the node [35] and the set

$$12, \quad 18, \quad 17, \quad 46$$

with the node [46].

The lack of symmetry is also indicated by the corresponding configuration in the X plane, for one double point corresponds to a pair of complementary points M, M' and the other to a node P on the original quartic.

To find the condition that a quartic may touch OA, OB, OC at A, B, C, pass through M, M' and have a node at P, we must find the condition corresponding to

$$x'(Y'z'-Z'y')^2+y'(Z'x'-X'y')^2+z'(X'y'-Y'x')^2=0.$$

Putting

$$\frac{x'}{x^2(y-z)}=\frac{y'}{y^2(z-x)}=\frac{z'}{z^2(x-y)},$$

$$\frac{X'}{X^2(Y-Z)}=\frac{Y'}{Y^2(Z-X)}=\frac{Z'}{Z^2(X-Y)},$$

the equation takes the form

$$(y-z)(z-x)(x-y)[X'yz+Y'zx+Z'xy][X'yz(yz-zx-xy) + Y'zx(zx-xy-yz)+Z'xy(xy-yz-zx)]=0.$$

Hence, when M and M' are given, the locus of P is a conic * and a quartic.

---

*If the conic pass through a fixed point (a, b, c), the cubic (2) will also pass through a fixed point (a', b', c') and its node lie on the fixed line

$$\frac{X'}{a}+\frac{Y'}{b}+\frac{Z'}{c}=0.$$

There is a $(1, 2)$ correspondence between the point (a', b', c') and this line, in which corresponding elements are incident.

When P is given the locus of M and M' consists of two cubics, each of which is described by pairs of complementary points. If we are given two double points and three asyzygetic double tangents, a fourth double tangent forming with the given ones an asyzygetic set can not be chosen arbitrarily. To find the envelope of the fourth double tangent, we write lx, my, nz in place of x', y', z'; we then have to find the envelope of the double tangent

$$lx + my + nz = 0,$$

when l, m, n are subject to the relation

$$mnx(Yz - Zy)^2 + nly(Zx - Xy)^2 + lmz(Xy - Yx)^2,$$

which is deduced from (2) by putting

$$lX = X', \; mY = Y', \; nZ = Z'.$$

The envelope of the fourth double tangent is thus a conic which touches the sides of the triangle $xyz = 0$. This conic passes through the point (x, y, z) and also through the point $\left(\frac{X^2}{x}, \frac{Y^2}{y}, \frac{Z^2}{z}\right)$. The tangents at these two points meet at the point (X, Y, Z). The conic, therefore, touches the line joining the nodes at one of the nodes.* In reality there are two conics, one being associated with each node.

5. Turning now to the second case when we are given three double tangents forming a syzygetic set, we shall next consider whether a fourth double tangent can be chosen arbitrarily when the two double points are given, and whether the double points can be chosen arbitrarily when a fourth double tangent is given. Let the three given bitangents be taken as sides of the triangle of reference; then there are three possible forms for the equation of a binodal quartic, viz.:

$$[yz + x(lx + my + nz) + (px + qy + rz)^2]^2 = 4\, xyz(lx + my + nz),$$
$$[zx + y(lx + my + nz) + (px + qy + rz)^2]^2 = 4\, xyz(lx + my + nz),$$
$$[xy + z(lx + my + nz) + (px + qy + rz)^2]^2 = 4\, xyz(lx + my + nz).$$

In the first case the double points are given by the equations

$$px + qy + rz = 0,$$
$$yz - x(lx + my + nz) = 0.$$

*This might have been deduced from the fact that a Brianchon set of six bitangents of a general quartic touch a conic. When four asyzygetic bitangents and one node A are given, the locus of the other node consists either of the tangents to the two conics through A which touch the four lines, or else of the rational cubic which is the locus of the points of contact of the tangents from A to conics touching the four lines.

Let the coordinates of the double points be (a, b, c), (α, β, γ) respectively; then

$$a\beta\gamma(la + mb + nc) = \alpha bc(l\alpha + m\beta + n\gamma).$$

Hence the double tangent $lx + my + nz = 0$ passes through a fixed point on the line of nodes. This agrees with the result of § 3. Writing the equation of the quartic in the form

$$[yz - x(lx + my + nz)]^2 + (px + qy + rz)^2[(px + qy + rz)^2 + 2yz + 2x(lx + my + nz)] = 0, \quad (1)$$

we may transform it into

$$[yz - x(lx + my + nz) + \theta(px + qy + rz)^2]^2 = (px + qy + rz)^2[(\theta^2 - 1)(px + qy + rz)^2 + 2(\theta - 1)yz - 2(\theta + 1)x(lx + my + nz)].$$

Now let θ be chosen so that

$$(\theta^2 - 1)(px + qy + rz)^2 + 2(\theta - 1)yz - 2(\theta + 1)x(lx + my + nz) = 0$$

breaks up into factors; then the lines represented by these equations will be double tangents. Let $x = \xi y + \eta z$ be one of these lines. Then we must have identically

$$(\theta^2 - 1)[(p\xi + q)y + (p\eta + r)z]^2 + 2(\theta - 1)yz - 2(\theta + 1)(\xi y + \eta z)[(l\xi + m)y + (l\eta + n)z] \equiv 0;$$

$$\therefore \quad (\theta^2 - 1)(p\xi + q)^2 - 2(\theta + 1)(\xi l + m)\xi = 0,$$

$$2(\theta^2 - 1)(p\xi + q)(p\eta + r) + 2(\theta - 1) - 2(\theta + 1)[\xi(l\eta + n) + \eta(l\xi + m)] = 0,$$

$$(\theta^2 - 1)(p\eta + r)^2 - 2(\theta + 1)(l\eta + n)\eta = 0.$$

These equations give

$$\theta = 1 + 2\frac{\xi(\xi l + m)}{(p\xi + q)^2} = 1 + 2\frac{\eta(\eta l + n)}{(p\eta + r)^2}.$$

Substituting these values of $\xi l + m$, $\eta l + n$ in the second equation, we get

$$2(\theta + 1)(p\xi + q)(p\eta + r) + 2 - (\theta + 1)\left[\frac{\xi}{\eta}(p\eta + r)^2 + \frac{\eta}{\xi}(p\xi + q)^2\right] = 0,$$

or

$$\frac{2\xi\eta}{\theta + 1} = [\xi(p\eta + r) - \eta(p\xi + q)]^2 = (\xi r - \eta q)^2.$$

Hence we have the equations

$$\frac{\xi\eta}{(\xi r - \eta q)^2} = 1 + \frac{\xi(\xi l + m)}{(p\xi + q)^2} = 1 + \frac{\eta(\eta l + n)}{(p\eta + r)^2}. \quad (2)$$

Now, if the double points are given, the equations

$$pa + qb + rc = 0,$$
$$p\alpha + q\beta + r\gamma = 0$$

give

$$\frac{p}{b\gamma - c\beta} = \frac{q}{c\alpha - a\gamma} = \frac{r}{a\beta - b\alpha} = \lambda,$$

where λ is an arbitrary parameter, and the equations

$$bc - a(la + mb + nc) = 0,$$
$$\beta\gamma - \alpha(l\alpha + m\beta + n\gamma) = 0$$

give

$$m(b\gamma - c\beta) = \frac{c\gamma}{a\alpha}(b\alpha - a\beta) + l(\alpha c - a\gamma),$$

$$n(b\gamma - c\beta) = \frac{b\beta}{a\alpha}(\gamma a - c\alpha) - l(b\alpha - a\beta).$$

Hence p, q, r, l, m, n can be expressed in terms of two arbitrary parameters l and λ. These parameters both occur in equations (2) and these equations can be solved for l and λ; consequently there are two arbitrary parameters in the determination of ξ and η, and so the double tangent $x = \xi y + \eta z$ can be chosen arbitrarily. It follows, then, that when this double tangent is given, the two double points can be chosen arbitrarily.

To complete the proof we must show that the binodal quartic represented by equation (1) is not degenerate. If it consisted of a straight line and a cubic, the line would have to be the line joining the double points, viz., $px+qy+rz=0$. This is not generally a factor of (1), so this possibility may be rejected. If the curve consisted of two conics, there would only be four constants in the equation instead of six, so this possibility may also be rejected.

6. The theorems of §§ 3, 4 and 5 may also be proved geometrically by considering the case of a bicircular quartic.

It is known that in the case of a general quartic the twelve bitangents of a Steiner system, *e. g.*,

12	27	28	35	36	34
78	18	17	46	56	54

touch a curve of the third class,* and that the six points of intersection of the pairs of bitangents lie on a conic.† Now, when the quartic has double points

*Salmon: Higher plane curves, 3rd edition (1879), pp. 150, 226.

†Baker: *Proc. London Math. Soc.*, Ser. 2, Vol. IX (1911), p. 145.

at the circular points, the four bitangents 36, 34, 56, 54 coincide with the line at infinity, and so the curve of the third class has this line as a double tangent. The points of intersection of 34, 54 and 36, 56 become the points of contact of the double tangent; these points lie on a conic through F_1, F_2, F_3, F_4, and this conic is a rectangular hyperbola since F_1, F_2, F_3, F_4 form a triangle and its orthocenter. Hence, the points of contact of the line at infinity with the curve are harmonic with the circular points. They are also the points at infinity on the bisectors of the angles between pairs of bitangents. To prove this let $\alpha\alpha', \beta\beta', \gamma\gamma'$ be three pairs of bitangents; then the curve of the third class is the Cayleyan of the net of conics

$$\lambda\alpha\alpha' + \mu\beta\beta' + \nu\gamma\gamma' = 0.$$

But since the circle through the four points of intersection of $\alpha\alpha'$ with $\beta\beta'$ is concentric with the circle through the four points of intersection of $\alpha\alpha'$ with $\gamma\gamma'$, the problem of finding the Cayleyan may be simplified by regarding the net of conics as built up from

$$ax^2 + by^2 = 0, \quad (x-\alpha_0)^2 + (y-\beta_0)^2 = r^2, \quad (x-\alpha_0)^2 + (y-\beta_0)^2 = \rho^2.$$

The tangential equation of the Cayleyan is easily found to be

$$(b\xi^2 + a\eta^2)(\beta_0\xi - \alpha_0\eta) = (a-b)\xi\eta\zeta.$$

Its form indicates at once that the line at infinity ($\xi = \eta = 0$) is a double tangent and that the points of contact are on the axes, *i. e.*, on the bisectors of the angles between the bitangents $ax^2 + by^2 = 0$.

By using the equation of the Cayleyan we can express the equations of the eight bitangents in the form

$$\begin{aligned} y + xt_s &= \frac{(a + bt_s^2)(\beta_0 t_s - \alpha_0)}{(a-b)t_s}, \\ y + xt_s &= \frac{(a + bt_s^2)(\beta_0 t_s + \alpha_0)}{(a-b)t_s}, \end{aligned} \qquad (s = 1, 2, 3, 4)$$

where

$$t_1^2 t_2^2 t_3^2 t_4^2 = -\frac{a^2\alpha_0^2}{b^2\beta_0^2}.$$

The coordinates of the points of intersection of pairs of bitangents are

$$x_s = \frac{\alpha_0}{b-a}\left(b + \frac{a}{t_s^2}\right), \quad y_s = \frac{\beta_0}{a-b}(a + bt_s^2), \quad (s = 1, 2, 3, 4)$$

and these form a triangle and its orthocenter on account of the relation

between t_1, t_2, t_3, t_4. The equation of the circle through the four points of intersection of the bitangents with parameters $t_r, \ldots, t_r$ with those with parameters $t_s, \ldots, t_s$ is

$$(x-\alpha_0)^2+(y-\beta_0)^2=(a^2\alpha_0^2+b^2\beta_0^2 t_r^2 t_s^2)(1+t_r^2)(1+t_s^2).$$

Now let us see how the Cayleyan helps us in the problem of determining the bicircular quartic by means of four bitangents. If four syzygetic bitangents such as

$$\begin{matrix} 12 & 27 \\ 78 & 18 \end{matrix}$$

are given, the bisectors of the angles between each pair must be parallel, otherwise there would be no quartic. These four bitangents determine the Cayleyan completely; a third bitangent, such as 28, may be taken to be an arbitrary tangent of the Cayleyan, and then the configuration can be determined completely, for 17 is the tangent making the same angles with the axes as 28, and three points such as F_1, F_2, F_3 determine the fourth on account of the property of the orthocenter.

If a set of bitangents such as

$$\begin{matrix} 12 & 27 & 28 \\ 78 & & \end{matrix}$$

be given, the Cayleyan can be determined, for 12 and 78 determine the origin of coordinates, the directions of the axes and the ratio a/b, while 27 and 28 give the constants α and β. The whole configuration is now determined uniquely; hence four double tangents of this kind can be chosen arbitrarily, and there is only one bicircular quartic.

The case when four bitangents such as

$$12 \cdot 27 \quad 28 \quad 35$$

are given can not be discussed very easily with the aid of the Cayleyan.

7. The special (2, 1) transformation used in this paper is capable of so many useful applications * that it seems worth while to work out its properties in detail. The equations

$$\frac{x'}{x^2(y-z)}=\frac{y'}{y^2(z-x)}=\frac{z'}{z^2(x-y)}$$

give

$$x'(y+z)+y'(z+x)+z'(x+y)=0.$$

*I have used it in a former paper, *Quarterly Journal*, No. 147 (1906), to prove an interesting property of a uninodal quartic.

This is the equation of the line joining two complementary points. On the portion $x=0$ of the Jacobian, we have

$$x'=0, \quad \frac{y'}{y}=-\frac{z'}{z};$$

hence the line joining two consecutive corresponding points is in this case

$$x(Y-Z)-yZ+zY=0.$$

This passes through the point $(-1,1,1)$. The "principal lines" at points of $y=0$, $z=0$ can be shown to pass through $(1,-1,1)$, $(1,1,-1)$ respectively.

A straight line $lx+my+nz=0$ corresponds to a rational cubic, for the equations

$$lx+my+nz=0,$$
$$x(y'+z')+y(z'+x')+z(x'+y')=0$$

give

$$\frac{x}{n(z'+x')-m(x'+y')}=\frac{y}{l(x'+y')-n(y'+z')}=\frac{z}{m(y'+z')-l(z'+x')}.$$

Substituting in the identical relation

$$\frac{x'}{x}+\frac{y'}{y}+\frac{z'}{z}=0,$$

we obtain

$$\frac{x'}{n(z'+x')-m(x'+y')}+\frac{y'}{l(x'+y')-n(y'+z')}+\frac{z'}{m(y'+z')-l(z'+x')}=0.$$

This equation represents a cubic having a node at the point given by

$$\frac{y'+z'}{l}=\frac{z'+x'}{m}=\frac{x'+y'}{n}.$$

The curve touches the three lines $x'=0$, $y'=0$, $z'=0$ and meets them again at the points where they are met by the line $x'+y'+z'=0$. The cubic has a cusp if the line $lx+my+nz=0$ passes through one of the points $(-1,1,1)$, $(1,-1,1)$, $(1,1,-1)$.

If the line passes through the point $(1,1,1)$, the cubic breaks up into the line $x'+y'+z'=0$ and a conic touching the four lines

$$x'=y'=z'=x'+y'+z'=0.$$

For if the equation of the straight line be written in the form

$$l(y-z)+m(z-x)+n(x-y)=0,$$

we have

$$\frac{lx'}{x^2}+\frac{my'}{y^2}+\frac{nz'}{z^2}=0.$$

But

$$\frac{x'}{x^2}+\frac{y'}{y^2}+\frac{z'}{z^2}=0,$$

$$\frac{\frac{x'}{x^2}}{m-n}=\frac{\frac{y'}{y^2}}{n-l}=\frac{\frac{z'}{z^2}}{l-m}.$$

Substituting in the equation

$$\frac{x'}{x}+\frac{y'}{y}+\frac{z'}{z}=0,$$

we get

$$\sqrt{x'(m-n)}+\sqrt{y'(n-l)}+\sqrt{z'(l-m)}=0.$$

The complete curve in the plane X corresponding to this conic consists of a straight line and a quintic curve whose equation is

$$\begin{aligned}(m-n)x[xy+yz-zx][yz+zx-xy]\\ +(n-l)y[zx+xy-yz][yz+zx-xy]\\ +(l-m)z[xy+yz-zx][zx+xy-yz]=0.\end{aligned}$$

This curve passes through the point O (1, 1, 1) and has tacnodes at A, B, C, the tangents at these tacnodes being OA, OB, OC. We shall now show that a straight line through the point O is cut in a constant anharmonic ratio by this quintic curve. Consider two conics Q and Q' touching the four lines

$$x'=y'=z'=x'+y'+z'=0.$$

By a known theorem,* the cross-ratio of the four points on Q in which it is cut by Q' is equal to the cross-ratio of the four points on Q' at which it is touched by the common tangents of Q and Q'.

Now let L and H be the line and quintic corresponding to Q, L' and H' the line and quintic corresponding to Q'. The four points of intersection other than O of L and H' correspond to the four points of intersection of Q and Q', and so the cross-ratio of these four points on L is equal to the cross-ratio of the four points on Q, for when there is a (1, 1) correspondence between the points of a line and conic, cross-ratios are unaltered by the transformation.

*This is proved by reciprocating one conic into the other.

Again, the cross-ratio of the four points in which Q' touches the lines

$$x' = y' = z' = x' + y' + z' = 0$$

is equal to the cross ratio $(Oabc)$, where a, b, c are the points in which L' meets the lines BC, CA, AB. Hence, it follows that the cross-ratio in which L is cut by H' is equal to $(Oabc)$ and is independent of the position of L.

To verify this theorem analytically, we write

$$Y \equiv x - y, \quad Z \equiv x - z.$$

The equation of the quintic then becomes

$$\begin{aligned}(m-n)\,x[x^2 - 2xY + YZ]\,[x^2 - 2xZ + YZ] \\ + (n-l)\,(x-Y)\,(x^2 - YZ)\,(x^2 - 2xZ + YZ) \\ + (l-m)\,(x^2 - 2xY + YZ)\,(x^2 - YZ) = 0,\end{aligned}$$

or

$$\begin{aligned}x^4(bY + cZ) - 4x^3(b+c)YZ + 2x^2[YZ^2(2b+c) + Y^2Z(b+2c)] \\ - 4x(b+c)Y^2Z^2 + Y^2Z^2(bY + cZ) = 0,\end{aligned}$$

where

$$b = n - l, \quad c = l - m.$$

Calculating the invariants of this quartic in x, we have,

$$\begin{aligned}3I &= 4Y^2Z^2(Y-Z)^2(b^2 + bc + c^2), \\ 27J &= 4Y^3Z^3(Y-Z)^3(2b+c)\,(b+2c)\,(b-c).\end{aligned}$$

Hence, $\dfrac{I^3}{27J^2}$ is independent of the ratio of Y to Z. Since the cross-ratios of the four values of x depend only on the ratio $\dfrac{I^3}{J^2}$, it follows that the anharmonic ratio is the same for all straight lines through O.

Involutorial Transformations.

By Frank M. Morgan.

It has been proved by Professor Bertini* that there exist four distinct types of involutorial transformations in a ternary field.

By distinct types he means that no type can be reduced to any other type by means of a series of quadratic transformations.

In the present article I have set up the equations of these four types. The first type, harmonic homology, can be reduced to the form, $\rho x_1 = x_1'$, $\rho x_2 = x_2'$, $\rho x_3 = -x_3'$. It will not be considered further. I have then shown how the transformations of class one and two considered by Bertini,† those of class three and four considered by Martinetti,‡ and those of class five considered by Berzolari,§ may be reduced to some one of the four types. I have then shown by the same process how a transformation of any class may be built.

The four types are:

1) Harmonic homology.

2) Involutorial perspective Jonquières transformations of degree n, with a fixed curve of degree n, which contains an $(n-2)$-fold point.

3) The Geiser transformation of degree eight with seven fundamental points of the third order, and a fixed curve of order six which has double points at the seven fundamental points.

4) The Bertini transformation of degree seventeen with eight fundamental points of the sixth order, and a fixed curve of degree nine, having triple points at the eight fundamental points.

Type II.

To set up a Jonquières transformation, consider a pencil of conics having four distinct basis points, and a pencil of cubics four of whose basis points coincide with the basis points of the conics. This configuration sets up an

* *Annali di Mat.*, Ser. 2, T. 8, pp. 244–286.

† "Sopra alcune involutioni piane," *Lomb. Ist. Rend.*, Ser. 2, Vol. XVI (1883), pp. 89–101, 190–208.

‡ "Le involuzioni di 3a e 4a classe," *Annali di Mat.*, Ser. 2, T. 12 (1883–1884), pp. 73–106.

§ "Ricerche sulle trasformazioni piane, univoche, involutorie e loro applicazione delle involuzioni di quinta classe," *Annali di Mat.*, Ser. 2, T. 16 (1888–1889), pp. 191–275.

involutorial transformation. For, choose a point P, and pass a conic and a cubic of the pencils through the point P. The conic and the cubic are fixed. Therefore the sixth point of intersection P' is fixed. Conversely, if we choose the point P', the same conic and cubic are fixed, and therefore the point P is determined. That is, the pencil of conics and the pencil of cubics determine an involutorial transformation.

Call the common basis points 1, 2, 3, 4, and the basis points of the cubics, which are not basis points of the conics, 5, 6, 7, 8, 9. The points are the fundamental points of the transformation. The fundamental curves corresponding to 5, 6, 7, 8, 9 are conics. For, take the point 6, with 1, 2, 3, 4. They determine a conic. Take P on this conic, and as the cubics vary for different points P, the remaining sixth point of intersection, which is the image of P, must necessarily be at 6. In the same way the fundamental curves corresponding to 5, 7, 8, 9 are conics.

To find the fundamental curves corresponding to the fundamental points 1, 2, 3, 4, take the pencil of conics $x_1 + u_2 + \lambda (x_2 + u_2') = 0$ and the pencil of cubics $x_1 + v_3 + \lambda (x_2 + v_3') = 0$. That is, we associate curves of the pencil which touch each other at $1 = (0, 0, 1)$.

Eliminating λ, we have

$$(x_1 + u_2)(x_2 + v_3') - (x_1 + v_3)(x_2 + u_2') = 0,$$

i. e.,

$$x_2 u_2 + x_1 v_3' + u_2 v_3' - x_2 v_3 - x_1 u_2' - u_2' v_3 = 0,$$

which is a C_5 having a triple point at 1, double points at 2, 3, 4 and simple points at 5, 6, 7, 8, 9. We will denote this by the following symbol:

$$C_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The fundamental curves corresponding to 2, 3, 4 are likewise quintics. The order of the transformation from the relation, order of Jacobian $= 3(n-1)$, gives $5 + 5 + 5 + 5 + 2 + 2 + 2 + 2 + 2 = 3(n-1)$.

The image of a straight line is a C_{11}, which may be written

$$C_{11} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 5 & 5 & 5 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

A C_{11} is fixed by 77 points. The four five-fold points count for 60 and the five double points for 15. Therefore there are two degrees of freedom. The maximum number of double points in a C_{11} is 45. The four five-fold points count as 40 and the five original double points make up the 45.

The whole problem may be approached in a different way. Take the points 2, 3, 4 as the vertices of a fundamental triangle with regard to a quadratic inversion Q. Transforming the pencils of conics and cubics through Q, we obtain a pencil of lines through 1′, the image of point one, and a pencil of cubics through $1', 2, 3, 4, 5', 6', 7', 8', 9'$. An involutorial transformation is again set up. For, take a point P, and this fixes a line and a cubic of each pencil which intersect in another point P'. Conversely, given P', the same C_1 and C_3 are fixed. Therefore the image of P' is P and the transformation is involutorial.

If the point P is on the line (15), the image of P is the point 5. Every point on (15) goes into 5. Therefore 5 is a fundamental point and (15) a fundamental line. There are therefore eight fundamental lines. The fundamental curve corresponding to the point 1 may be found in the following manner: Consider the C_3 whose equation is

$$X_1 X_3^2 + \Phi_2 (X_1 X_2) X_3 + \Phi_3 (X_1 X_2) = 0,$$

and the C_1

$$X_1 = 0.$$

Also the C_3 whose equation is

$$X_2 X_3^2 + \Phi_2 (X_1 X_2) X_3 + \Phi_3 (X_1 X_2) = 0,$$

and the C_1

$$X_2 = 0.$$

Then form the pencils

$$(X_1 + \lambda X_2) X_3^2 + (\Phi_2 + \lambda \psi_2) X_3 + (\Phi_3 + \lambda \psi_3) = 0$$

and

$$X_1 + \lambda X_2 = 0.$$

Eliminating λ, we get the locus of P,

$$(X_2 \Phi_2 - X_1 \psi_2) X_3 + (\Phi_3 X_2 - \psi_2 X_1) = 0.$$

Therefore the locus is a C_4 with a triple point at 1. The sum of the orders of the fundamental curves equals the order of the Jacobian, which is $3(n-1)$. Therefore

$$8 + 4 = 3(n-1),$$

or $n = 5$.

The C_5 has a four-fold point at 1, for the point P corresponds to P'. Therefore the line $(1P)$ can cut the C_5 only in one point outside of 1, *i. e.*, 1 is a fundamental point. The image of any curve, for example a straight line, will pass through 5. For the line must cut (15) in some point P, and the image of P we saw was 5. For the same reason the curve will pass

through 6, 7, 8, 9. There are certain points which are the images of themselves. These points form the curve of coincident points. It will be called the Γ curve.

To determine this curve we proceed as follows: Call the point $1 = (0,0,1)$ and take a point (ξ_1, ξ_2, ξ_3) on the curve. A line through $(0, 0, 1)$ and (ξ_1, ξ_2, ξ_3) is $\xi_1 X_2 - \xi_2 X_1 = 0$. A tangent to the curve at (ξ_1, ξ_2, ξ_3) is

$$X_1 \frac{\delta C_1}{\delta \xi_1} + X_2 \frac{\delta C_2}{\delta \xi_2} + X_3 \frac{\delta C_3}{\delta \xi_3} = 0.$$

These are necessarily identical, for geometrically we see that the curve of coincident points is the locus of the points of tangency of the tangents drawn from $(0, 0, 1)$ to the corresponding curve of the other pencil.

Therefore,

$$\frac{-\xi_2}{\frac{\delta C_3}{\delta \xi_1} + \lambda \frac{\delta C_3'}{\delta \xi_1}} = \frac{\xi_1}{\frac{\delta C_3}{\delta \xi_2} + \frac{\lambda \delta C_3'}{\delta \xi_2}}$$

and

$$\frac{\delta C_3}{\delta \xi_3} + \lambda \frac{\delta C_3'}{\delta \xi_3} = 0,$$

for no term X_3 can appear. Eliminating λ, we have

$$\xi_1 \left| \frac{\delta (C_3 C_3')}{\delta (\xi_1 \xi_2)} \right| = -\xi_2 \left| \frac{\delta (C_3 C_3')}{\delta (\xi_1 \xi_2)} \right|,$$

which is a curve of order five.

Going back to our original cubics

$$C_3 = X_1 X_3^2 + \Phi_2 (X_1 X_2) X_3 + \Phi_3 (X_1 X_2) = 0$$

and

$$C_3 = X_2 X_3^2 + \Phi_2 (X_1 X_2) X_3 + \Phi_3 (X_1 X_2) = 0,$$

getting the derivatives and substituting them in the above equation, gives

$$X_2 \left\{ (2 X_2 X_3 + \psi_2) (\Phi_2 X_3 + \Phi_3) - (2 X_1 X_3 + \Phi_2) \left(\frac{\delta \psi_2}{\delta X_2} + \frac{\delta \psi_3}{\delta X_3} \right) \right\}$$
$$= X_1 \left\{ \left(X_3^2 + \frac{\delta \Phi_2}{\delta X_1} X_2 + \frac{\delta \Phi_3}{\delta X_1} \right) (2 X_2 X_3 + \psi_2) - \left(\frac{\delta \Phi_2}{\delta X_1} X_3 + \frac{\delta \Phi_3}{\delta X_1} \right) \right\}.$$

This curve we see has a triple point at 1.

If we perform the operation $Q T_5 Q^{-1}$, we will have the transformation generated by the pencil of conics and cubics.

A C_1 by Q goes into a $C_2 (2, 3, 4)$. This by T_5 goes into a C_7, the three

fundamental curves corresponding to the points 2, 3, 4 dividing out as factors. This C_7 by Q now goes into a C_{11} with the symbol

$$C_{11}\begin{pmatrix}1&2&3&4&5&6&7&8&9\\5&5&5&5&2&2&2&2&2\end{pmatrix}.$$

The above results agree exactly with what we had before.

In general, if we have a pencil of lines through the point 1, and a pencil of curves of degree n having an $(n-2)$-fold point at 1, and passing through $2(n-1)$ simple points, it will define an involutorial Jonquières transformation. The curve of coincident points will also be of order n. Any two pencils of curves, one of which, by successive quadratic transformations, may be reduced to a pencil of lines, and the other to a pencil of curves having an $(n-2)$-fold point at the vertex of this pencil of lines, may be solved by the above method.

A very simple manner of obtaining the transformation is to consider it determined by Γ and the pencil of lines. If we choose a point X not on Γ and draw the line $(1X)$, it will cut the curve in two points X_1, X_2. To get the image of X, get the harmonic conjugate to X, X_1, X_2, and call it $\bar{X}$.

Since these four points are harmonic, we have

$$\frac{X-X_1}{X_2-X}=\frac{\bar{X}-X_1}{\bar{X}-X_2},$$

or

$$2X\bar{X}-(X+\bar{X})(X_1+X_2)+2X_1X_2=0. \qquad (A)$$

The equation of the general C_n having an $(n-2)$-fold point is

$$X^2\Phi_{n-2}(YZ)+X\Phi_{n-1}(YZ)+\Phi_n(YZ)=0,$$

or

$$X^2+X\frac{\Phi_{n-1}}{\Phi_{n-2}}+\frac{\Phi_n}{\Phi_{n-2}}=0.$$

Substituting now in relation (A) gives

$$2X\bar{X}+(X+\bar{X})\frac{\Phi_{n-1}}{\Phi_{n-2}}+2\frac{\Phi_n}{\Phi_{n-2}}=0,$$

or

$$2X\bar{X}\Phi_{n-2}(YZ)+(X+\bar{X})\Phi_{n-1}(YZ)+2\Phi_n(YZ)=0;$$

i. e.,

$$\bar{X}=\frac{X\Phi_{n-1}+2\Phi_n}{-(2X\Phi_{n-2}+\Phi_{n-1})}.$$

But $\bar{Y}=Y$ and $\bar{Z}=Z$; therefore we have

$$\begin{aligned}\bar{X}&=X\Phi_{n-1}+2\Phi_n,\\ \bar{Y}&=-(2X\Phi_{n-2}+\Phi_{n-1})Y,\\ \bar{Z}&=-(2X\Phi_{n-2}+\Phi_{n-1})Z,\end{aligned}$$

which are the equations of the general perspective Jonquières transformation.

A very interesting fact is that if the ratio of the four points is not taken to be harmonic, but is taken to be equal to λ, then the transformation is neither periodic nor birational. Then we have

$$\frac{(X_1-\bar{X})\,(X_2-X)}{(X_1-X)\,(X_2-\bar{X})}=\lambda,$$

or

$$(1-\lambda)\,(X_1X_1+\bar{X}X)-(X_1+X_2)\,(\bar{X}+X)+(1+\lambda)\,(X_1\bar{X}+X_2X)=0.$$

The curve Γ, we saw, was

$$X^2\Phi_{n-2}(YZ)+X\Phi_{n-1}+\Phi_n=0.$$

Therefore

$$X_1X_2=\frac{\Phi_n}{\Phi_{n-2}},$$

$$X_1+X_2=\frac{\Phi_{n-1}}{\Phi_{n-2}}.$$

It then follows that

$$(1-\lambda)\left(\frac{\Phi_n}{\Phi_{n-2}}+\bar{X}X\right)+\frac{\Phi_{n-1}}{\Phi_{n-2}}(\bar{X}+X)+(1+\lambda)\,(X_1\bar{X}+X_2X)=0.$$

If we solve this for X, we see it brings in radicals. Therefore the transformation is neither birational nor periodic.

I will now set up the equations of a Jonquières T_3 which is non-perspective. A non-perspective Jonquières transformation is one such that instead of having the image of a point A on the line connecting it to 1, it is a point B on a separate line $(1\,B)$. Conversely, the image of B is the point A. The rays through 1 are in involution and therefore contain two fixed rays. These constitute the curve Γ. If two of the fundamental points fall on one of these lines, the order of the transformation will be lowered one unit, and Γ will consist of a line of invariant points and two isolated invariant points. A still more particular case arises when we let two more fundamental points lie on the last fixed line. The curve Γ then consists of four isolated invariant points, and the order of the transformation is again lowered one unit.

In this T_3 whose equations I will derive, I will have one line of invariant points and two isolated invariant points. The invariant line becomes a tangent to the fundamental curve belonging to the point 1, for a tangent to an invariant curve at any fundamental point is tangent to the fundamental curve at that point. For the triangle of reference I will take the triangle whose vertices are $1, 2, 3$, calling $(1, 2) = X$, $(1, 3) = Y$, $(2, 3) = Z$. The image of a point

on X is on Y, and conversely, the image of a point on Y is on X. The image of the point 1 is evidently the C_2

$$(X-Y)\,Z+\lambda XY=0;$$

therefore

$$X'=Y\,[(X-Y)\,Z+\lambda XY]$$

and

$$Y=X\,[(X-Y)\,Z+\lambda XY].$$

It now remains to find the image of Z. We see 2 goes into Y and 3 into X. We now have to determine what is the image of $(4, 5)$. Let $Y=pX$ be the line connecting 1 to 4, and $Y=qX$ be the line connecting 1 to 5. The line $Y=pX$ is to go into the line $Y=qX$; *i. e.*, $pq=1$.

To find the lines through 1 consider the conic $Z(X-Y)+\lambda XY=0$ and an arbitrary line $AX+BY+CZ=0$. Solving gives

$$-(AX+BY)\,(X-Y)+\lambda CXY=0.$$

This represents the two lines through 1. We want them to be $Y=pX$ and $Y=qX$. Therefore

$$-AX^2-BXY+AXY+BY^2+\lambda CXY=Y^2-qXY-pXY+pqX^2;$$

i. e.,

$$A=-1 \quad \text{and} \quad B=1.$$

The value of C is indeterminate; so I will call it K. Then

$$Z=(X-Y+KZ)\,XY.$$

If we form the isologue, we know that $Y-X$ must be a factor. The isologue* is:

$$\begin{vmatrix} Y((X-Y)Z+\lambda YX) & X((X-Y)Z+\lambda XY) & XY(X-Y+KZ) \\ X & Y & Z \\ \alpha & \beta & \gamma \end{vmatrix}$$

If this is to have the factor $Y-X$, it is necessary that $K=\lambda$. Therefore the transformation may be written

$$\frac{X}{Y'Z'(X'-Y'+\lambda Y')}=\frac{Y}{X'Z'(X'-Y'+\lambda Y')}=\frac{Z}{X'Y'(X'-Y'+\lambda Z')}.$$

If one cares to find the two invariant points, he merely has to let $X=X'$, $Y=Y'$, $Z=Z'$ in the above equations, and then solve them, rejecting the factor $Y-X$.

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* The definitions of this and of other technical words are given in the long paper of Bertini mentioned above.

This non-perspective Jonquières transformation is immediately transformed into a perspective Jonquières transformation by the linear transformation

$$X = Y', \qquad Y = X', \qquad Z = Z'.$$

That is, we have

$$\frac{X}{X'Z'(X' - Y' + \lambda Y')} = \frac{Y}{Y'Z'(X' - Y' + \lambda Y')} = \frac{Z}{X'Y'(X' - Y' + \lambda Z')}.$$

The curve of coincident points is

$$X'Y'(X' - Y' + \lambda Z') = Z'((X' - Y')Z' + \lambda Y'Z'),$$

which agrees with what we had before.

So in general, any non-perspective Jonquières transformation can be transformed into a perspective Jonquières transformation by a linear transformation which interchanges the rays X and Y and leaves Z unchanged.*

TYPE III.

Let us start with seven basis points, no three of which are on a straight line. Through these seven points we can pass a net of cubic curves. If we choose another point P, this fixes a pencil. The curves of this pencil will intersect in another point Q. Conversely, if we had started with Q, the remaining intersection would have been P. Therefore an involutorial transformation is set up.

Consider the net $a\phi_1 + b\phi_2 + c\phi_3 = 0$ and the two pencils of the net

$$\phi_1 + k\phi_3 = 0, \qquad \phi_2 + l\phi_3 = 0.$$

If the point P is (ξ_1, ξ_2, ξ_3), the two cubics of the pencil become

$$\phi_1\phi_2(\xi) - \phi_2\phi_1(\xi) = 0,$$
$$\phi_2\phi_3(\xi) - \phi_3\phi_2(\xi) = 0.$$

To find the order of the transformation, let us consider what the image of the line $A_1X_1 + A_2X_2 + A_3X_3 = 0$ will be. Solving the last equation for X_1, we see $X_1 = mX_2 + nX_3$. Substituting this in the above equations, we have

$$\phi_1(X_2X_3)\phi_2(\xi) - \phi_2(X_2X_3)\phi_1(\xi) = 0,$$
$$\phi_2(X_2X_3)\phi_3(\xi) - \phi_3(X_2X_3)\phi_1(\xi) = 0.$$

Eliminating the X's by Sylvester's method we get a curve of order eight which has a triple point at each of the fundamental points.

The fundamental curves are cubics, for the Jacobian which is of order $3(n-1)$, where n is the order of the transformation, is equal to the sum of the degrees of the fundamental curves. Since all the fundamental curves are

* Dr. Karl Doehlemann: "Geometrische Transformationen," Teil II, p. 169.

associated with symmetrically situated fundamental points, the curves will be the same. Since fundamental curves are uniquely fixed by the fundamental points, the curves will be

$$1: \quad C_3\begin{pmatrix}1&2&3&4&5&6&7\\2&1&1&1&1&1&1\end{pmatrix},$$

$$2: \quad C_3\begin{pmatrix}1&2&3&4&5&6&7\\1&2&1&1&1&1&1\end{pmatrix},$$

$$\cdots\cdots\cdots\cdots,$$

$$7: \quad C_3\begin{pmatrix}1&2&3&4&5&6&7\\1&1&1&1&1&1&2\end{pmatrix}.$$

That is, the following transformation is set up:

$$T_8\begin{pmatrix}1&2&3&4&5&6&7\\3&3&3&3&3&3&3\end{pmatrix}.$$

Suppose the two cubics, instead of intersecting in the points P and Q, had the point $P=Q$, *i. e.*, were tangent to each other at P. Such a point as this would be an invariant point, *i. e.*, its image is itself. This means we have

$$a\frac{\delta\phi_1}{\delta\xi_1}+b\frac{\delta\phi_2}{\delta\xi_1}+c\frac{\delta\phi_3}{\delta\xi_1}=0,$$

$$a\frac{\delta\phi_1}{\delta\xi_2}+b\frac{\delta\phi_2}{\delta\xi_2}+c\frac{\delta\phi_3}{\delta\xi_2}=0,$$

$$a\frac{\delta\phi_1}{\delta\xi_3}+b\frac{\delta\phi_2}{\delta\xi_3}+c\frac{\delta\phi_3}{\delta\xi_3}=0.$$

For this to be true the determinant of the coefficients of a, b, c must vanish; *i. e.*,

$$\begin{vmatrix}\frac{\delta\phi_1}{\delta\xi_1}&\frac{\delta\phi_2}{\delta\xi_1}&\frac{\delta\phi_3}{\delta\xi_1}\\ \frac{\delta\phi_1}{\delta\xi_2}&\frac{\delta\phi_2}{\delta\xi_2}&\frac{\delta\phi_3}{\delta\xi_2}\\ \frac{\delta\phi_1}{\delta\xi_3}&\frac{\delta\phi_2}{\delta\xi_3}&\frac{\delta\phi_3}{\delta\xi_3}\end{vmatrix}=0.$$

Therefore the curve of coincident points is merely the Jacobian. Since every term is of order two, the determinant is of order six. The multiplicity at a fundamental point is known to be $3i-1$.* In our case $i=1$, or the multiplicity is 2. Therefore the curve of fixed points is

$$T_6\begin{pmatrix}1&2&3&4&5&6&7\\2&2&2&2&2&2&2\end{pmatrix}.$$

* Doehlemann: "Geometrische Transformationen," Teil II, p. 142.

According to our hypothesis no three of the fundamental points were on a straight line. I will now consider the case when three fundamental points are on a straight line. The curve of order eight, which is the image of a straight line, will necessarily degenerate, for we will have a straight line cutting it in three triple points. The transformation can then be written:

$$T_7 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 3 & 3 & 3 & 2 & 2 & 2 \end{pmatrix}.$$

The fundamental curves of the points 5, 6, 7 will also contain the line as a factor.

Consider the involutorial inversion whose triangle of reference is 1, 2, 3; then

$$C_1 I_{123} = C_2\ (1\ 2\ 3),$$
$$C_2 T_7 = C_5 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 2 & 2 & 3 & 1 & 1 & 1 \end{pmatrix},$$
$$C_5 I_{123} = C_4 \begin{pmatrix} 1 & 2 & 3 & 4' & 5' & 6' & 7' \\ 1 & 1 & 1 & 3 & 1 & 1 & 1 \end{pmatrix}.$$

If the cubics are $\phi_1 = 0$ and $C_1 (a C_2 + b C_2') = 0$, the Jacobian has the factor C_1, or

$$J = C_5 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 2 & 2 & 2 & 1 & 1 & 1 \end{pmatrix}.$$

Inverting this through I_{123} gives

$$C_4 \begin{pmatrix} 1 & 2 & 3 & 4' & 5' & 6' & 7' \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 \end{pmatrix}.$$

Therefore the first case reduces to a perspective Jonquières transformation.

Second, suppose that the points 3, 4, 7 were also on a straight line. Then we have a $T_6 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 3 & 2 & 2 & 2 & 2 & 1 \end{pmatrix}$. If the curves are $\phi_3 = 0$ and $X_1 X_2 (a X_1 + a X_2 - k X_3) = 0$, the Jacobian reduces to $C_4 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 2 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$. Then

$$C_1 I_{123} = C_2\ (1\ 2\ 3),$$
$$C_2 T_6 = C_4 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 2 & 1 & 2 & 1 & 1 & 0 \end{pmatrix},$$
$$C_4 I_{123} = C_3 \begin{pmatrix} 1 & 2 & 3 & 4' & 5' & 6' & 7' \\ 1 & 1 & 0 & 2 & 1 & 1 & 0 \end{pmatrix}.$$

Then Γ reduces to a $C_3 \begin{pmatrix} 1 & 2 & 3 & 4' & 5' & 6' & 7' \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$. Therefore this case reduces to a perspective Jonquières transformation.

Third, let the points 2, 4, 6 also be on a straight line. Then the trans-

formation is a $T_5\begin{pmatrix}1&2&3&4&5&6&7\\3&2&2&1&2&1&1\end{pmatrix}$, while I is $C_3\begin{pmatrix}1&2&3&4&5&6&7\\2&1&1&0&1&0&0\end{pmatrix}$. Then

$$C_1 I_{123} = C_2 (1\ 2\ 3),$$
$$C_2 T_5 = C_3\begin{pmatrix}1&2&3&4&5&6&7\\2&1&1&1&1&0&0\end{pmatrix},$$
$$C_3 I_{123} = C_2\begin{pmatrix}1&2&3&4'&5'&6'&7'\\1&0&0&1&1&0&0\end{pmatrix}.$$

Γ becomes $C_2\begin{pmatrix}1&2&3&4'&5'&6'&7'\\1&0&0&0&1&0&0\end{pmatrix}$. Therefore this may be generated from a perspective Jonquières transformation, which in this case is a perspective quadratic transformation.

Fourth, in the original T_8 let the points 1,2,7 and 3,4,7 and 5,6,7 lie on a straight line. Then we have a $T_5\begin{pmatrix}1&2&3&4&5&6&7\\2&2&2&2&2&2&0\end{pmatrix}$, while Γ is $C_3\begin{pmatrix}1&2&3&4&5&6&7\\1&1&1&1&1&1&0\end{pmatrix}$. Then

$$C_1 I_{123} = C_2 (1\ 2\ 3),$$
$$C_2 T_5 = C_4\begin{pmatrix}1&2&3&4&5&6&7\\2&2&1&2&1&1&0\end{pmatrix},$$
$$C_4 I_{123} = C_3\begin{pmatrix}1&2&3&4'&5'&6'&7'\\1&1&0&2&1&1&0\end{pmatrix}.$$

The curve Γ remains the same; therefore the transformation may be built from a perspective Jonquières transformation.

Fifth, let the points 1, 2, 7; 3, 4, 7; 5, 6, 7; 2, 4, 6 be on a straight line. Then we have a $T_4\begin{pmatrix}1&2&3&4&5&6&7\\2&1&2&1&2&1&0\end{pmatrix}$, and Γ is $C_2(1\ 3\ 5)$. Now

$$C_1 I_{123} = C_2 (1\ 2\ 3),$$
$$C_2 T_4 = C_3\begin{pmatrix}1&2&3&4&5&6&7\\2&1&1&1&1&0&0\end{pmatrix},$$
$$C_3 I_{123} = C_2\begin{pmatrix}1&2&3&4'&5'&6'&7'\\1&0&0&1&1&0&0\end{pmatrix}.$$

The curve Γ becomes $C_2(1\ 3\ 5')$. Therefore the transformation may be built from a non-perspective quadratic.

Sixth, let the points 3, 4, 7; 5, 6, 7; 2, 4, 6; 2, 3, 5 be on straight lines. Then we have a $T_4\begin{pmatrix}1&2&3&4&5&6&7\\3&1&1&1&1&1&1\end{pmatrix}$, where Γ is $C_2\begin{pmatrix}1\\2\end{pmatrix}$. This, we see, is a non-perspective Jonquières transformation in which the curve of coincident points consists of two fixed rays through the point 1.

Seventh, let the points 1, 2, 7; 3, 4, 7; 5, 6, 7: 2, 4, 6; 2, 3, 5 be on straight lines. Then we have a $T_3\begin{pmatrix}1&2&3&4&5&6&7\\2&0&1&1&1&1&0\end{pmatrix}$, and Γ is $C_2(1)$. This is the non-perspective Jonquières transformation whose equations I derived.

The eighth and last possibility is to have 1, 2, 7; 3, 4, 7; 5, 6, 7; 2, 4, 6; 2, 3, 5; 1, 3, 7 on straight lines. We then have a quadratic transformation whose triangle of reference is 1, 4, 5 and whose coincident points are 2, 3, 6, 7.*

TYPE IV.

A curve of order six may have ten double points, and is completely determined by twenty-seven arbitrarily chosen points or by nine arbitrarily chosen double points. If we choose nine double points, the sextic will break up into a cubic counted twice. It is therefore apparent that if nine points are double points, there is some relation between them. Therefore, if eight arbitrarily chosen points are double points on sextic curves, how may the ninth point be situated so that these nine points are on a proper sextic. There can be no proper sextic having the ninth point at the intersection of the two cubics through the eight points, for we would then have a cubic and a sextic intersecting in the nine points and any other point, which makes nineteen intersections of a cubic and a sextic.

Let $\phi(X) = 0$ and $f(X) = 0$ be the equation of two cubics through eight arbitrarily chosen points, and $\theta(X) = 0$ be a proper sextic having a double point at each of these points; then all sextics having these eight double points are of the form

$$\phi^2(X) + \lambda\phi(X) f(X) + \mu f^2(X) + \nu\theta(X) = 0.$$

If we impose the condition that the above curve have a ninth double point, we may still have one other point arbitrarily chosen, through which it may pass.

If we pass this curve through a point P', it will be of the form

$$[\phi^2(X)\theta'(X) - \phi'^2(X)\theta(X)] + \mu[f^2(X)\theta'(X) - f'^2(X)\theta(X)] + \lambda[\phi(X) f(X)\theta'(X) - \phi'(X) f'(X)\theta(X) = 0,$$

where $\phi'(X)$, $f'(X)$, $\theta'(X)$ are the results of substituting the coordinates of the point P' in $\phi(X)$, $f(X)$, $\theta(X)$.

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* de Paolis: "Le trasformazioni piane doppia," *Atti d. r. Accad. d. Lincei,* Series 3ª, Vol. I (1877), pp. 511–544; "La trasformazione piana doppia di secondo ordine, e la sua applicazione alla geometria non euclidea," *Atti d. r. Accad. d. Lincei,* Series 3ª, Vol. II (1878), pp. 31–50; "La trasformazione piana doppia di terzo ordine, primo genere e la sua applicazione alle curve del quarto ordine," *Atti d. r. Accad. d. Lincei,* Series 3ª, Vol. II (1878), pp. 851–878.

Sturm: "Lehre von den geometrischen Verwandtschaften," Vol. IV, pp. 120–122.

Snyder: "The Involutorial Birational Transformation of the Plane, of Order 17," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIII (1911), No. 4, pp. 327–336.

This is the equation of a net, and the locus of the double points must be the Jacobian

$$\begin{vmatrix} \Phi'^2\theta_1 - 2\Phi\Phi_1\theta' & f'^2\theta_1 - 2ff_1\theta' & \Phi' f'\theta_1 - \Phi f_1\theta' - \Phi_1 f\theta' \\ \Phi'^2\theta_2 - 2\Phi\Phi_2\theta' & f'^2\theta_2 - 2ff_2\theta' & \Phi' f'\theta_2 - \Phi f_2\theta' - \Phi_2 f\theta' \\ \Phi'^2\theta_3 - 2\Phi\Phi_3\theta' & f'^2\theta_3 - 2ff_3\theta' & \Phi' f'\theta_3 - \Phi f_3\theta' - \Phi_3 f\theta' \end{vmatrix} = 0,$$

i. e.,

$$\theta'(\Phi f' - \Phi' f)^2 \begin{vmatrix} \Phi_1 & f_1 & \theta_1 \\ \Phi_2 & f_2 & \theta_2 \\ \Phi_3 & f_3 & \theta_3 \end{vmatrix} = 0.$$

We may always choose θ so it does not pass through P'. Therefore θ' does not equal zero. Also no proper sextic could have a ninth double point on the cubic $\Phi f' - \Phi' f = 0$; for if it did, this cubic would have nineteen intersections with the sextic, eighteen at the nine double points and one at the point P'. The proper locus is therefore

$$K = \begin{vmatrix} \Phi_1 & f_1 & \theta_1 \\ \Phi_2 & f_2 & \theta_2 \\ \Phi_3 & f_3 & \theta_3 \end{vmatrix} = 0.$$

The Jacobian K is of order nine and has a triple point at each of the original points. To show that each of these points is a double point on $K = 0$, we must show that at these points $K_i = 0$ $(i = 1, 2, 3)$. Performing the differentiation, we have

$$K_i = \begin{vmatrix} \Phi_{1i} & f_1 & \theta_1 \\ \Phi_{2i} & f_2 & \theta_2 \\ \Phi_{3i} & f_3 & \theta_3 \end{vmatrix} + \begin{vmatrix} \Phi_1 & f_{1i} & \theta_1 \\ \Phi_2 & f_{2i} & \theta_2 \\ \Phi_3 & f_{3i} & \theta_3 \end{vmatrix} + \begin{vmatrix} \Phi_1 & f_1 & \theta_{1i} \\ \Phi_2 & f_2 & \theta_{2i} \\ \Phi_3 & f_3 & \theta_{3i} \end{vmatrix}.$$

But at these points $\theta_i = 0$; therefore the first two determinants disappear, and we have left the last determinant. Multiplying the three rows by X_1, X_2, X_3 respectively, and adding, we have

$$X_3 K_i = \begin{vmatrix} \Phi_1 & f_1 & \theta_{1i} \\ \Phi_2 & f_2 & \theta_{2i} \\ 3\Phi & 3f & 5\theta_i \end{vmatrix}.$$

At each of these points considered, $\Phi = 0$, $f = 0$, $\theta_i = 0$. Therefore, K_i vanishes, *i. e.*, $K = 0$ has a double point. Let us now represent the second derivative of K with respect to X_i and X_k by $K_{ik} = 0$ $(i, k =$ any two of the numbers 1, 2, 3). We have, neglecting the determinants that have columns vanishing at the points in question,

$$K_{ik} = \begin{vmatrix} \Phi_{1i} & f_1 & \theta_{1k} \\ \Phi_{2i} & f_2 & \theta_{2k} \\ \Phi_{3i} & f_3 & \theta_{3k} \end{vmatrix} + \begin{vmatrix} \Phi_1 & f_{1i} & \theta_{1k} \\ \Phi_2 & f_{2i} & \theta_{2k} \\ \Phi_3 & f_{3i} & \theta_{3k} \end{vmatrix} + \begin{vmatrix} \Phi_1 & f_1 & \theta_{1ik} \\ \Phi_2 & f_2 & \theta_{2ik} \\ \Phi_3 & f_3 & \theta_{3ik} \end{vmatrix} + \begin{vmatrix} \Phi_{1k} & f_1 & \theta_{1i} \\ \Phi_{2k} & f_2 & \theta_{2i} \\ \Phi_{3k} & f_3 & \theta_{3i} \end{vmatrix} + \begin{vmatrix} \Phi_1 & f_{1k} & \theta_{1i} \\ \Phi_2 & f_{2k} & \theta_{2i} \\ \Phi_3 & f_{3k} & \theta_{3i} \end{vmatrix}.$$

Multiplying the rows in each determinant by X_1, X_2 and X_3 respectively, and adding, remembering at these points $\phi = 0$, $f = 0$, $\theta_j = 0$ $(j = 1, 2, 3)$, we have

$$X_3 K_{ik} = 2\phi_i \begin{vmatrix} f_1 & \theta_{1k} \\ f_2 & \theta_{2k} \end{vmatrix} - 2 f_i \begin{vmatrix} \phi_1 & \theta_{1k} \\ \phi_2 & \theta_{2k} \end{vmatrix} + 4\theta_{ik} \begin{vmatrix} \phi_1 & f_1 \\ \phi_2 & f_2 \end{vmatrix} + 2\phi_k \begin{vmatrix} f_1 & \theta_{1i} \\ f_2 & \theta_{2i} \end{vmatrix} - 2 f_k \begin{vmatrix} \phi_1 & \theta_{1i} \\ \phi_2 & \theta_{2i} \end{vmatrix};$$

$$\therefore X_3 K_{ik} = 2 \begin{vmatrix} \phi_1 & f_1 & \theta_{1k} \\ \phi_2 & f_2 & \theta_{2k} \\ \phi_3 & f_3 & \theta_{ik} \end{vmatrix} + 2 \begin{vmatrix} \phi_1 & f_1 & \theta_{1i} \\ \phi_2 & f_2 & \theta_{2i} \\ \phi_3 & f_k & \theta_{ik} \end{vmatrix}.$$

But at each of the original eight points, both of these determinants are equal to zero, because either there are two rows alike or, multiplying the rows by X_1, X_2 and X_3 respectively, and adding, we have a row 3ϕ, $3f$, $5\theta_j$ $(j = i$ or $k)$, which vanishes at these points. Therefore, since the second derivative of K vanishes, $K = 0$ has a triple point on this curve.

If we consider the net $\phi^2 + \lambda\phi f + \mu\theta = 0$ passing through eight double points and one simple point P, this point being the residual intersection of all the cubics of the pencil passing through the eight aforenamed points, we can set up an involutorial transformation, as this accounts for $8 \cdot 3 + 1 = 25$ constants and 27 constants determine a sextic curve. Therefore take any point Q and pass a sextic of each pencil through it. These will intersect in another point R. Conversely, starting with R, the remaining intersection will be Q, *i. e.*, we have an involutorial transformation.

The fundamental curves may be found as follows: Suppose Q, the arbitrarily chosen point, to coincide with a fundamental point. Then $\phi = 0$ and $f = 0$ would be fixed cubics, and the corresponding point P would be any point on the fixed curve which is a C_6 having the symbol

$$C_6 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

Similarly, for 2 we have

$$C_6 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix},$$

. . . . . .,

and for 8 we have

$$C_6 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 \end{pmatrix}.$$

The order of the transformation is therefore easily obtained by remembering that $3(n-1)$ equals the sum of the orders of the fundamental curves. Therefore $3(n-1) = 8 \cdot 6$ or $n = 17$.

The curve of coincident points is evidently the Jacobian which we have found. Therefore we have an involutorial transformation of order seventeen

with eight fundamental points, which have corresponding fundamental curves of order six, that have a triple point at the fundamental point to which they correspond, and have double points at the other fundamental points. There is, however, a ninth fundamental point of a different type. If we try to find the image of the point which is the residual intersection of all the cubics through the points 1, 2, 3, 4, 5, 6, 7, 8, we see it is itself. But, however, we know that the curve of coincident points does not go through this point. We have then an isolated invariant point. The transformation is therefore

$$T_{17}\begin{pmatrix}1&2&3&4&5&6&7&8&9\\6&6&6&6&6&6&6&6&0\end{pmatrix}.$$

The particular cases of three points on a line, or five points on a conic, may be reduced to a Geiser transformation by the following scheme:

Let the points 1, 2, 3 be on a straight line. Since in T_{17} a general line not passing through a fundamental point goes into a C_{17} with 8 P_6, it follows that if 1, 2, 3 lie on a line, it must divide out as a factor. The symbol of the transformation may now be written

$$C_1 T_{16} = C_{16}\begin{pmatrix}1&2&3&4&5&6&7&8&9\\5&5&5&6&6&6&6&6&0\end{pmatrix}.$$

The fundamental curves of 1, 2, 3 must also contain the line 1, 2, 3 as a factor, becoming for 1, $C_5\begin{pmatrix}1&2&3&4&5&6&7&8\\2&1&1&2&2&2&2&2\end{pmatrix}$, and similarly for the others. To see if T_{16} can be reduced to a simpler form, consider a set of points $1', 2', 3', 4', 5'$, 6, 7, 8 such that $1', 2', 3', 6, 7, 8$ lie on a conic. Consider the involutorial inversion I_{678}. The conic will go into a straight line through 1, 2, 3. Let $4', 5'$ go into 4, 5. Then, by I_{678}, an arbitrary straight line will go into a conic through 6, 7, 8; *i. e.*,

$$C_1 I_{678} = C_2\ (6\ 7\ 8).$$

Now transform this conic by T_{16},

$$C_2 T_{16} = C_{14}\begin{pmatrix}1&2&3&4&5&6&7&8\\4&4&4&6&6&5&5&5\end{pmatrix}.$$

Then transform this C_{14} by I_{678} backwards,

$$C_{14} I_{678}^{-1} = C_{13}\begin{pmatrix}1'&2'&3'&4'&5'&6&7&8&9\\4&4&4&6&6&4&4&4&0\end{pmatrix}.$$

Therefore

$$I\, T_{16}\, I^{-1} = T_{13}.$$

It now is necessary to obtain the configuration of the fundamental curves of T_{13}. No new fundamental points were introduced. We proceed as follows:

The point $1'$ goes into 1 by I. Then 1 by T_{16} goes into

$$C_5\begin{pmatrix}1&2&3&4&5&6&7&8\\2&1&1&2&2&2&2&2\end{pmatrix}.$$

This by I goes into

$$C_4\begin{pmatrix}1'&2'&3'&4'&5'&6&7&8\\2&1&1&2&2&1&1&1\end{pmatrix}.$$

Thus

$$1:\ C_4\begin{pmatrix}1'&2'&3'&4'&5'&6&7&8\\2&1&1&2&2&1&1&1\end{pmatrix}. \qquad 2:\ C_4\begin{pmatrix}1'&2'&3'&4'&5'&6&7&8\\1&2&1&2&2&1&1&1\end{pmatrix}.$$

$$3:\ C_4\begin{pmatrix}1'&2'&3'&4'&5'&6&7&8\\1&1&2&2&2&1&1&1\end{pmatrix}. \qquad 4:\ C_6\begin{pmatrix}1'&2'&3'&4'&5'&6&7&8\\2&2&2&3&2&2&2&2\end{pmatrix}.$$

$$5:\ C_6\begin{pmatrix}1'&2'&3'&4'&5'&6&7&8\\2&2&2&2&3&2&2&2\end{pmatrix}. \qquad 6:\ C_4\begin{pmatrix}1'&2'&3'&4'&5'&6&7&8\\1&1&1&2&2&2&1&1\end{pmatrix}.$$

$$7:\ C_4\begin{pmatrix}1'&2'&3'&4'&5'&6&7&8\\1&1&1&2&2&1&2&1\end{pmatrix}. \qquad 8:\ C_4\begin{pmatrix}1'&2'&3'&4'&5'&6&7&8\\1&1&1&2&2&1&1&2\end{pmatrix}.$$

The next step will be to reduce T_{13}. It will be recalled that $1', 2', 3', 6, 7, 8$ lie on a conic. Take $4', 5', 6$ as the vertices of an inverting triangle, and five other points, $1'', 2'', 3'', 7'', 8''$, which will go into $1', 2', 3', 7', 8'$ by $I_{4'5'6}$. An arbitrary line will go into the conic $C_2\,(4', 5', 6)$; *i. e.*,

$$C_1 I_{456} = C_2\,(4\ 5\ 6).$$

This curve will go into a curve of order ten by T_{13} having the signature

$$C_{10}\begin{pmatrix}1'&2'&3'&4'&5'&6&7&8&9\\3&3&3&5&5&2&3&3&0\end{pmatrix}.$$

This curve is now to be transformed by $I^{-1}_{4'5'6}$. It becomes

$$C_8\begin{pmatrix}1''&2''&3''&4'&5'&6&7''&8''&9\\3&3&3&3&3&0&3&3&0\end{pmatrix}.$$

To get the configuration of the fundamental points proceed as before. $1''$ by $I_{4'5'6}$ goes into $1'$. Then $1'$ by T_{13} is $C_4\begin{pmatrix}1'&2'&3'&4'&5'&6&7&8\\2&1&1&2&2&1&1&1\end{pmatrix}$. This by $I_{4'5'6}$ becomes $C_3\begin{pmatrix}1''&2''&3''&4'&5'&6&7''&8''\\2&1&1&1&1&0&1&1\end{pmatrix}$, and similarly for the others. But this is exactly the configuration of the general Geiser which we have considered. Having considered the special cases of the Geiser, there is nothing more to be considered in this transformation.*

Bertini† has considered the transformations of Classes I and II. By

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* Sturm: "Lehre von den geometrischen Verwandtschaften," Vol. IV, pp. 120–122.

Snyder: "The Involutorial Birational Transformation of the Plane, of Order 17," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIII (1911), No. 4, pp. 327–336.

† "Sopra alcune involutioni piane," *Lomb. Ist. Rend.*, Ser. 2, Vol. XVI (1883), pp. 89–101.

class is merely meant this: Let n be the order of the transformation and m the degree of Γ; then the class of the transformation is $\frac{n-m}{2}$. The class is evidently not an invariant property, for we may transform so as to keep Γ invariant and not the order of the transformation. The transformations of Class I are the first five special cases of the Geiser which have already been considered. By the same process the transformations of Class II all reduce to one of the four types.

To amplify this method, I will consider a few transformations of Classes III and IV studied by Martinetti * and one of the class studied by Berzolari. †

Consider a pencil of lines 1 and a pencil of conics 2, 3, 4, 5. Let a line of 1 and a conic 2, 3, 4, 5 be determined by a point P. They intersect in one other point Q. Conversely, fix Q and you get the point P. Therefore we have a one-to-one correspondence. The fundamental curves are

1) $C_2(12345)$, 2) $C_1(12)$, 3) $C_1(13)$, 4) $C_1(14)$, 5) $C_1(15)$.

Using the relation $3(n-1) =$ sum of orders of fundamental curves, we have $n = 3$. Therefore we have an involutorial transformation $T_3\binom{1\,2\,3\,4\,5}{2\,1\,1\,1\,1}$. Performing the operation $I_{236}\,T_3\,I_{236}^{-1}$, we have

$$C_1 I_{236} = C_2(2\,3\,6),$$
$$C_2 T_3 = C_4\binom{1\,2\,3\,4\,5\,6'}{2\,1\,1\,2\,2\,1},$$
$$C_4 I_{236}^{-1} = C_6\binom{1'\,2\,3\,4'\,5'\,6\,6''}{2\,3\,3\,2\,2\,2\,1}.$$

The pencil of lines becomes a pencil of conics, and the pencil of conics remains a pencil of conics. We now have, then, an involutorial transformation defined by two pencils of conics having two basis points 1 and 2 in common, and the curves of the first pencil also passing through 3, 4, while those of the second pencil pass through 5, 6. Rewriting, we will call the transformation

$$T_6\binom{1\,2\,3\,4\,5\,6\,7}{3\,3\,2\,2\,2\,2\,1}.$$

Then

$$C_1 I_{189} = C_2(1\,8\,9),$$
$$C_2 T_6 = C_9\binom{1\,2\,3\,4\,5\,6\,7\,8'\,9'}{4\,5\,3\,3\,3\,3\,1\,1\,1},$$
$$C_9 I_{189}^{-1} = C_{14}\binom{1\,2\,3\,4\,5\,6\,7\,8\,9\,8''\,9''}{9\,5\,3\,3\,3\,3\,1\,5\,5\,1\,\;1}.$$

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* "Le involutioni di 3a e 4a classe," *Annali di Mat.*, Ser. 2, T. 12 (1883, 1884), pp. 73–106.

† "Ricerche sulle trasformazioni piane, univoche, involutorie e loro applicazione alla determinazione delle involuzioni di quinta classe, *Annali di Mat.*, Ser. 2, T. 16 (1888, 1889), pp. 191–275.

The Γ of the T_3 is $C_3\binom{1\,2\,3\,4\,5}{1\,1\,1\,1\,1}$. Transforming this by I_{236} gives $C_4\binom{1\,2\,3\,4\,5\,6}{1\,2\,2\,1\,1\,1}$. Thus we see that the T_6 is of Class I. Rewriting, as we did with the transformation, we have Γ equal to $C_4\binom{1\,2\,3\,4\,5\,6}{2\,2\,1\,1\,1\,1}$. Transforming this by I_{189}, we have $C_6\binom{1\,2\,3\,4\,5\,6\,7\,8\,9}{4\,2\,1\,1\,1\,1\,0\,2\,2}$. Therefore the T_{14} is of Class IV.

By letting certain points lie on straight lines, we can have particular cases of the above transformation. For example, if $2, 3, 5$ are on a line in the T_6, we have a T_5. If this is transformed by I_{189}, as before, we get a T_{10}, which is of Class III.

Martinetti,* in discussing the transformations of Class III, gives as his last species $T_6\binom{1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11}{3\,3\,2\,2\,2\,2\,1\,0\,0\,\;0\,\;0}$, which is generated, he says, by $C_2\,(1\,2\,3\,4)$ and $C_2\,(1\,2\,3\,5)$. Berzolari in the *Annali di Mat.*, Ser. 2, Tomo 16, p. 201, shows that this transformation does not exist. This can easily be shown by the method used in the previous problems. If we invert the two pencils by I_{123}, we get two pencils of lines. These generate a T_2. If this is transformed by I_{123}, we ought to arrive at the above transformation. But $I_{123}\,T_2\,I_{123}^{-1}$ gives T_8.

A very interesting transformation of Class II may be set up in the following manner: Consider the T_3 generated by $C_1\,(5)$ and $C_2\,(1\,2\,3\,4)$.

$$C_1\,I_{167} = C_2\,(1\,6\,7),$$
$$C_2\,T_3 = C_5\binom{1\,2\,3\,4\,5\,6\,7}{1\,2\,2\,2\,3\,1\,1},$$
$$C_5\,I_{167}^{-1} = C_7\binom{1\,2\,3\,4\,5\,6\,7}{3\,2\,2\,2\,3\,3\,3}.$$

The fundamental curve corresponding to

$$1:\ C_3\binom{1\,2\,3\,4\,5\,6\,7}{1\,1\,1\,1\,2\,1\,1}, \qquad 5:\ C_3\binom{1\,2\,3\,4\,5\,6\,7}{2\,1\,1\,1\,1\,1\,1},$$
$$6:\ C_3\binom{1\,2\,3\,4\,5\,6\,7}{1\,1\,1\,1\,1\,1\,2}, \qquad 7:\ C_3\binom{1\,2\,3\,4\,5\,6\,7}{1\,1\,1\,1\,1\,2\,1}.$$

It will be noticed that the image of the lines (1 7) (5 6) (7 5) (1 6) are the lines themselves. Therefore the points of intersection of (1 7) (5 6) and (1 6) (7 5) are invariant. The transformation therefore is

$$T_7\binom{1\,2\,3\,4\,5\,6\,7\,8\,9}{3\,2\,2\,2\,3\,3\,3\,0\,0},$$

while Γ is $C_3\,(1\,2\,3\,4\,5\,6\,7)$.

If T_7 is transformed by I_{2310}, we get a T_{14} of Class V.

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* "Le involuzioni di 3ª e 4ª classe, *Annali di Mat.*, Ser. 2, T. 12 (1883–1884), pp. 73–106.

I will now show how to build transformations of Class v from the general Jonquières transformation. The same procedure may be applied to the Geiser and to the Bertini. In fact, it may be applied to any involutorial transformation already found, so as to get one of higher class. The general Jonquières transformation is

$$T_n\begin{pmatrix}1 & 2\,3\,4\,5\,\ldots.\,2n-1\\ n-1 & 1\,1\,1\,1\,\ldots. & 1\end{pmatrix}.$$

I. Consider the quadratic inversion I_{234}. Then

$$C_1 I_{234} = C_2\,(2\,3\,4),$$

$$C_2 T_n = C_{2n-3}\begin{pmatrix}1 & 2\,3\,4\,5 & \ldots. & 2n-1\\ 2n-1 & 1\,1\,1\,1 & \ldots. & 2\end{pmatrix},$$

$$C_{2n-3} I_{234}^{-1} = C_{4n-9}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \ldots. & 2n-1\\ 2n-5 & 2n-5 & 2n-5 & 2n-5 & 2 & \ldots. & 2\end{pmatrix}.$$

Γ becomes

$$C_{2n-3}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \ldots. & 2n-1\\ n-2 & n-2 & n-2 & n-2 & 1 & \ldots. & 1\end{pmatrix}.$$

Therefore $\nu = n-3$. The fundamental curves are

$$1:\ C_{2n-5}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \ldots. & 2n-1\\ n-2 & n-3 & n-3 & n-3 & 1 & \ldots. & 1\end{pmatrix},$$

$$2:\ C_{2n-5}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \ldots. & 2n-1\\ n-3 & n-2 & n-3 & n-3 & 1 & \ldots. & 1\end{pmatrix},$$

. . . . . .,

$$5:\ C_2\,(1\,2\,3\,4\,5),$$

. . . . . .,

$$2n-1:\ C_2\,(1\,2\,3\,4\,2n-1).$$

Writing this transformation in terms of v, we have

$$T_{4v+3} = \begin{pmatrix}1 & 2 & 3 & 4 & 5\,6 & \ldots. & 2n-1\\ 2v+1 & 2v+1 & 2v+1 & 2v+1 & 2\,2 & \ldots. & 2\end{pmatrix}.$$

Γ is

$$C_{2v+3}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \ldots. & 2n-1\\ v+1 & v+1 & v+1 & v+1 & 1 & \ldots. & 1\end{pmatrix}.$$

The transformation has the following fundamental curves:

$$\begin{pmatrix}1 & 2\,3\,4\,5\,\ldots.\,2v+5\\ v+1 & v\,v\,v\,1\,\ldots. & 1\end{pmatrix}^{1}_{2v+1}\quad\begin{pmatrix}1 & 2 & 3\,4\,5\,\ldots.\,2v+5\\ v & v+1 & v\,v\,1\,\ldots. & 1\end{pmatrix}^{2}_{2v+1},$$

. . . . . .,

$$(1\,2\,3\,4\,5)^{5}_{2}\,(1\,2\,3\,4\,6)^{6}_{2}\,\ldots.\,(1\,2\,3\,4\,2v+5)^{2v+5}_{2}.$$

The number to the right and above the parenthesis is the fundamental point to which the curve belongs, while the number below the last named number gives the order of the fundamental curve.

II. Transform the general Jonquières transformation by I_{23P}, where the point P is any point except a point on Γ. Then

$$C_1 I_{23P} = C_2 (2\ 3\ P),$$

$$C_2 T_n = C_{2n-2} \begin{pmatrix} 1 & 2\ 3\ 4\ 5\ \ldots\ 2n-1 & P' \\ 2n-4 & 1\ 1\ 2\ 2\ \ldots\ 2 & 1 \end{pmatrix},$$

$$C_{2n-2} I_{23P}^{-1} = C_{4n-6} \begin{pmatrix} 1 & 2 & 3 & 4\ 5\ \ldots\ 2n-1 & P'' & P \\ 2n-4 & 2n-3 & 2n-3 & 2\ 2\ \ldots\ 2 & 1 & 2n-4 \end{pmatrix}.$$

Γ becomes

$$C_{2n-2} \begin{pmatrix} 1 & 2 & 3 & 4\ 5\ \ldots\ 2n-1 & P \\ n-2 & n-1 & n-1 & 1\ 1\ \ldots\ 1 & n-2 \end{pmatrix}.$$

Therefore the class $v = n - 2$.

Writing everything in terms of v and calling the points $1\ 2\ 3\ 4\ 5\ \ldots\ 2v+5$, we have

$$T_{4v+2} \begin{pmatrix} 1 & 2 & 3 & 4 & 5\ \ldots\ 2v+4 & 2v+5 \\ 2v+1 & 2v+1 & 2v & 2v & 2\ \ldots\ 2 & 1 \end{pmatrix},$$

Γ is

$$C_{2v+2} \begin{pmatrix} 1 & 2 & 3\ 4\ 5\ \ldots\ 2v+4 \\ v+1 & v+1 & v\ v\ 1\ \ldots\ 1 \end{pmatrix}.$$

The fundamental curves are

$$\begin{pmatrix} 1 & 2\ 3\ 4\ 5\ \ldots\ 2v+5 \\ v+1 & v\ v\ v\ 1\ \ldots\ 1 \end{pmatrix}^1_{2v+1} \quad \begin{pmatrix} 1 & 2 & 3\ 4\ 5\ \ldots\ 2v+5 \\ v & v+1 & v\ v\ 1\ \ldots\ 1 \end{pmatrix}^2_{2v+1},$$

$$\begin{pmatrix} 1\ 2\ 3 & 4 & 5\ \ldots\ 2v+4 \\ v\ v\ v-1 & 1\ \ldots\ 1 \end{pmatrix}^3_{2v} \quad \begin{pmatrix} 1\ 2 & 3 & 4\ 5\ \ldots\ 2v+4 \\ v\ v & v-1 & v\ 1\ \ldots\ 1 \end{pmatrix}^4_{2v},$$

$$(1\ 2\ 3\ 4\ 5)^5_2\ (1\ 2\ 3\ 4\ 6)^6_2\ \ldots\ (1\ 2\ 3\ 4\ 2v+4)^{2v+4}_2\ (1\ 2)^{2v+5}_1.$$

It will be noticed that this is equivalent to the first transformation when the points $3, 4, 2v+5$ are on a straight line.

III. Transform the Jonquières transformation by I_{2QP}, where Q and P are points that are not on Γ. Then

$$C_1 I_{2QP} = C_2 (2\ Q\ P),$$

$$C_2 T_n = C_{2n-1} \begin{pmatrix} 1 & 2\ 3\ 4\ \ldots\ 2n-1 & Q'\ P' \\ 2n-3 & 1\ 2\ 2\ \ldots\ 2 & 1\ 1 \end{pmatrix},$$

$$C_{2n-1} I_{2QP} = C_{4n-3} \begin{pmatrix} 1 & 2 & 3\ 4\ \ldots\ 2n-1 & Q'' & P'' & Q & P \\ 2n-3 & 2n-1 & 2\ 2\ \ldots\ 2 & 1 & 1 & 2n-2 & 2n-2 \end{pmatrix}.$$

Γ is

$$C_{2n-1} \begin{pmatrix} 1 & 2\ 3\ 4\ \ldots\ 2n-1 & Q & P \\ n-2 & n\ 1\ 1\ \ldots\ 1 & n-1 & n-1 \end{pmatrix}.$$

Therefore $v = n - 1$.

In terms of v we have, rearranging the order of the points,

$$T_{4v+1}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+3 & 2v+4 & 2v+5 \\ 2v+1 & 2v & 2v & 2v-1 & 2 & \dots & 2 & 1 & 1 \end{pmatrix}.$$

Γ is

$$C_{2v+1}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+3 \\ v+1 & v & v & v-1 & 1 & \dots & 1 \end{pmatrix}.$$

The fundamental curves are

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+5 \\ v+1 & v & v & v & 1 & \dots & 1 \end{pmatrix}^{1}_{2v+1} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+3 & 2v+4 \\ v & v & v & v-1 & 1 & \dots & 1 & 1 \end{pmatrix}^{2}_{2v}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+3 & 2v+5 \\ v & v & v & v-1 & 1 & \dots & 1 & 1 \end{pmatrix}^{3}_{2v} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+3 \\ v & v-1 & v-1 & v-1 & 1 & \dots & 1 \end{pmatrix}^{4}_{2v-1}$$

$$(1\,2\,3\,4\,5)^{5}_{2}\ (1\,2\,3\,4\,6)^{6}_{2}\ \dots\ (1\,2\,3\,4\,2v+3)^{2v+3}_{2}\ (1\,2)^{2v+4}_{1}\ (1\,3)^{2v+5}_{1}.$$

Notice that this transformation is equivalent to having $(3\ 4\ 2v+5)$ $(2\ 4\ 2v+4)$ of the original transformation, in a line.

IV. Transform the Jonquières transformation now through $I_{23\bar{P}}$, where $\bar{P}$ is a point on Γ. This transformation is of interest, for we will see that taking P on the invariant curve is equivalent to taking out two linear factors of I. Furthermore, we will see that the transformation will have an invariant point which is not a point on Γ. The location of this point will also be shown. Now

$$C_1 I_{23\bar{P}} = C_2\,(2\ 3\ \bar{P}),$$

$$C_2 T_n = C_{2n-2}\begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2n-1 & \bar{P} \\ 2n-4 & 1 & 1 & 2 & \dots & 2 & 1 \end{pmatrix},$$

$$C_{2n-2} I_{23\bar{P}} = T_{4n-7}\begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2n-1 & \bar{P} \\ 2n-4 & 2n-4 & 2n-4 & 2 & \dots & 2 & 2n-4 \end{pmatrix}.$$

Then Γ is

$$C_{2n-3}\begin{pmatrix} 1 & 2 & 3 & 4\ 5 & \dots & 2n-1 & \bar{P} \\ n-2 & n-2 & n-2 & 1\ 1 & \dots & 1 & n-2 \end{pmatrix}$$

and $v = n-2$.

Writing the transformation in terms of v, we have

$$T_{4v+1}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \dots & 2v+4 \\ 2v & 2v & 2v & 2v & 2 & 2 & \dots & 2 \end{pmatrix}.$$

Γ is

$$C_{2v+1}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+4 \\ v & v & v & v & 1 & \dots & 1 \end{pmatrix}.$$

The fundamental curves are

$$\begin{pmatrix} 1 & 2 & 3\ 4\ 5 & \dots & 2v+4 \\ v & v-1 & v\ v & 1 & \dots & 1 \end{pmatrix}^{1}_{2v} \quad \begin{pmatrix} 1 & 2\ 3\ 4\ 5 & \dots & 2v+4 \\ v-1 & v\ v\ v & 1 & \dots & 1 \end{pmatrix}^{2}_{2v}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5\ 6 & \dots & 2v+4 \\ v & v & v & v-1 & 1\ 1 & \dots & 1 \end{pmatrix}^{3}_{2v} \quad \begin{pmatrix} 1 & 2 & 3 & 4\ 5 & \dots & 2v+4 \\ v & v & v-1 & v\ 1 & \dots & 1 \end{pmatrix}^{4}_{2v}.$$

$$(1\,2\,3\,4\,5)^{5}_{2}\ \dots\ (1\,2\,3\,4\ 2v+4)^{2v+4}_{2}.$$

It will be noticed that the image of (1 2) is itself. The same is true of (3 4). Therefore the point of intersection of these two lines is invariant. The transformation therefore is

$$T_{4v+1}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \dots & 2v+4 & 2v+5\\ 2v & 2v & 2v & 2v & 2 & \dots & 2 & 0\end{pmatrix}.$$

This is equivalent to I where $(3\,4\,2v+5)$ $(1\,2\,2v+5)$ are on a line.

V. Transform by $I_{2\bar{Q}P}$, where Q is a point on Γ and P is not. Then

$$C_1 I_{2\bar{Q}P} = C_2\,(2\;\bar{Q}\;P),$$

$$C_2 T_n = C_{2n-1}\begin{pmatrix}1 & 2\,3\,4 & \dots & 2n-1 & \bar{Q} & P'\\ 2n-3 & 1\,2\,2 & \dots & 2 & 1 & 1\end{pmatrix},$$

$$C_{2n-1} I_{2\bar{Q}P} = C_{4n-4}\begin{pmatrix}1 & 2 & 3\,4 & \dots & 2n-1 & \bar{Q} & P'' & P\\ 2n-3 & 2n-2 & 2\,2 & \dots & 2 & 1 & 2n-2 & 2n-3\end{pmatrix}.$$

Γ is $$C_{2n-2}\begin{pmatrix}1 & 2\,3\,4\,5 & \dots & 2n-1 & \bar{Q} & P\\ n-2 & n\,1\,1\,1 & \dots & 1 & n-1 & n-1\end{pmatrix}$$

and $v = n-1$.

In terms of v, we have

$$T_{4v}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \dots & 2v+3 & 2v+4\\ 2v & 2v & 2v-1 & 2v-1 & 2 & \dots & 2 & 1\end{pmatrix}.$$

Γ is $$C_{2v}\begin{pmatrix}1\,2 & 3 & 4 & 5 & \dots & 2v+3\\ v\;v & v-1 & v-1 & 1 & \dots & 1\end{pmatrix}.$$

The fundamental curves are

$$\begin{pmatrix}1\,2 & 3 & 4\,5 & \dots & 2v+4\\ v\;v & v-1 & v\,1 & \dots & 1\end{pmatrix}^{1}_{2v} \quad \begin{pmatrix}1\,2\,3 & 4 & 5 & \dots & 2v+4\\ v\;v\;v & v-1 & 1 & \dots & 1\end{pmatrix}^{2}_{2v}$$

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \dots & 2v+3\\ v-1 & v & v-1 & v-1 & 1 & \dots & 1\end{pmatrix}^{3}_{2v-1} \begin{pmatrix}1 & 2 & 3 & 4 & 5 & \dots & 2v+3\\ v & v-1 & v-1 & v-1 & 1 & \dots & 1\end{pmatrix}^{4}_{2v-1}$$

$$(1\,2\,3\,4\,5)^{5}_{2} \dots (1\,2\,3\,4\,2v+3)^{2v+3}_{2}\,(1\,2)^{2v+4}_{1}.$$

It will be noticed that the lines (1 3) and (2 4) are invariant. Therefore the transformation is

$$T_{4v}\begin{pmatrix}1 & 2 & 3 & 4 & 5 & \dots & 2v+3 & 2v+4 & 2v+5\\ 2v & 2v & 2v-1 & 2v-1 & 2 & \dots & 2 & 1 & 0\end{pmatrix}.$$

It will be noticed that this is equivalent to I with three factors taken out. One goes out on account of the choice of P and two for $\bar{Q}$, since $\bar{Q}$ is on Γ.

VI. Invert by I_{PQR}, where PQR are points not on Γ. Then

$$C_1 I_{PQR} = C_2\,(P\,Q\,R),$$

$$C_2 T_n = C_{2n}\begin{pmatrix}1 & 2\,3 & \dots & 2n-1 & P' & Q' & R'\\ 2n-2 & 2\,2 & \dots & 2 & 1 & 1 & 1\end{pmatrix},$$

$$C_{2n} I_{PQR} = C_{4n}\begin{pmatrix}1 & 2\,3 & \dots & 2n-1 & P'' & Q'' & R'' & P & Q & R\\ 2n-2 & 2\,2 & \dots & 2 & 1 & 1 & 1 & 2n & 2n & 2n\end{pmatrix}.$$

Γ is $$C_{2n}\begin{pmatrix} 1 & 2\,3\,4 & \dots & 2n-1 & P\,Q\,R \\ n-2 & 1\,1\,1 & \dots & 1 & n\;n\;n \end{pmatrix}$$

and $v = n$.

Rearranging the terms and writing them in terms of v, we have

$$T_{4v}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+2 & 2v+3 & 2v+4 & 2v+5 \\ 2v & 2v & 2v & 2v-2 & 2 & \dots & 2 & 1 & 1 & 1 \end{pmatrix},$$

and Γ is

$$C_{2v}\begin{pmatrix} 1\,2\,3 & 4 & 5\,6 & \dots & 2v+2 \\ v\;v\;v & v-2 & 1\,1 & \dots & 1 \end{pmatrix}.$$

The fundamental curves are

$$\begin{pmatrix} 1\,2\,3 & 4 & 5\,6 & \dots & 2v+2 & 2v+4 & 2v+5 \\ v\;v\;v & v-1 & 1\,1 & \dots & 1 & 1 & 1 \end{pmatrix}^{1}_{2v}$$

$$\begin{pmatrix} 1\,2\,3 & 4 & 5 & \dots & 2v+3 & 2v+5 \\ v\;v\;v & v-1 & 1 & \dots & 1 & 1 \end{pmatrix}^{2}_{2v}$$

$$\begin{pmatrix} 1\,2\,3 & 4 & 5 & \dots & 2v+4 \\ v\;v\;v & v-1 & 1 & \dots & 1 \end{pmatrix}^{3}_{2v} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+2 \\ v-1 & v-1 & v-1 & v-2 & 1 & \dots & 1 \end{pmatrix}^{4}_{2v+2}$$

$$(1\,2\,3\,4\,5)^{5}_{2} \dots (1\,2\,3\,4\,2v+2)^{2v+2}_{2}\,(1\,2v+3)^{2v+3}_{1}\,(1\,2v+4)^{2v+4}_{1}$$
$$(1\,2v+5)^{2v+5}_{1}.$$

This transformation is equivalent to having in I three sets of three points on a line.

VII. Transform by $I_{\bar{P}QR}$, where $\bar{P}$ is on Γ and Q and R are not on Γ. Then

$$C_1\, I_{\bar{P}QR} = C_2\,(\bar{P}\;Q\;R).$$

$$C_2\, T_n = C_{2n}\begin{pmatrix} 1 & 2\,3\,4 & \dots & 2n-1 & \bar{P}\;Q'\;R' \\ 2n-2 & 2\,2\,2 & \dots & 2 & 1\;1\;1 \end{pmatrix},$$

$$C_{2n}\, I_{\bar{P}QR} = C_{4n-1}\begin{pmatrix} 1 & 2\,3 & \dots & 2n-1 & \bar{P} & Q'' & R'' & Q & R \\ 2n-2 & 2\,2 & \dots & 2 & 2n & 1 & 1 & 2n-1 & 2n-1 \end{pmatrix};$$

and Γ is

$$C_{2n-1}\begin{pmatrix} 1 & 2\,3\,4\,5 & \dots & 2n-1 & \bar{P} & Q & R \\ n-2 & 1\,1\,1\,1 & \dots & 1 & n & n-1 & n-1 \end{pmatrix}.$$

The class $v = n$.

The transformation in terms of v is

$$T_{4v-1}\begin{pmatrix} 1 & 2 & 3 & 4 & 5\,6 & \dots & 2v+2 & 2v+3 & 2v+4 \\ 2v & 2v-1 & 2v-1 & 2v & 2\,2 & \dots & 2 & 1 & 1 \end{pmatrix},$$

and Γ is

$$C_{2v-1}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+2 \\ v & v-1 & v-1 & v-2 & 1 & \dots & 1 \end{pmatrix}.$$

The fundamental curves are

$$\begin{pmatrix}1&2&3&4&5&\dots.&2v+4\\ v&v&v&v-1&1&\dots.&1\end{pmatrix}^{1}_{2v}\quad\begin{pmatrix}1&2&3&4&5&\dots.&2v+3\\ v&v-1&v-1&v-1&1&\dots.&1\end{pmatrix}^{2}_{2v-1}$$

$$\begin{pmatrix}1&2&3&4&5&\dots.&2v+2&2v+4\\ v&v-1&v-1&v-1&1&\dots.&1&1\end{pmatrix}^{3}_{2v-1}$$

$$\begin{pmatrix}1&2&3&4&5&\dots.&2v+2\\ v-1&v-1&v-1&v-2&1&\dots.&1\end{pmatrix}^{4}_{2v-2}$$

$$(1\,2\,3\,4\,5)^{5}_{2}\ \dots.\ (1\,2\,3\,4\,2v+2)^{2v+2}_{2}\ (1\,2)^{2v+2}_{1}\ (1\,3)^{2v+3}_{1}.$$

It will be noticed that the point of intersection of the lines (14) and (23) is invariant; therefore the transformation is

$$T_{4v-1}\begin{pmatrix}1&2&3&4&5&\dots.&2v+2&2v+3&2v+4&2v+5\\ 2v&2v-1&2v-1&2v-2&2&\dots.&2&1&1&0\end{pmatrix}.$$

This is equivalent to having four linear factors taken out of I.

VIII. Transform through $I_{2\bar{Q}\bar{P}}$, where $\bar{Q}$ and $\bar{P}$ are points on Γ. Then

$$C_1 I_{2\bar{Q}\bar{P}} = C_2\,(2\ \bar{Q}\ \bar{P}),$$

$$C_2 T_n = C_{2n-1}\begin{pmatrix}1&2&3&4&5&\dots.&2n-1&\bar{Q}&\bar{P}\\ 2n-3&1&2&2&2&\dots.&2&1&1\end{pmatrix},$$

$$C_{2n-1} I_{2\bar{Q}\bar{P}} = C_{4n-5}\begin{pmatrix}1&2&3&\dots.&2n-1&\bar{Q}&\bar{P}\\ 2n-3&2n-3&2&\dots.&2&2n-3&2n-3\end{pmatrix}.$$

Γ is $$C_{2n-3}\begin{pmatrix}1&2&3&4&\dots.&2n-1&\bar{Q}&\bar{P}\\ n-2&n-2&1&1&\dots.&1&n-2&n-2\end{pmatrix}.$$

Therefore $v = n-1$.

In terms of v, we have for the transformation

$$T_{4v-1}\begin{pmatrix}1&2&3&4&5&\dots.&2v+3\\ 2v-1&2v-1&2v-1&2v-1&2&\dots.&2\end{pmatrix},$$

while Γ is

$$C_{2v-1} = \begin{pmatrix}1&2&3&4&5&\dots.&2v+3\\ v-1&v-1&v-1&4&1&\dots.&1\end{pmatrix}.$$

The fundamental curves are

$$\begin{pmatrix}1&2&3&4&5&\dots.&2v+3\\ v-1&v-1&v-1&v&1&\dots.&1\end{pmatrix}^{1}_{2v-1}$$

$$\begin{pmatrix}1&2&3&4&5&\dots.&2v+3\\ v-1&v-1&v&v-1&1&\dots.&1\end{pmatrix}^{2}_{2v-1}$$

$$\begin{pmatrix}1&2&3&4&5&\dots.&2v+3\\ v-1&v&v-1&v-1&1&\dots.&1\end{pmatrix}^{3}_{2v-1}$$

$$\begin{pmatrix}1&2&3&4&5&\dots.&2v+3\\ v&v-1&v-1&v-1&1&\dots.&1\end{pmatrix}^{4}_{2v-1}$$

$$(1\,2\,3\,4\,5)^{5}_{2}\ \dots.\ (1\,2\,3\,4\,2v+3)^{2v+3}_{2}.$$

It will be noticed that (1 2), (3 4), (1 3), (2 4) are invariant lines. This determines two invariant points. The transformation therefore is

$$T_{4v-1}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \ldots. & 2v+3 & 2v+4 & 2v+5 \\ 2v-1 & 2v-1 & 2v-1 & 2v-1 & 2 \ldots. & 2 & 0 & 0 \end{pmatrix},$$

which is equivalent to having three factors divided out of I.

IX. Transform by $I_{\bar{P}\bar{Q}R}$, where $\bar{P}$ and $\bar{Q}$ are on Γ and R is not. Then

$$C_1 I_{\bar{P}\bar{Q}R} = C_2 (\bar{P}\ \bar{Q}\ R),$$

$$C_2 T_n = C_{2n}\begin{pmatrix} 1 & 2\,3 \ldots. & 2n-1 & \bar{P}\ \bar{Q}\ R' \\ 2n-2 & 2\,2 \ldots. & 2 & 1\ 1\ 1 \end{pmatrix},$$

$$C_{2n} I_{\bar{P}\bar{Q}R} = C_{4n-2}\begin{pmatrix} 1 & 2\,3 \ldots. & 2n-1 & \bar{P} & \bar{Q} & R'' & R \\ 2n-2 & 2\,2 \ldots. & 2 & 2n-1 & 2n-1 & 1 & 2n-2 \end{pmatrix},$$

and Γ is

$$C_{2n-2}\begin{pmatrix} 1 & 2 \ldots. & 2n-1 & \bar{P} & \bar{Q} & R \\ n-2 & 1 \ldots. & 1 & n-1 & n-1 & n-1 \end{pmatrix}.$$

Therefore $v = n$ and the transformation in terms of v is

$$T_{4v-2}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \ldots. & 2v+2 & 2v+3 \\ 2v-1 & 2v-1 & 2v-2 & 2v-2 & 2 \ldots. & 2 & 1 \end{pmatrix}.$$

The fundamental curves are

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \ldots. & 2v+3 \\ v-1 & v & v-1 & v-1 & 1 \ldots. & 1 \end{pmatrix}^{1}_{2v-1}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \ldots. & 2v+3 \\ v & v-1 & v-1 & v-1 & 1 \ldots. & 1 \end{pmatrix}^{2}_{2v-1}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \ldots. & 2v+2 \\ v-1 & v-1 & v-2 & v-1 & 1 \ldots. & 1 \end{pmatrix}^{3}_{2v-2}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \ldots. & 2v+2 \\ v-1 & v-1 & v-1 & v-2 & 1 \ldots. & 1 \end{pmatrix}^{4}_{2v-2}$$

$$(1\,2\,3\,4\,5)^{5}_{2} \ldots. (1\,2\,3\,4\,2v+2)^{2v+2}_{2}\,(1\,2)^{2v+3}_{1}.$$

The lines (1 3) (1 4) (2 4) (2 4) are invariant. Therefore there are two invariant points, and the transformation is

$$T_{4v-2}\begin{pmatrix} 1 & 2 & 3 & 4 & 5\,6 \ldots. & 2v+2 & 2v+3 & 2v+4 & 2v+5 \\ 2v-1 & 2v-1 & 2v-2 & 2v-2 & 2\,2 \ldots. & 2 & 1 & 0 & 0 \end{pmatrix}.$$

This is equivalent to dividing out five linear factors out of I.

X. Transform by $I_{\bar{P}\bar{Q}\bar{R}}$, where $\bar{P}\ \bar{Q}\ \bar{R}$ are points on Γ. Then

$$C_1 I_{\bar{P}\bar{Q}\bar{R}} = C_2\,(\bar{P}\ \bar{Q}\ \bar{R}),$$

$$C_2 T_n = C_{2n}\begin{pmatrix} 1 & 2\,3 & \dots & 2n-1 & \bar{P}\ \bar{Q}\ \bar{R} \\ 2n-2 & 2\,2 & \dots & 2 & 1\ 1\ 1 \end{pmatrix},$$

$$C_{2n} T_{\bar{P}\bar{Q}\bar{R}} = C_{4n-3}\begin{pmatrix} 1 & 2\,3 & \dots & 2n-1 & \bar{P} & \bar{Q} & \bar{R} \\ 2n-2 & 2\,2 & \dots & 2 & 2n-2 & 2n-2 & 2n-2 \end{pmatrix}.$$

Γ becomes

$$C_{2n-3}\begin{pmatrix} 1 & 2\,3 & \dots & 2n-1 & \bar{P}\ Q\ \bar{R} \\ n-2 & 1\,1 & \dots & 1 & n\ n\ n \end{pmatrix}.$$

Therefore $n = v$.

In terms of v, we have

$$T_{4v-3}\begin{pmatrix} 1 & 2 & 3 & 4 & 5\,6 & \dots & 2v+2 \\ 2v-2 & 2v-2 & 2v-2 & 2v-2 & 2\,2 & \dots & 2 \end{pmatrix}.$$

Γ becomes

$$C_{2v-3}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+2 \\ v-2 & v-2 & v-2 & v-2 & 1 & \dots & 1 \end{pmatrix}.$$

The fundamental curves are

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+2 \\ v-2 & v-1 & v-1 & v-1 & 1 & \dots & 1 \end{pmatrix}^1_{2v-2}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 2v+2 \\ v-1 & v-1 & v-1 & v-1 & 1 & \dots & 1 \end{pmatrix}^2_{2v-2}$$

$$\dots\ (1\,2\,3\,4\,5)^5_2 \dots\ (1\,2\,3\,4\,2v+2)^{2v+2}_2.$$

But the lines (1 2) (1 3) (1 4) (2 3) (2 4) are invariant. Therefore there are three invariant points. The transformation therefore is

$$T_{4v-3}\begin{pmatrix} 1 & 2 & 3 & 4\,5\,6 & \dots & 2v+2 & 2v+3 & 2v+4 & 2v+5 \\ 2v-2 & 2v-2 & 2v-2 & 2v-2\ 2\,2 & \dots & 2 & 0 & 0 & 0 \end{pmatrix}.$$

This is equivalent to dividing out six linear factors out of I.

By applying this process to any transformation which is involutorial, we can build involutorial transformations of all the different classes.

CORNELL UNIVERSITY, *February*, 1912.

A Theorem for the Development of a Function as an Infinite Product.

By A. F. Carpenter.

The usual conception of change of any kind as being brought about by an additive process, is purely arbitrary. So far as quantitative relations are concerned, the multiplicative, or indeed, any one of the extensions—in the sense of De Morgan*—of the ordinary algebraic processes, may be substituted for the additive. Corresponding to each of these conceptions, a consistent calculus may be built up† involving theorems which, by analogy, should be extensions of, and indeed, should be reducible to, the theorems of the ordinary calculus. For example, the Quotiential Calculus, based upon the conception of variation as taking place through successive multiplication by a quantity $h \lesssim 1$ and whose limit is 1, should produce a set of theorems analogous to those of the ordinary calculus, the exact relation being that the processes of addition, multiplication, ..., of the ordinary calculus are respectively replaced in the new theorems by multiplication, involution,..... As a specific instance, Taylor's Theorem should have, corresponding to it, in the Quotiential Calculus, a theorem whereby a function may be expanded into an infinite product. This is exactly the case, as the following development serves to illustrate.

I.

To conform with the new conception of variation, it will be well to restate the definition for continuity at a point.

Definition: *The function* $f(x)$ *is continuous for* $x = x_0$, *if* $\frac{f(x_0 h)}{f(x_0)}$ *approaches the limit* 1 *as* h *approaches* 1 *through values either less than, or greater than,* 1.

*Christine Ladd: "De Morgan's Extension of the Algebraic Processes," American Journal of Mathematics, Vol. III (1880).

†Moritz: "Generalization of the Differentiation Process," American Journal of Mathematics, Vol. XXIV, No. 3.

A function is continuous over an interval if it is continuous for every point in that interval. It is clear that the values $\pm \infty$ and 0 must be excluded from consideration, since for each, $x_0 h$ will not differ from x_0 and hence, every function would be continuous at these points. Any interval under consideration will then be finite and exclude the origin.

In addition to the definition of continuity, the proof of the principle theorem makes use of the following auxiliary theorem:

THEOREM I: *If $F(x)$ and $QF(x)$* are continuous in the closed interval (a, b), and $F(a) = F(b)$, then for some value x_1 of x, such that $a < x_1 < b$, will $QF(x_1) = 0$.*

Proof: If $F(x)$ is constant in the interval (a, b), then will $QF(x) = 0$ over the whole interval. If $F(x)$ is not constant, then will there be some value x_1 of x, $(a \leqq x_1 \leqq b)$ for which $F(x_1)$ is a maximum (or minimum). Suppose $F(x_1)$ is a maximum and let,

1st, $h > 1$;

then $\dfrac{F(x_1 h)}{F(x_1)} \leqq 1$ and $\log_h \dfrac{F(x_1 h)}{F(x_1)} \leqq 0$, hence $QF(x_1) = \lim\limits_{h=1} \log_h \dfrac{F(x_1 h)}{F(x_1)} \leqq 0$:

2nd, $h < 1$;

then $\dfrac{F(x_1 h)}{F(x_1)} \leqq 1$ and $\log_h \dfrac{F(x_1 h)}{F(x_1)} \geqq 0$, hence $QF(x_1) = \lim\limits_{h=1} \log_h \dfrac{F(x_1 h)}{F(x_1)} \geqq 0$:

therefore, $QF(x_1) = 0$.

A similar course of reasoning for $F(x_1)$, a minimum, will result in the same conclusion.

II.

Let $f(x)$ and $Qf(x)$ be continuous in the closed interval (a, b) and define a quantity P_1 by means of the equation

$$\frac{f(b)}{f(a)\left(\frac{b}{a}\right)^{P_1}} = 1, \qquad (a < b) \tag{1}$$

and write

$$F_1(x) = \frac{f(x)}{f(a)\left(\frac{x}{a}\right)^{P_1}}. \tag{2}$$

Then $F_1(b) = 1$, by (1), and $F_1(a) = 1$, by (2).†

---

* $QF(x)$ = the first quotiential coefficient of $F(x)$ with respect to x. In like manner $Q^n F(x)$ = the nth quotiential coefficient.

† Neither $f(a) = 0$ nor $f(b) = 0$, else (1) would fail to define P_1.

Now* $QF_1(x) = Qf(x) - P_1$, whence $QF_1(x_1) = Qf(x_1) - P_1 = 0$, where x_1 is some value of x such that $a < x_1 < b$. (Theorem I.)

Then $P_1 = Qf(x_1)$, and (1) becomes

$$\frac{f(b)}{f(a)\left(\frac{b}{a}\right)^{Qf(x_1)}} = 1. \tag{3}$$

Equation (3) may be said to express the law of the mean for Quotientiation.

To obtain a first extension of (3), define a quantity P_2 by means of the equation

$$\frac{f(b)}{f(a)\left(\frac{b}{a}\right)^{Qf(a)+P_2 \log \frac{b}{a}}} = 1, \tag{4}$$

and write

$$F_2(x) = \frac{f(x)}{f(a)\left(\frac{x}{a}\right)^{Qf(a)+P_2 \log \frac{x}{a}}}. \tag{5}$$

Then $F_2(b) = 1$, by (4), and $F_2(a) = 1$, by (5).

Now

$$QF_2(x) = \phi(x) = Qf(x) - Qf(a) - 2P_2 \log \frac{x}{a}, \tag{6}$$

so that for some value x_1, of x, such that $a < x_1 < b$, $\phi(x_1) = 0$ (Theorem I); also $\phi(a) = 0$, by (6). Again,

$$Q\phi(x) = \frac{Qf(x)Q^2f(x) - 2P_2}{\phi(x)},$$

and for some value x_2, of x, such that $a < x_2 < x_1$,

$$Q\phi(x_2) = \frac{Qf(x_2)Q^2f(x_2) - 2P_2}{\phi(x_2)} = 0.$$

Hence,† $Qf(x_2)Q^2f(x_2) - 2P_2 = 0$, and $P_2 = \frac{1}{2!} Qf(x_2)Q^2f(x_2)$, so that (4) becomes

$$\frac{f(b)}{f(a)\left(\frac{b}{a}\right)^{Qf(a)+\frac{1}{2!}Qf(x_2)Q^2f(x_2) \log \frac{b}{a}}} = 1. \tag{7}$$

---

*All the Quotientiation formulas involved in this discussion are easily developed without the use of Taylor's Theorem, or, indeed, without recourse to the differentiation process.

† $\varphi(x_2)$ is not infinite, else at least one of the quantities $Qf(x_2)$, $Qf(a)$, $\log \frac{x_2}{a}$ would be infinite, by (6). But (a, b) is a finite interval excluding the origin, and by hypothesis, $Qf(x)$ is continuous.

A second extension of (3) is made by defining a quantity P_3 by means of the equation

$$\frac{f(b)}{f(a)\left(\frac{b}{a}\right)^{Qf(a)+\frac{1}{2!}Qf(a)Q^2f(a)\log\frac{b}{a}+P_3\left(\log\frac{b}{a}\right)^2}}=1, \tag{8}$$

and proceeding as before, the resulting value of P_3 being

$$P_3=\frac{1}{3!}Qf(x_3)Q^2f(x_3)\left[Q^2f(x_3)+Q^3f(x_3)\right],$$

where x_3 is some value of x, such that $a<x_3<b$.

By assuming the existence of the first n quotiential coefficients of $f(x)$, and writing ak for $b\left(k=\frac{b}{a}\right)$, the $(n-1)$th extension of (3) gives

$$\begin{aligned} f(ak) &= f(a)\,(k)^{C_1+C_2\log k+C_3(\log k)^2+\ldots.C_n(\log k)^{n-1}} \\ &= f(a)\,(e)^{C_1\log k+C_2(\log k)^2+C_3(\log k)^3+\ldots.C_n(\log k)^n}, \end{aligned}$$

where, by the nature of process used, it is clear that

$$\begin{aligned} C_1&=Qf(a),\\ C_2&=\tfrac{1}{2}C_1QC_1,\\ C_3&=\tfrac{1}{3}C_2QC_2,\\ &\ldots\ldots\ldots,\\ C_n^*&=\tfrac{1}{n}C_{n-1}QC_{n-1}. \end{aligned}$$

If any coefficient, say C_n, proves constant when a (or x_n) is replaced by x, then will C_{n+1} and all succeeding coefficients be zero† and the expansion stops of itself. If this be not the case, then in order to find the proper value to assign to this development, it will be necessary to find an expression for the quotient

$$\frac{f(ak)}{f(a)\,(e)^{C_1\log k+C_2(\log k)^2+C_3(\log k)^3+\ldots.C_n(\log k)^n}},$$

assuming the existence of $f(x)$ and its first $n+1$ quotiential coefficients in the interval (a, ak). To do this, set

$$\frac{f(ak)}{f(a)\,(e)^{C_1\log k+C_2(\log k)^2+C_3(\log k)^3+\ldots.C_n(\log k)^n+P(\log k)^p}}=1, \tag{9}$$

---

* In C_n, a is replaced by x_n, where x_n is some value of x such that $a<x_n<b$.

† This is not equivalent to $Q^{n+1}f(x)=0$. The vanishing of any q. c. does not bring the development to an end, but leads only to a special case.

where p is any positive integer and P, a quantity defined by the equation itself, and consider the function

$$F(x) = \frac{f(ak)}{f(x)\,(e)^{C_1' \log \frac{ak}{x} + C_2' \left(\log \frac{ak}{x}\right)^2 + C_3' \left(\log \frac{ak}{x}\right)^3 + \ldots . C_n' \left(\log \frac{ak}{x}\right)^n + P \left(\log \frac{ak}{x}\right)^p}}, \tag{10}$$

where

$$\begin{aligned} C_1' &= Qf(x), \\ C_2' &= \tfrac{1}{2} C_1' Q C_1', \\ C_3' &= \tfrac{1}{3} C_2' Q C_2', \\ &\ldots\ldots\ldots, \\ C_n' &= \tfrac{1}{n} C_{n-1}' Q C_{n-1}'. \end{aligned}$$

$F(a) = 1$, by (9), and $F(ak) = 1$, by (10), and upon quotientiating after a simple reduction,

$$QF(x) = Pp \left(\log \frac{ak}{x}\right)^{p-1} - C_n' Q C_n' \left(\log \frac{ak}{x}\right)^n, \tag{11}$$

so that, by Theorem I,* $QF(x_0) = 0$, where x_0 is some value of x such that $a < x_0 < ak$. Replacing x by $x_0 = ak^{\theta_1}$, where $\theta_1 =$ a positive proper fraction, and solving the resulting equation for P, gives

$$P = \frac{1}{p} \overline{C}_n Q \overline{C}_n (\log k)^{n-p+1}\, \theta^{n-p+1}, \tag{12}$$

where $\overline{C}_n$ is the result of replacing x with ak^{θ_1} in C_n', and $\theta = 1 - \theta_1 =$ a positive proper fraction.

By replacing P with its value from (12), equation (9) gives rise to the following:

THEOREM II. *If $f(x)$ and its first $n+1$ quotiential coefficients are continuous in the interval (a, ak), then will*

$$f(ak) = f(a)\,(e)^{C_1 \log k + C_2 (\log k)^2 + C_3 (\log k)^3 + \ldots . C_n (\log k)^n + P_n}, \tag{13}$$

where

$$P_n = \frac{1}{p} \overline{C}_n Q \overline{C}_n \theta^{n-p+1} (\log \mathrm{k})^{n+1},$$

p being any positive integer and θ a positive proper fraction. The values which suggest themselves for p are readily seen to be

$$\begin{aligned} p &= 1, \qquad \text{whence } P_n = \overline{C}_n Q \overline{C}_n \theta^n (\log \mathrm{k})^{n+1}; \\ p &= n+1, \text{ whence } P_n = \frac{1}{n+1} \overline{C}_n Q \overline{C}_n (\log k)^{n+1}. \end{aligned}$$

---

* $F(x)$ and $QF(x)$ are continuous in (a, ak) through the conditions imposed on $f(x)$.

It is also to be noted that if P_n approaches 0 as n approaches ∞, (13) may be written

$$f(ak) = f(a) \prod_{n=1}^{\infty} (e)^{C_n(\log k)^n}, \tag{14}$$

or, placing x for a,

$$f(xk) = f(x) \prod_{n=1}^{\infty} (e)^{C'_n(\log k)^n}. \tag{15}$$

Interchanging x and k in (15), gives

$$f(kx) = f(k) \prod_{n=1}^{\infty} (e)^{C_n(\log x)^n}, \tag{16}$$

and, when $k = 1$, (16) becomes

$$f(x) = f(1) \prod_{n=1}^{\infty} (e)^{C_n(\log x)^n}. \tag{17}$$

Equations (15) and (16) at once suggest Taylor's Theorem, and (17), MacLaurin's Theorem.

III.

It will be observed that the coefficients $C_1, \ldots, C_n$, in (15), (16), (17), are not as readily expressed in terms of successive quotiential coefficients as are the coefficients in Taylor's Theorem, in terms of successive derivatives. But if the derivative notation be substituted for the quotiential, this difficulty is removed. For this reason, the extension of Theorem II, to functions of two or more independent variables, will be considered subsequent to the change of notation.

Now,

$$Qy = \frac{xdy}{ydx} = \frac{d \log y}{d \log x} = \frac{dy}{yd \log x},^* \tag{18}$$

so that

$$yQy = \frac{dy}{d \log x}. \tag{19}$$

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*Moritz, "Generalization of the Differentiation Process," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXIV, No. 3.

From (18) and (19) the coefficients $C_1, C_2, \ldots, C_n, \ldots$, may now be written

$$C_1 = \frac{d \log f(a)}{d \log a},$$

$$C_2 = \frac{1}{2!} \frac{d^2 \log f(a)}{(d \log a)^2},$$

$$\ldots\ldots\ldots\ldots\ldots,$$

$$C_n = \frac{1}{n!} \frac{d^n \log f(a)}{(d \log a)^n},$$

$$\ldots\ldots\ldots\ldots\ldots,$$

and equations (14) to (17) become

$$f(ak) = f(a) \prod_{n=1}^{\infty} (e)^{\frac{1}{n!} \frac{d^n \log f(a)}{(d \log a)^n} (\log k)^n}, \tag{20}$$

$$f(xk) = f(x) \prod_{n=1}^{\infty} (e)^{\frac{1}{n!} \frac{d^n \log f(x)}{(d \log x)^n} (\log k)^n}, \tag{21}$$

$$f(kx) = f(k) \prod_{n=1}^{\infty} (e)^{\frac{1}{n!} \frac{d^n \log f(k)}{(d \log k)^n} (\log x)^n}, \tag{22}$$

$$f(x) = f(1) \prod_{n=1}^{\infty} (e)^{\frac{1}{n!} \frac{d^n \log f(1)}{(d \log 1)^n} (\log x)^n}. \tag{23}$$

Consider now, the function

$$\phi(\gamma) = f(a^{\log \gamma} x,\ b^{\log \gamma} y) = f(\alpha, \beta), \tag{24}$$

where $\alpha = a^{\log \gamma} x$, $\beta = b^{\log \gamma} y$.

From (24) $\phi(1) = f(x, y)$; also

$$\gamma \frac{\partial \alpha}{\partial \gamma} = x a^{\log \gamma} \log a \text{ and } \gamma \frac{\partial \beta}{\partial \gamma} = y b^{\log \gamma} \log b. \tag{25}$$

Again

$$\frac{d\phi}{d\gamma} = \frac{\partial \phi}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial \gamma} + \frac{\partial \phi}{\partial \beta} \cdot \frac{\partial \beta}{\partial \gamma}, \tag{26}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial x} = \frac{\partial \phi}{\partial \alpha} \cdot a^{\log \gamma}, \tag{27}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial \beta} \cdot \frac{\partial \beta}{\partial y} = \frac{\partial \phi}{\partial \beta} \cdot b^{\log \gamma}. \tag{28}$$

Whence, by (25), (26), (27) and (28),

$$\frac{d\log\phi}{d\log\gamma}=\frac{\gamma\dfrac{d\phi}{d\gamma}}{\phi}=\log a\frac{x\dfrac{\partial\phi}{\partial x}}{\phi}+\log b\frac{y\dfrac{\partial\phi}{\partial y}}{\phi}=\log a\frac{\partial\log\phi}{\partial\log x}+\log b\frac{\partial\log\phi}{\partial\log y},$$

that is

$$\frac{d}{d\log\gamma}=\log a\frac{\partial}{\partial\log x}+\log b\frac{\partial}{\partial\log y}. \tag{29}$$

From (29)

$$\frac{d^2\log\phi}{(d\log\gamma)^2}=\frac{d}{d\log\gamma}\left(\frac{d\log\phi}{d\log\gamma}\right)=\left[\log a\frac{\partial\log\phi}{\partial\log x}+\log b\frac{\partial\log\phi}{\partial\log y}\right]^{(2)},$$

and, in general

$$\frac{d^n\log\phi}{(d\log\gamma)^n}=\left[\log a\frac{\partial\log\phi}{\partial\log x}+\log b\frac{\partial\log\phi}{\partial\log y}\right]^{(n)}. \tag{30}$$

From (30) and (23), since $\phi(1)=f(x,y)$, there results the equation

$$\phi(\gamma)=f(a^{\log\gamma}x,\,b^{\log\gamma}y)=f(x,y)\prod_{n=1}^{\infty}(e)^{\frac{1}{n!}\left[\log a\frac{\partial\log f(x,y)}{\partial\log x}+\log b\frac{\partial\log f(x,y)}{\partial\log y}\right]^{(n)}(\log\gamma)^n}. \tag{31}$$

When $\gamma=e$, (31) becomes

$$f(ax,by)=f(x,y)\prod_{n=1}^{\infty}(e)^{\frac{1}{n!}\left[\log a\frac{\partial\log f(x,y)}{\partial\log x}+\log b\frac{\partial\log f(x,y)}{\partial\log y}\right]^{(n)}}. \tag{32}$$

The above extension to a function of two independent variables is general and will apply equally well to functions of any number of variables.

IV.

It remains to exhibit the exact nature of the relation between the theorems just established and certain theorems of the ordinary calculus.

Consider the function

$$F(y)=e^{f(\log y)}.$$

Applying (22) to $F(ky)$,

$$F(ky)=e^{f(\log ky)}=e^{f(k)}\prod_{n=1}^{\infty}(e)^{\frac{1}{n!}\frac{d^nf(\log k)}{(d\log k)^n}(\log y)^n}. \tag{33}$$

Now, writing $k=e^a$ and $y=e^x$, and taking the logarithm of each member, (33) reduces to

$$f(a+x)=f(a)+\sum_{n=1}^{\infty}\frac{x^n}{n!}f^{(n)}(a),$$

one of the common forms of Taylor's Series for a function of a single variable.

Again, consider the function

$$F(u, v, w, \ldots) = e^{f(\log u, \log v, \log w, \ldots)}.$$

Applying (32) to $F(gu, kv, lw, \ldots)$,

$$F(gu, kv, lw, \ldots) = e^{f(\log gu, \log kv, \log lw, \ldots)}$$
$$= e^{f(\log u, \log v, \log w, \ldots)} \prod_{n=1}^{\infty} (e)^{\frac{1}{n!}\left[\log g \frac{\partial f}{\partial \log u} + \log k \frac{\partial f}{\partial \log v} + \log l \frac{\partial f}{\partial \log w} + \ldots\right]^{(n)}}. \quad (34)$$

Now writing

$$g = e^a, \; k = e^b, \; l = e^c, \ldots,$$
$$u = e^x, \; v = e^y, \; w = e^z, \ldots,$$

and taking the logarithm of each member, (34) reduces to

$$f(a + x, b + y, c + z, \ldots) = f(x, y, z, \ldots)$$
$$+ \sum_{n=1}^{\infty} \frac{1}{n!}\left[a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y} + c \frac{\partial f}{\partial z} + \ldots\right]^{(n)},$$

Taylor's Series for a function of any number of variables.

Or, again, writing

$$g = e^0, \; k = e^{ah}, \; l = e^{2bh + ah^2}, \ldots,$$
$$u = e^a, \; v = e^b, \; w = e^c, \ldots,$$

and taking the logarithm of each member, (34) reduces to

$$f(a, b + ah, c + 2bh + ah^2, \ldots)$$
$$= f(a, b, c, \ldots) + \sum_{n=1}^{\infty} \frac{1}{n!}\left[ah \frac{\partial}{\partial b} + h(2b + ah) \frac{\partial}{\partial c}\right.$$
$$\left. + h(3c + 3bh + ah^2) \frac{\partial}{\partial d} + \ldots\right]^n f$$
$$= f(a, b, c, \ldots) + \sum_{n=1}^{\infty} \frac{1}{n!}\left[h\left(a \frac{\partial}{\partial b} + 2b \frac{\partial}{\partial c} + 3c \frac{\partial}{\partial d} + \ldots\right)\right.$$
$$\left. + h^2\left(a \frac{\partial}{\partial c} + 3b \frac{\partial}{\partial d} + \ldots\right) + h^3\left(a \frac{\partial}{\partial d} + 4b \frac{\partial}{\partial e} + \ldots\right)\right]^n f, \quad (35)$$

where it is to be noted that the nth power of the operator is no longer

equivalent to n successive applications of its first power. If, following Sylvester,* we write

$$f(a, b+ah, c+2bh+ah^2, \ldots.) = f_1,$$

$$a\frac{\partial}{\partial b}+2b\frac{\partial}{\partial c}+3c\frac{\partial}{\partial d}+\ldots\ldots = \Omega,$$

$$a\frac{\partial}{\partial c}+3b\frac{\partial}{\partial d}+\ldots\ldots\ldots\ldots = \Omega_1,$$

$$a\frac{\partial}{\partial d}+4b\frac{\partial}{\partial e}+\ldots\ldots\ldots\ldots = \Omega_2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

(35) becomes

$$f_1 = f+\sum_{n=1}^{\infty}\frac{1}{n!}(h\Omega+h^2\Omega_1+h^3\Omega_2+\ldots.)^n,$$

or, more concisely,

$$f_1 = e^{h\Omega+h^2\Omega_1+h^3\Omega_2+\ldots.}f. \tag{36}$$

Equation (36) is one form of what Sylvester chooses to call Salmon's Theorem.

V.

Theorem II can easily be established for functions of complex variables, the work necessitating, of course, the defining of quotientiation and its inverse, for such functions, and involving also the proof of a theorem expressed by the equation

$$f(a) = \left[\Pi e^{\frac{\operatorname{Log} f(z)}{\operatorname{Log} z/a}(qz-1)}\right]^{\frac{1}{2\Pi i}}.\dagger$$

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* Sylvester, *Mathematical Papers*, Vol. III, pp. 88–92.

† The inverse of the quotientiation process is $Q^{-1}f(x) = \Pi e^{f(x)(qx-1)}$.

The Reducibility of Maps.

By George D. Birkhoff.

Introduction.

It is known that any map on a surface of genus zero, such as the sphere, can be so colored in at most five colors that no two regions with a common boundary line are of the same color; but it is not known whether or not only four colors always suffice.* In the present paper we restrict ourselves to maps on such a surface and to the use of four colors, a, b, c, d.

Substantially all that has been done toward the solution of this four-color problem is contained in the following four reductions:

If more than three boundary lines meet at any vertex of a map, the coloring of the map may be reduced to the coloring of a map of fewer regions.† For, not every pair of regions meeting at such a vertex have a boundary line in common, as is the case when three boundary lines meet at a vertex. Thus we may join two of these regions without a common boundary line, by opening the vertex, so as to form a map in one less region. When this simple map is colored, the original map is obtained, properly colored, by merely restoring the vertex to its original form.

If any region of a map is multiply-connected, the coloring of the map may be reduced to the coloring of maps of fewer regions. For we may color the partial maps which arise when all but one of the parts into which such a multiply-connected region separates the surface are erased, and afterward give the same color to this region in all of the partial maps by a permutation of the colors. By effecting a superposition of the partial maps, we obtain a coloring of the original map in the same colors employed in the partial maps.

If two or three regions of a map form a multiply-connected region, the coloring of the map may be reduced to the coloring of maps of fewer regions.

* A set of references may be found in the *Encyklopädie der mathematischen Wissenschaften*, III A, B 3, pp. 177–178. The question admits of transformation to several remarkable forms; in particular see the recent paper by O. Veblen, *Annals of Mathematics*, Ser. 2, Vol. XIV (1912), pp. 86–94.

† Throughout this article we use the term *boundary line* or *side* to mean one of the continuous lines in common to two distinct regions, and the term *vertex* to mean a point in common to three or more distinct regions. A region may have only a single boundary line or side.

In fact, these regions separate the surface into two or more parts. If we color the partial maps which arise when all but one of these parts are contracted to a point, and by a permutation then make the colors of the two or three regions the same on each of the partial maps, the original map may be colored by a superposition of the partial maps.

If the map contains any 1-, 2-, 3- *or* 4-*sided region, the coloring of the map may be reduced to the coloring of a map of fewer regions.* If a 1-, 2- or 3-sided region be present, we may shrink this region to a point, color the resulting map in one less region, and then introduce this region again in a color different from the colors of the regions to which it is adjacent along a boundary line. If a 4-sided region be present, at least one of the two pairs of regions which abut on opposite sides are distinct and have no boundary line in common. Let two opposite regions of this kind coalesce with the 4-sided region. If the resulting map in two less regions is colored, the original map may be colored by inserting the 4-sided region in a color different from that of the two or three colors of the regions adjacent to it along a boundary line.

By a succession of reductions of the above types it is clear that we are brought either to a map of one region consisting of the entire surface, which can be colored in one color, or to *regular maps* in which (1) all vertices are triple (*i. e.*, belong to three regions), (2) every region is simply-connected, (3) no two or three regions form a multiply-connected region, (4) every region has at least five sides.*

The purpose of the present paper is to show that there exist a number of further reductions which may be effected with the aid of the notion of *chains* due to A. B. Kempe. †

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* In the case where five colors are given, an additional reduction of 5-sided regions is possible. Namely, let coalesce with any 5-sided region two distinct regions having a boundary line in common with this region but themselves without a common boundary line, and color the resulting map which has two less regions. Then introduce the 5-sided region in a color different from the four or fewer colors of the regions which are adjacent to it along a boundary line.

This reduction shows that the coloring in five colors can always be effected since there can not exist a regular map in which there are no 5-sided regions. For, let $f_5, f_6, \ldots$. denote the number of 5-sided, 6-sided, ... regions respectively in a regular map. By Euler's theorem on polyhedra we have for this map

$$\text{number of regions} + \text{number of vertices} = \text{number of sides} + 2,$$

or

$$f_5 + f_6 + \ldots. + \tfrac{1}{3}[5f_5 + 6f_6 + \ldots.] = \tfrac{1}{2}(5f_5 + 6f_6 + \ldots.) + 2,$$

whence

$$f_5 = f_7 + 2f_8 + \ldots. + 12,$$

so that there are at least twelve 5-sided regions in any regular map.

† Kempe employed this notion in attempting to prove that every map on the sphere can be colored in only four colors. See AMERICAN JOURNAL OF MATHEMATICS, Vol. II (1879), pp. 193–200. The error was pointed out later by P. J. Heawood, *Quarterly Journal of Mathematics*, Vol. XXIV (1890), pp. 332–338.

§ 1. *The Kempe Chains and Reducibility.*

Suppose that a map M is colored in four colors, a, b, c, d, and select a pair of these as a, b. Consider any region colored in one of this pair of colors together with all of the regions in these two colors, adjacent to it, or connected with it by a set of regions in the two colors. Such a set of regions will be called an *a, b chain*. Obviously, the same a, b chain is defined by any region in the chain.

A fundamental property of the chain is that if the two colors on the regions of a single chain, or of any set of these chains in the same colors, be all interchanged, a new coloring of the map results, since no region in one of these two colors becomes adjacent to another of the same color by such a transposition of colors; the colors on any set of the chains in the complementary pair of colors may be interchanged at the same time.

Every region on the map lies on an a, b chain or c, d chain, an a, c chain or b, d chain, an a, d chain or b, c chain. By successive permutations of the colors on the chains a number of different colorings may be obtained from a single one. It is only when there is but one chain of each description that the new colorings obtained differ from the original one by a mere permutation of the colors throughout the map.

Consider now in a regular map a cyclical arrangement of $n\,(n>3)$ regions $\mu_1, \mu_2, \ldots, \mu_n$ such that each of these regions has a boundary line in common with the one preceding and following it in cyclical order, but with no other region of the set.

A *ring* R of regions of this kind divides the map into two sets of regions M_1 and M_2 which together with R make up all the regions of the map M.

The partial maps $M_1 + R$ and $M_2 + R$ are bordered by the ring R of n regions. If it is possible to color $M_1 + R$ and $M_2 + R$ so that the arrangement of colors on R is the same in both cases except for a mere permutation of the colors, it is clearly possible to take the coloring on R the same in both the partial maps and by a superposition color the map $M = M_1 + M_2 + R$.

Take now one of the partial maps $M_1 + R$, and consider any pair of its regions on the ring R, either region having one of the colors a, b, say. If these two regions are on the same a, b chain, this fact may be indicated by joining the two regions by an *a, b line*, marked a, b and lying within the chain. Let all such lines be drawn and marked with their pairs of colors, and at the same time let the colors on R be indicated: thus a *scheme* on the ring R is formed.

No two lines marked in complementary pairs of colors will cross since lines in complementary pairs of colors lie in different regions.

If a chain (found in $M_1 + R$) joins a region α of R to a region β of R, and if also a chain in the same colors joins the region β to the region γ of R, then a chain in these colors joins α to γ. Consequently, if in any scheme a line marked in two of the colors joins α to β, and a line marked in the same colors joins β to γ, such a line will also join α to γ.

Two regions α and β of R in one or both of any pair of colors are either joined by a chain in these colors, or else a pair of regions γ and δ of R, such that $\alpha, \gamma, \beta, \delta$ occur in cyclical order on R, are joined by a chain in the complementary pair of colors. For, if the chain in the first pair of colors which contains α does not contain β, this chain abuts on a chain in the complementary pair of colors which joins a region γ to a region δ. Consequently, either a line marked with this pair of colors connects α and β, or a line in the complementary pair of colors connects a pair of regions γ and δ.

These three properties of schemes will serve to define them completely for our purposes.

As we have observed, it is possible to derive from the given coloring of $M_1 + R$ a second coloring by transposing the pairs of colors at pleasure on any two complementary sets of chains. Returning now to the corresponding scheme, we see that we effect a transposition of complementary pairs of colors as a, b and c, d on a set of regions of R and that we transpose together the colors of all regions of R connected by a, b and c, d lines respectively. The a, b and c, d lines of the new scheme will be the same as in the old, but the lines of the four other types may be altered.

We are thus led to the following formal principle: *A given scheme on R gives rise to a new scheme in which two complementary sets of lines are unaltered and the corresponding colors are permuted on R in any way so that all those connected by these lines are transposed together.*

Suppose that in M we replace $M_2 + R$ by a set of regions $\mathfrak{M}_2$ of not more than a certain number k of regions but with the same n peripheral boundary lines as R; in this manner we form a map $M_1 + \mathfrak{M}_2$ which we assume to be colored. For each of the finite number of essentially different choices of $\mathfrak{M}_2$ we get a certain number of choices of colors on the periphery of R, which are possible for M_1, and thence, by permutation, a certain number of sets I of schemes for $M_1 + R$ with these same colorings on R.

Likewise, by forming a map $\mathfrak{M}_1$ and then $M_2 + \mathfrak{M}_1$ we get a certain number of sets II of schemes for $M_2 + R$.

These two sets are closed under the application of the above formal principle. If it can be shown that for some k any two sets of schemes I and II

have at least one coloring in common, we will say that the ring of n regions is *reducible*; and if k is the least integer for which this is true, we will say that $\phi(n) = k - n$ is the *reducing number* of R.

It is clear that if in the map M there exists a reducible ring of n regions having more than $\phi(n)$ regions on both sides of it, then, by coloring the maps $M_1 + \mathfrak{M}_2$ and $M_2 + \mathfrak{M}_1$ of fewer regions than M in all possible ways, one will always be led to two such sets of schemes, at least unless one of these maps can not be colored. By definition, one coloring at least in each of the two sets will be essentially the same. Hence, by a superposition one may color M, provided all of the auxiliary maps employed in this process can be colored. Since these auxiliary maps each contain fewer regions than M, the question of the coloring of M has been reduced to that of coloring maps in fewer regions. This fact justifies the use of the word *reducible* here.

The definition of reducible rings is entirely independent of whether or not a coloring of the map M is possible. We note, however, that if there is a map M which can not be colored but which contains a reducible ring of n regions having more than $\phi(n)$ regions on both sides of it, then one of the auxiliary maps $M_1 + \mathfrak{M}_2$ or $M_2 + \mathfrak{M}_1$ can not be colored; and that the map of the least number of regions which can not be colored, if there is such a map, contains no ring of this kind.

It is proved in the present paper that rings for $n = 4$ or 5 are reducible. However, for $n = 6$ and greater values of n the question is left unsettled.

Suppose now that a ring R of n regions is given, which is such that any possible set of schemes deduced from $M_2 + \mathfrak{M}_1$ as before, where $\mathfrak{M}_1$, however, has fewer regions than $M_1 + R$, always contains at least one coloring for R suitable for $M_1 + R$.

The question of coloring a map M of this kind reduces to that of coloring maps $M_2 + \mathfrak{M}_1$ in fewer regions than M, and in these circumstances the ring R will be termed *reducible with respect to* M_1.

It is clear that in this case the map M can be colored if all the maps $M_2 + \mathfrak{M}_1$ in fewer regions can be colored, and that the map of the least number of regions which can not be colored contains no reducible ring of this type.

§ 2. *Rings of Four Regions.*

We can prove at once that in a regular map *a ring of four regions is reducible and the reducing number is* 0.*

* In this proof and later proofs we use the notation of § 1 in our consideration of rings; in particular, R denotes the ring, M_1 and M_2 the two aggregates of regions on opposite sides of the ring, and M denotes the map $M_1 + M_2 + R$.

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the four regions of the ring taken in cyclical order. Consider the maps $M_1 + \mathfrak{M}_2$ and $M_2 + \mathfrak{M}_1$, where $\mathfrak{M}_2$ and $\mathfrak{M}_1$ are formed from M_2+R and M_1+R by shrinking M_2 and M_1, respectively, to a point and joining α_1 and α_3. The possibilities for the colors are essentially either

$$a, b, a, b \quad \text{or} \quad a, b, a, c,$$

for $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, respectively. If the colors are the same in both sets, we have a coloring for R the same in both sets of schemes. Otherwise we have

$$a, b, a, b \text{ for } M_1 + R \quad \text{and} \quad a, b, a, c \text{ for } M_2 + R, \text{ say.}$$

Now form a second choice of $\mathfrak{M}_1$ by shrinking M_1 to a point and joining α_2 to α_4, giving

$$a, b, a, b \quad \text{or} \quad a, b, c, b \quad \text{for} \quad M_2 + R.$$

The only case necessary for us to consider is the second of these two since the first is listed for $M_1 + R$. Thus we have the set of colors a, b, a, b in the schemes for $M_1 + R$; and the sets a, b, a, c and a, b, c, b in the schemes for $M_2 + R$.

In the scheme a, b, a, b for $M_1 + R$ either an a, d line connects the regions α_1, α_3, in which case we obtain a, b, a, c by a permutation, or else a b, c line connects the regions colored in b, and we get a, b, d, b by a permutation. In either case we get an arrangement of colors already essentially found in the set of schemes for $M_2 + R$.

Hence, the ring of four regions is reducible. The reducing number is 0 since $\mathfrak{M}_1$ and $\mathfrak{M}_2$ did not contain in any case more than four regions.

§ 3. *Rings of Five Regions.*

Any regular map contains a 5-sided region* and, after the reductions of § 2, contains a ring of five regions about each such region. If rings of five regions were reducible with a reducing number 0, one could therefore conclude that the coloring of every map is reducible to that of maps of fewer regions, and hence that all maps could be colored in four colors. As a matter of fact, however, we can prove that *a ring of five regions is reducible and has a reducing number* 1, *that is* $\phi(5) = 1$.

Let us first prove that $\phi(5)$ does not exceed 1. We need to show that any two possible sets of schemes for $M_1 + R$ and $M_2 + R$ have at least one common coloring for R, if for $\mathfrak{M}_2$ and $\mathfrak{M}_1$ are permitted all maps of less than six regions.

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* See foot-note, p. 116.

We can then take $\mathfrak{M}_1$ and $\mathfrak{M}_2$ to consist of the five regions of R and a single contained region. This leads us to schemes on R which are in at most three colors (those different from the color of the contained region); and, since the colors can not alternate on five regions, there are precisely three colors, at most two of any color; we have then essentially two regions in a, two regions in b and one in c. This gives arrangements c, a, b, a, b for the regions in R taken in cyclical order. We will call the region colored c the *marked region;* it may be any one of the five regions $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ of R.

If the marked region is the same in a scheme of each set, we can make the colors on the ring the same for $M_1 + R$ and $M_2 + R$ by a permutation, so that this case is disposed of.

Now in every case either a marked region in set I, of schemes for $M_1 + R$, and set II, of schemes for $M_2 + R$, must fall at adjacent places, or a coloring for R in I and II is the same. For suppose that this is not so. It is then no restriction to assume that α_1 and α_3 are marked regions for I and II, so that we have I$(c\,a\,b\,a\,b)$ [*i.e.*, c, a, b, a, b for $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ in I] and II$(a\,b\,c\,a\,b)$ [*i.e.*, a, b, c, a, b for $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ in II].

In the second of these schemes we may suppose that a b, c line joins the regions α_2, α_3 to the region α_5, for otherwise we would be led to II$(a\,c\,b\,a\,b)$ by a transposition. This is a scheme for II in which the marked region is on α_2 adjacent to α_1, contrary to assumption. Hence a b, c line joins these regions and we may change a on α_4 to d, obtaining II$(a\,b\,c\,d\,b)$.

Now, form a second map $\mathfrak{M}_2$ by shrinking M_2 to a point and joining α_2, α_5, so that $\mathfrak{M}_2$ consists of four regions. We are then led to a scheme I$(*\,a\,*\,*\,a)$ where the $*$-colors are different from a. The last two $*$-colors may be taken to be b and c, and thus the cases arise:

$$\text{I}(b\,a\,b\,c\,a),\quad \text{I}(c\,a\,b\,c\,a),\quad \text{I}(d\,a\,b\,c\,a).$$

The first of these has a marked region α_4 adjacent to α_3, a marked region in II. Hence this case is excluded. The second case is of the same type but with marked region α_3. This is the case disposed of at the outset, namely that of the same marked region in I and II. The last case I$(d\,a\,b\,c\,a)$ must then arise, giving essentially the same coloring as II$(a\,b\,c\,d\,b)$ obtained earlier. Thus a contradiction arises and the statement is proved.

If, however, marked regions in I and in II are adjacent, the two schemes again have a coloring for R in common.

To prove this, let us take these marked elements to be α_1 and α_2 respectively so that we have I$(c\,a\,b\,a\,b)$ and II$(b\,c\,a\,b\,a)$. Consider the regions in the colors

b, c in $\mathrm{II}(b\,c\,a\,b\,a)$. If a b, c line does not connect the regions α_1, α_2, and α_4, we obtain $\mathrm{II}(c\,b\,a\,b\,a)$ with marked region at α_1, a type already listed in I. In this case the statement is true. When a b, c line does connect these regions we obtain $\mathrm{II}(b\,c\,d\,b\,a)$ by changing a on α_3 to d.

Now, form $\mathfrak{M}_2$ by shrinking M_2 to a point and joining α_1 to α_4 in R. This gives $\mathrm{I}(b * * b *)$, which leads to one of the types of schemes:

$$\mathrm{I}(b\,c\,d\,b\,a),\quad \mathrm{I}(b\,c\,d\,b\,c),\quad \mathrm{I}(b\,c\,d\,b\,d).$$

The first of these has just been listed in II, and the third has a marked region at α_2 so that this type also appears in II. Thus we are led to $\mathrm{I}(b\,c\,d\,b\,c)$ with marked region at α_3.

Consequently, if the two sets of schemes have no common coloring for R, we infer that I and II have a marked element at α_1 and α_2 respectively, and thence that I has a marked region at α_3.

By a repetition of this argument, starting with a marked region α_2 of II and α_3 of I, we prove that α_4 is a marked region for II. By another repetition we prove that α_5 is a marked region for I, and by yet another repetition we prove that α_1 is a marked region for II as well as for I.

Hence, in every case there is a coloring common to the two sets of schemes, and this shows that the ring is reducible, with a reducing number which does not exceed 1.

To show that the reducing number $\phi(5)$ is 1, it is required to exhibit two sets of schemes containing essentially different colorings for R, and such that all the permutations of either set of schemes yield another scheme of the same set. These sets of schemes must have the further property that some coloring in each set is consistent with any possible choice of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of less than six regions.

A simple reckoning shows that all the schemes a, b, a, c, d for R in which the regions in b, c and b, d lie on b, c and b, d lines respectively, suffice for one of these sets of schemes; and that all the schemes c, a, b, a, b on R in which the regions in a and b lie on an a, d and b, d line suffice for the other set of schemes. Hence we must have $\phi(5) = 1$.

§4. *Regular Maps under the Reductions of §2 and §3.*

It is interesting to inquire as to the kind of map to which we are led by the above reductions. What is the nature of regular maps M containing no rings of four regions, or of five regions except about a single region?

To answer this question we consider an arbitrary k-sided ($k \geqq 5$) region A of the map adjacent to the regions $B_1, B_2, \ldots, B_k$, where these regions are taken in cyclical order, one for each boundary line of A. These regions are all distinct, since otherwise a region B_i and A would together form a multiply-connected region, a case excluded by the definition of a regular map. Likewise, no region B_i has a side in common with any region B_j except the two which precede and follow it in cyclical order, for otherwise B_i, B_j, A together form a multiply-connected region. Consequently the regions $B_1, \ldots, B_k$ form a ring about A.

Consider next the regions $C_1, \ldots, C_l$ which abut on the outer edge of the ring $B_1, \ldots, B_k$. These are taken in cyclical order, one for each boundary line directed outward from the ring $B_1, \ldots, B_k$ and l is at least as great as k since each region B_i has at least five sides. All of the regions $C_1, \ldots, C_l$ are distinct; otherwise we should have a ring of four regions formed by regions C_i, B_j, A, B_k, or a multiply-connected region formed by C_i, B_j, B_k.

If two regions C_i, not adjacent along a boundary line directed outward from a region B, had a side in common, we would then have a ring of five regions of the form C_i, C_j, B_k, A, B_l, or a ring of four regions C_i, C_j, B_k, B_l, or a multiply-connected region C_i, C_j, B_k. The last two cases are excluded by our hypothesis about M. In the first case, such a ring of five regions would contain one region only on one side by our hypothesis, and this region is a region B, of course. The regions C_i, C_j would then be adjacent to each other along a boundary line directed outward from a region B. Thus the first case is excluded as well as the other two.

Therefore we conclude that *in a regular map not subject to the reductions of* § 2, § 3 *the neighborhood of any region of the map is formed by a ring* $B_1, B_2, \ldots, B_k$ ($k \geqq 5$) *enclosing* A *and a second ring* $C_1, C_2, \ldots, C_l$ ($l \geqq k$) *enclosing* $A, B_1, B_2, \ldots, B_k$. Conversely, it is obvious that if a regular map has this property it will contain no rings of four regions, or rings of five regions except about a single region.

A simple example of such a map is the map of twelve 5-sided regions formed by the faces of a dodecahedron.

§ 5. *Rings and the Four-Color Problem.*

After the preceding reductions and the application of them to normalizing the form of the map, it is natural to inquire whether or not rings of more than five regions are reducible and have such reducing numbers as to enable one to extend the preceding conclusions about the structure of a completely reduced map.

For example, if rings of six regions were reducible with a reducing number 3 we could infer at once that in a map not subject to reduction there was a third set $D_1, \ldots, D_m$ of cyclically ordered regions about $C_1, C_2, \ldots, C_l$, *all of which were distinct.**

It is apparent, however, that the ring of six regions is of a decidedly different character from the ring of five regions. In fact, we may have a regular map, each region of which is surrounded by three successive rings of regions, in which, however, a ring of six regions containing an arbitrary number of 5-sided regions may be found (see fig. 1, p. 125, in which the dotted line indicates the ring). It will be seen later, that if there are more than three such regions, the ring of six regions is reducible. In so far as there is an analogy between the ring of five regions and that of six regions, the number 3 may be expected to play the same rôle for the latter as the reducing number 1 did for the former.

In all the cases which I have considered, the ring R of six regions containing more than three regions in M_1 is reducible with respect to M_1.

It may easily be seen that if a ring R of six regions, or more generally of n regions, contains a sufficiently large number of regions in M_1, it will necessarily be reducible with respect to M_1. In fact, all the possible choices of regions $\mathfrak{M}_1$ to replace $M_1 + R$ (where we do not now limit the number of regions in $\mathfrak{M}_1$) only yield a finite number of different aggregates of colorings for R. Let $t_1, t_2, \ldots, t_k$ be the number of regions in maps furnishing respectively each of the different aggregates of colorings. Clearly if t be the greatest of these numbers, the ring R of n regions will be reducible with respect to M_1 if $M_1 + R$ contains more than t regions.

The importance of reduction by means of rings may be explained as follows: In any map the direct and successful method of coloring is to start with a single region and color the regions adjoining, in succession, by trial. If a ring of k regions, where k is small, occurs somewhere in the map, and one comes to the regions of the ring from one side, it is impossible to determine what coloring to give to the regions of the ring unless one has considered the very complex map that may be on the other side.

On account of the curious results which a careful analysis of the ring of six regions gives, it is not probable that the ring of more than five regions can be eliminated as has been done with the ring of five or fewer regions, and this fact seems to me to make plausible any one of the following three alternatives:

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* See § 4.

1. There exist maps which can not be colored in four colors, a leading feature of the simplest one of them perhaps being a ring of six regions with more than three regions on either side. By the method of reduction one will always be led either to a coloring of the given map, or to one or more maps that can not be colored.

2. All maps can be colored in four colors and a set of reducible rings can be found, one of which exists in every map.

3. All maps can be colored in four colors, but only by means of reductions of a more extensive character, applicable to sets of regions bounded by any number of rings.

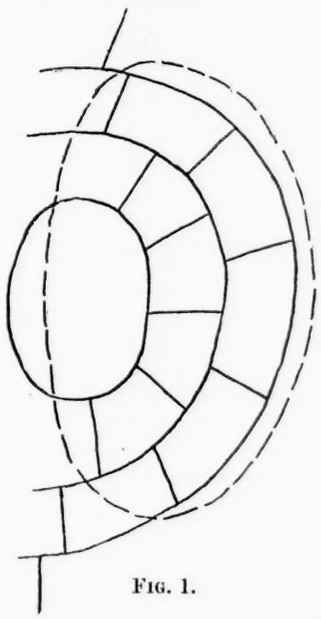


FIG. 1.

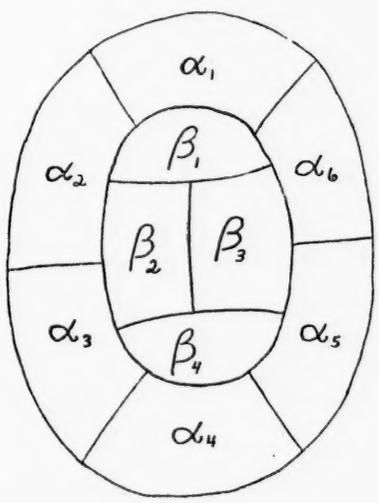


FIG. 2.

§ 6. *Rings of Six Regions.*

I shall not attempt to give an exhaustive analysis of the ring of six regions, such as I have actually carried out, but will confine attention to the more interesting special results that arise.

To begin with, let us show that *a ring R of six regions is reducible with respect to M_1 if M_1 consists of four 5-sided regions* (fig. 2).

Let β_1, β_2, β_3, β_4 be the 5-sided regions of M_1, taken to abut on the regions $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ of the ring as indicated in the figure.

Our method of proof is to assume R not reducible with respect to M_1, so that by hypothesis one has not necessarily one of the sets of schemes for $M_2 + R$ suitable for coloring R of $M_1 + R$, and to show that a contradiction results.

Let $\beta_1, \beta_2, \beta_3, \beta_4, \alpha_3, \alpha_5$ of $M_1 + R$ coalesce to form $\mathfrak{M}_1$. With this choice of $\mathfrak{M}_1$ we obtain a coloring $b, *, a, *, a, *$ of R in the scheme for $M_2 + R$.

The only essentially different cases are then the six following:

$$b, c, a, b, a, c; \quad b, c, a, c, a, c; \quad b, c, a, d, a, c;$$
$$b, c, a, b, a, d; \quad b, c, a, c, a, d; \quad b, c, a, d, a, d.$$

Direct trial shows that it is possible to color $M_1 + R$ starting with any of these colorings for R except b, c, a, c, a, c which must then be listed in the set of schemes for $M_2 + R$.

A c, d line will connect all of the regions colored c in b, c, a, c, a, c, for otherwise we should be led to one of the schemes just excluded by a transposition of the colors c, d. Hence any permutation of the colors a, b is permissible, and one of these gives b, c, b, c, a, c, which is a suitable arrangement of colors on R for $M_1 + R$.

Thus in every case we are led to a scheme for $M_2 + R$ having a coloring on R suitable for $M_1 + R$, so that the initial hypothesis is not correct.

We may generalize the result as follows: *If all the regions M_1 interior to a ring R of six regions are 5-sided, and if there are more than three such regions then R is reducible.* For $M_1 + R$ then contains a configuration of the kind just seen to be reducible. This follows from the fact that the interior arrangement of 5-sided regions is of the simple nature indicated in fig. 1. This suffices to show that R is reducible as stated.

We can show immediately that *a ring R of six regions is reducible with respect to M_1 if M_1 consists of six 5-sided regions about a 6-sided region.* In fact coalesce M_1 and alternate regions of R into a single region, thus forming a possible choice of $\mathfrak{M}_1$ and a scheme $a, *, a, *, a, *$. Taking account of the circular symmetry, we obtain only the cases a, b, a, b, a, b; a, b, a, b, a, c; a, b, a, c, a, d, all of which are suitable colorings for $M_1 + R$.

These are the simplest types of reducible rings of six regions containing more than three regions in M_1.

The first of the above reductions is interesting since it has the consequence that *in a completely reduced map no boundary line can be enclosed by four 5-sided regions.* In fact, four such regions necessarily are surrounded by a ring of six regions in the map if the reductions of § 2, § 3 have been made. If we recall (foot-note, p. 116) that the number of 5-sided regions in a regular map exceeds the number of 7-, 8-, ...-sided regions combined, we see that this reduction applies to a variety of regular maps.

If the ring of six regions is reducible, the reducing number exceeds 3. To prove this statement it is necessary to show that there exist two sets of schemes I and II without a common coloring for R, such that a suitable coloring for every $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of nine or fewer regions exists in both schemes and such that every permutation of either scheme leads to another of the same scheme.

The colorings of R may be taken to be

$$\text{I.}\begin{cases} a,b,a,b,a,b; & a,b,a,b,a,c; & a,b,a,c,a,b; & a,c,a,b,a,b; & a,b,c,a,c,b; \\ c,b,a,b,c,a; & c,a,c,b,a,b; & a,b,c,a,d,b; & d,b,a,b,c,a; & c,a,d,b,a,b; \\ b,c,d,a,b,a; & a,b,a,b,c,d; & a,b,c,a,b,d; & a,d,b,a,c,b; & a,b,c,a,b,c. \end{cases}$$

$$\text{II.}\begin{cases} a,b,a,c,a,d; & b,a,c,a,c,a; & b,a,c,a,b,a; & b,a,b,a,c,a; & b,a,c,a,d,a; \\ a,c,b,a,b,d; & b,d,a,c,b,a; & b,a,b,d,a,c; & c,d,a,b,a,b; & d,a,b,a,b,c; \\ a,b,c,d,a,b; & b,a,b,c,d,a; & a,b,c,d,c,b; & a,c,b,d,c,b. & \end{cases}$$

It is not necessary to specify the lines in the schemes but only to verify that for each coloring in I and II and for some arrangement of the lines all the permutations deduced by the formal principle of § 1 lead to colorings in I and II respectively.

We will not verify this property of I and II, or that colorings in I and II exist suitable for all choices of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of not more than nine regions. It may be assumed that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are not subject to the earlier reductions, and this limits the total set of possibilities for $\mathfrak{M}_1$ and $\mathfrak{M}_2$ to 38.

§ 7. *Rings of n Regions.*

It is possible to find reducible rings of n regions. I shall only consider two of the simplest cases.

A ring R of n regions in a map $M_1 + R + M_2$, where M_1 consists of a ring of 5*-sided regions about a single region, is reducible with respect to M_1.* Let $\alpha_1, \ldots, \alpha_n$ be the n regions of R in cyclical order, and let $\beta_1, \ldots, \beta_n$ be the 5-sided regions of M_1 which abut on (α_1, α_2), (α_2, α_3), $\ldots$, (α_n, α_1) respectively, and let γ be the remaining region of M_1. To form $\mathfrak{M}_1$, let alternate regions $\alpha_1, \alpha_3, \ldots, \alpha_{n-1}$ coalesce with M_1 if n is even, and alternate regions $\alpha_1, \alpha_3, \ldots, \alpha_{n-2}$ with M_1 if n is odd. We thus obtain a coloring in a scheme for $M_2 + R$

$$a, *, a, *, \ldots, a, b, \quad \text{or} \quad a, *, a, \ldots, a, b, c.$$

In either case we can color $M_1 + R$ with such a coloring for R. In the first case if the $*$-colors are all one color b we can color $\beta_1, \ldots, \beta_n$ in c and d

and then γ in a. If this is not so, by taking account of the circular symmetry we may assume that we have

$$a, c, a, *, \ldots, a, b;$$

i. e., we may assume α_2 and α_n are not of the same color. Now color β_1 which abuts on α_1, α_2 in b, β_2 in d, β_3 in a color different from that of α_3, β_2, α_4, and so on, until we reach β_n, which must be colored in a different color than that of α_n, β_{n-1}, α_1 and β_1. But α_n and β_1 are both colored in b, so that this is possible. Having colored $\beta_1, \ldots, \beta_n$ in the three colors different from a, we may color γ in a. In the second case we can employ a similar process as follows. First color β_n in b. If then α_2 is also colored in b, we can color $\beta_{n-1}, \ldots, \beta_1$ and γ as in the first case. Otherwise α_2 is in c or d, and β_1 may be colored in d or c, and β_2 in b. Here again, if α_4 is colored in b, we may color $\beta_{n-1}, \ldots, \beta_3$ as in the first case. Otherwise α_4 is in c or d, and β_3 may be colored in d or c, and β_4 in b. In this way we either color $\beta_1, \ldots, \beta_n$ in b, c, d and γ in a or we come to a coloring

$$c \text{ or } d, b, c \text{ or } d, b, \ldots, b, *, *, b,$$

where the $*$-colors may be taken to be c and d, and γ may be colored in a.

When combined with the reductions of § 2, § 3 of a regular map, this reduction obviously eliminates any region of n sides surrounded by a ring of 5-sided regions, for this region is surrounded by a second ring. The simplest example not previously covered is the case $n = 7$.

A ring R of $4n$ regions in a map $M_1 + R + M_2$, where M_1 consists of a set of $2n$ 6-sided regions about a single region, is reducible with respect to M_1. Let $\alpha_1, \alpha_2, \ldots, \alpha_{4n}$ be the regions of R, and $\beta_1, \ldots, \beta_{2n}$ the 6-sided regions, so that $\alpha_1, \ldots, \alpha_{4n}$ abut on β_1, (β_1, β_2), β_2, (β_2, β_3), $\ldots$, (β_{2n}, β_1), respectively, and let γ be the inner region. To form $\mathfrak{M}_1$ let $\alpha_2, \alpha_4, \ldots, \alpha_{4n}$ coalesce with M_1 and obtain a coloring $*, a, *, a, *, \ldots, a$, for R. The regions β each touch two regions α_{2i} and one $*$-region. If the $*$-regions are all in one color b, we may color the regions β_i in c and d, alternately, and the n-sided region in a or b. If this is not the case, we may assume that we have $c, a, \ldots, b, a$ for $\alpha_1, \alpha_2, \ldots, \alpha_{4n}$ and proceed to color β_1 (in b), $\beta_2, \ldots, \beta_{2n}$, successively, and then γ just as we did in the preceding reduction.

The simplest application is to a 6-sided region surrounded by a ring of six 6-sided regions.

*The Highest Common Factor of a System of Polynomials in One Variable.**

By Lloyd L. Dines.

As is well known, the condition for the existence of a common factor of first or higher degree of two polynomials

$$F_1(x) \equiv c_{10}x^{m_1} + c_{11}x^{m_1-1} + \ldots + c_{1m_1},$$
$$F_2(x) \equiv c_{20}x^{m_2} + c_{21}x^{m_2-1} + \ldots + c_{2m_2}$$

is the vanishing of a certain rational integral function of the coefficients called the *resultant*. This resultant of two polynomials has been expressed in many different forms, and exhaustively studied.†

The object of the present paper is to consider the more general problem: the conditions which characterize the existence of a common factor of first or higher degree of a system

$$F_j(x) \equiv c_{j0}x^{m_j} + c_{j1}x^{m_j-1} + \ldots + c_{jm_j} \quad (j = 1, 2, \ldots, n),$$

consisting of any finite number of polynomials in one variable. This problem has been treated briefly by Meyer,‡ and under certain restrictions by Foethke.§

Meyer showed by a generalization of the Euclidian algorithm, that there always exist $n-1$ rational integral functions of the coefficients whose vanishing constitutes necessary and sufficient conditions for the existence of a common factor of first or higher degree of the polynomials $F_1, F_2, \ldots, F_n$. These functions appear as remainders in a process of repeated division. Foethke, by a careful analysis of the Meyer process, showed the existence of $n-1$ *determinants* such that if certain restrictions be made upon the coefficients of the polynomials $F_1, F_2, \ldots, F_n$, the vanishing of these $n-1$ determinants is equivalent to the vanishing of the $n-1$ remainder functions of Meyer, and therefore constitutes necessary and sufficient conditions.

* Read before the American Mathematical Society, October 28, 1911.

† By Euler, Bezout, Sylvester, Scheibner, Meyer and others. For bibliography, see *Encyclopédie des Sciences Math.*, I, 2, 73.

‡ Franz Meyer: *Jahresbericht der Math. Ver.*, Vol. XVI (1907), p. 16.

§ Foethke: Dissertation, Königsberg (1907), and *Archiv der Math. und Physik*, Vol. XVII (1910), p. 170.

In the present paper (Chapter II), a matrix is constructed whose elements are the coefficients c_{jk} of the given polynomials, arranged according to a simple rule, and which for $n-2$ reduces to the matrix of the ordinary resultant determinant of Euler and Sylvester for two polynomials. This matrix has the following properties:

(1) Its vanishing* constitutes necessary and sufficient conditions for the existence of a common factor of first or higher degree (Theorem I).

(2) Its rank determines the degree of the highest common factor (Theorem III).

(3) The coefficients of the highest common factor are determinants of this matrix, easily characterized (Theorem IV).

In Chapter III it is shown that in any numerical case there are $n-1$ "essential" determinants of this resultant matrix, whose simultaneous vanishing is equivalent to the vanishing of the matrix (Theorem VII). In the case $n=3$, the two $(=n-1)$ "essential" determinants are easily characterized (Theorem V). For *any* value of n, the "essential" determinants are easily characterized whenever there is a pair of the given polynomials whose coefficients satisfy a certain condition (the analytic condition that the degree of the highest common factor of the two be not higher than *one*). (Theorem VI.)

Chapter I contains two simple lemmas concerning matrices, which facilitate the succeeding proofs; and Chapter IV gives as an illustrative example a detailed treatment of the case of three cubics.

CHAPTER I.

A CERTAIN TRANSFORMATION OF MATRICES INTO EQUIVALENT MATRICES.

§ 1. *Equivalent Matrices. The Transformation* T_a.

Two matrices are said to be equivalent if one of them can be obtained from the other by a transformation which does not change the rank of a matrix. The elementary transformations which do not change the rank of a matrix are:

(1) The interchange of two rows (or columns).

(2) The multiplication of all the elements of one row (or column) by the same quantity different from zero.

(3) The addition of the elements of one row (or column) to the corresponding elements of another row (or column).

---

* Throughout this paper a matrix of μ rows and ν columns ($\mu \leqq \nu$) will be said to "vanish" when all the determinants of order μ of the matrix vanish. This definition, given by Pascal ("Die Determinanten," p. 192), though not in as general use as another which is current in the theory of matrices, is more convenient for present purposes.

In the proofs of theorems which follow, considerable use will be made of a transformation which is a repeated product of the elementary transformations (2) and (3), and which we now define.

Definition. If $\alpha_1, \alpha_2, \ldots, \alpha_h$ are any h quantities, and α denotes the sequence $(\alpha_1, \alpha_2, \ldots, \alpha_h)$, then T_α is that transformation which changes a matrix

$$K = \begin{Vmatrix} a_{11} & a_{21} & \ldots & a_{\nu 1} \\ a_{12} & a_{22} & \ldots & a_{\nu 2} \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ a_{1\mu} & a_{2\mu} & \ldots & a_{\nu\mu} \end{Vmatrix} \quad \text{into} \quad K' = \begin{Vmatrix} a'_{11} & a'_{21} & \ldots & a'_{\nu 1} \\ a'_{12} & a'_{22} & \ldots & a'_{\nu 2} \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ a'_{1\mu} & a'_{2\mu} & \ldots & a'_{\nu\mu} \end{Vmatrix}$$

where the rows of K' are found successively from the rows of K by means of the recursion formulas

$$a'_{jk} = a_{jk} - \sum_{i=1}^{h} \alpha_i a'_{jk-i} \quad (j = 1, 2, \ldots, \nu),$$

$a'_{jk-i}\,(i \geqq k)$ being interpreted as zero. The relation existing between K and K' is expressed symbolically by

$$T_\alpha K = K'.$$

Evidently K' can be obtained from K by repetition of the transformations (2) and (3), and we therefore have

Lemma I. *If two matrices M and M' are so related that there exists a sequence of quantities $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_h)$ such that $T_\alpha M = M'$, then the two matrices are equivalent.*

§ 2. *Application of the Transformation T_α to a Particular Type of Matrices.*

In what follows, the transformation T_α is applied to a particular type of matrices, namely those of the form *

$$M = \begin{Vmatrix} c_0 & & & & & \\ c_1 & c_0 & & & & \\ \cdot & c_1 & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & c_0 & \\ \cdot & \cdot & \cdot & \cdot & c_1 & \\ c_m & \cdot & \cdot & \cdot & \cdot & \\ & c_m & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \\ & & & & & c_m \end{Vmatrix} \begin{matrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix} \begin{matrix} \overbrace{\qquad\qquad\qquad\qquad} \\ N - m \text{ columns} \end{matrix} \quad \Big\} N \text{ rows} \tag{1}$$

* In the representation of this matrix, and likewise of others which follow, the portions of the array in which no elements are indicated, either by letters or by dots, contain only zero elements.

in which each column contains as elements the sequence of quantities $c_0, c_1, \ldots, c_m$, and zeros.

Frequent use is made of the following

LEMMA II. *If for any sequence* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_h)$, *the matrix* M *of* (1) *is changed by the transformation* T_α *into*

$$M' = \left\| \begin{array}{cccccc} c'_0 & & & & & \\ c'_1 & c'_0 & & & & \\ . & c'_1 & . & & & \\ . & . & . & . & & \\ . & . & . & . & c'_0 & \\ . & . & . & . & c'_1 & \\ . & . & . & . & . & \\ . & . & . & . & . & \\ c'_{N-h} & . & . & . & c'_{m-h+1} & \\ . & . & . & . & . & \\ c'_{N-1} & c'_{N-2} & . & . & c'_m & \end{array} \right\| \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} N \text{ rows}$$

$N - m$ columns

then the matrix consisting of the last h *rows of* M' *vanishes, if the determinant*

$$\begin{vmatrix} c'_m & c'_{m-1} & . & c'_{m-h+1} \\ c'_{m+1} & c'_m & . & . \\ . & . & . & . \\ c'_{m+h-1} & . & . & c'_m \end{vmatrix} \tag{2}$$

consisting of the elements common to the last h *rows and last* h *columns of* M' *vanishes.*

To prove this lemma, we note that by the definition of the transformation T_α we have the relations

$$c'_k = c_k - \sum_{i=1}^{h} \alpha_i c'_{k-i} \qquad (k = 0, 1, \ldots, N-1), \tag{3}$$

where the notations c_k $(k > m)$ and c'_{k-i} $(i > k)$ are interpreted as zero.

Now since the elements of any one of the last h rows of M' constitute some segment of the sequence

$$c'_{N-1}, c'_{N-2}, c'_{N-3}, \ldots, c'_{m-h+1},$$

it is evident from (3) that if to one of the first $N - m - h$ elements, say the

p-th element, of such a row, there be added the $(p+1)$-th, $(p+2)$-th, ..., $(p+h)$-th elements multiplied respectively by $\alpha_1, \alpha_2, \ldots, \alpha_h$, the result is the corresponding element of the matrix M. Therefore, by processes which do not change the rank of a matrix, the matrix consisting of the last h rows of M' can be transformed into one in which the elements of the last h columns remain unchanged, while the elements of the other columns are the corresponding elements of the matrix consisting of the last h rows of M. But these elements of M are all zero, and from this it follows that the matrix consisting of the last h rows of M' vanishes if the determinant (2) vanishes.

CHAPTER II.

The Resultant Matrix.

§ 3. *Conditions for the Existence of a Common Factor.*

Given the system of polynomials

$$F_j(x) = c_{j0}x^{m_j} + c_{j1}x^{m_j-1} + \ldots + c_{jm_j} \quad (j = 1, 2, \ldots, n), \qquad (4)$$

where the coefficients c_{jk} are independent of x, and the leading coefficients c_{j0} are all different from zero. It will further be supposed that the polynomials F_j are arranged according to their degrees, so that $m_i \geqq m_{i-1}$.

Of fundamental importance in the theory of the highest common factor of the polynomials (4) is a matrix constructed as follows:

$$M_N(F_1 \ldots F_n) = \left\| \begin{array}{cccccccccccc}
\overbrace{\quad}^{N-m_1\text{ columns}} & & & & \overbrace{\quad}^{N-m_j\text{ columns}} & & & & \overbrace{\quad}^{N-m_n\text{ columns}} & & & \\
c_{10} & & & \ldots & c_{j0} & & & \ldots & c_{n0} & & & \\
c_{11} & c_{10} & & \ldots & c_{j1} & c_{j0} & & \ldots & c_{n1} & c_{n0} & & \\
\cdot & c_{11} & \cdot & \ldots & \cdot & c_{j1} & \cdot & \ldots & \cdot & c_{n1} & \cdot & \\
\cdot & \cdot & \cdot & \ldots & \cdot & \cdot & \cdot & \ldots & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & c_{10} \ldots & \cdot & \cdot & \cdot & c_{j0} \ldots & \cdot & \cdot & \cdot & c_{n0} \\
c_{1m_1} & \cdot & \cdot & c_{11} \ldots & c_{jm_j} & \cdot & \cdot & c_{j1} \ldots & c_{nm_n} & \cdot & \cdot & c_{n1} \\
 & c_{1m_1} & \cdot & \cdot \ldots & & c_{jm_j} & \cdot & \cdot \ldots & & c_{nm_n} & \cdot & \cdot \\
 & & \cdot & \cdot \ldots & & & \cdot & \cdot \ldots & & & \cdot & \cdot \\
 & & & \cdot \ldots & & & & \cdot \ldots & & & & \cdot \\
 & & & c_{1m_1} \ldots & & & & c_{jm_j} \ldots & & & & c_{nm_n}
\end{array} \right\| \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} N \text{ rows}$$

where N is any integer satisfying the conditions

$$N \geqq m_i + m_j \quad (i \neq j). \qquad (5)$$

The matrix $M_N(F_1 \ldots F_n)$ has N rows and $\sum_{j=1}^{n}(N - m_j)$ columns, the latter being arranged in n groups, each group containing elements which are the

coefficients of one polynomial F_j. The group corresponding to the polynomial F_j has $N - m_j$ columns and will be denoted by G_j.

There are evidently an infinite number of such matrices corresponding to the infinite number of possible values of N, the matrix of lowest possible order being given by $N = m_1 + m_2$. Any matrix $M_N(F_1 \ldots . F_n)$ constructed in the manner indicated is called a *resultant matrix* of the system (4), The theory which follows is developed for the *general* resultant matrix $M_N(F_1 \ldots . F_n)$, N being any integer satisfying the relations (5).

THEOREM I. *The vanishing* of the resultant matrix $M_N(F_1 \ldots . F_n)$ is necessary and sufficient for the existence of a common factor of first or higher degree of the polynomials F_j.*

The necessity of the condition may be seen as follows: If the polynomials F_j have a common factor containing x, then the n equations $F_j = 0$ $(j = 1, 2, \ldots, n)$ have at least one common root, and therefore the system of equations

$$\left.\begin{aligned}
x^{N-m_j-1} \cdot F_j &\equiv c_{j0}x^{N-1} + c_{j1}x^{N-2} + \ldots + c_{jm_j}x^{N-m_j-1} &&= 0,\\
x^{N-m_j-2} \cdot F_j &\equiv c_{j0}x^{N-2} + c_{j1}x^{N-3} + \ldots + c_{jm_j}x^{N-m_j-2} &&= 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &&\ldots,\\
x \cdot F_j &\equiv c_{j0}x^{m_j+1} + c_{j1}x^{m_j} + \ldots + c_{j0}x &&= 0,\\
F_j &\equiv c_{j0}x^{m_j} + c_{j1}x^{m_j-1} + \ldots + c_{jm_j} &&= 0\\
&(j = 1, 2, \ldots, n),
\end{aligned}\right\} \quad (6)$$

considered as a system of linear homogeneous equations in the quantities $x^{N-1}, x^{N-2}, \ldots, x, 1$, has a solution. From the theory of linear equations it follows that the matrix of coefficients of this system of equations must vanish, and that is exactly the matrix $M_N(F_1 \ldots . F_n)$.

To prove the sufficiency of the condition, we employ a device used by Kronecker in his general theory of elimination. If u_k $(k = 1, 3, 4, \ldots, n)$ and u'_l $(l = 2, 3, \ldots, n)$ are two sets of arbitrary parameters, then the two polynomials

$$\sum_k F_k u_k, \quad \sum_l F_l u'_l$$

have a common factor containing x, only if the polynomials F_j have such a common factor. But, as is well known, these two polynomials have a common factor containing x if their resultant determinant vanishes, and therefore the vanishing of this determinant forms a sufficient condition for the existence of a common factor of the polynomials F_j. Since the polynomials $\sum_k F_k u_k$ and $\sum_l F_l u'_l$ are respectively of degrees m_1 and m_2 in x, their resultant determinant

* See foot-note, p. 130, concerning definition of term "vanish."

is of order $m_1 + m_2$. It is furthermore a rational integral function of the parameters u_k and u'_l, and can therefore be expanded in the form

$$\rho_0 U_0 + \rho_1 U_1 + \ldots\ldots,$$

where $U_0, U_1, \ldots\ldots$ are different power products of the parameters u_k and u'_l; and $\rho_0, \rho_1, \ldots\ldots$ are functions of the coefficients of the polynomials F_j. The polynomials F_j have therefore a common factor containing x if the functions $\rho_0, \rho_1, \ldots\ldots$ all vanish.

Now if the resultant determinant of the two polynomials $\sum_k F_k u_k$ and $\sum_l F_l u'_l$ be constructed, it is seen that the functions $\rho_0, \rho_1, \ldots\ldots$ are linear combinations of determinants of order $m_1 + m_2$ of a matrix

$$\begin{array}{c}
\overbrace{}^{m_2 \text{ columns}} \quad \overbrace{}^{m_1 \text{ columns}} \quad \overbrace{}^{m_1 \text{ columns}} \\
\left\|\begin{array}{ccccccccccccccc}
c_{10} & & & & \ldots\ldots & & & & \ldots\ldots & & & & \\
c_{11} & c_{10} & & & \ldots\ldots & c_{j0} & & & \ldots\ldots & & & & \\
\cdot & c_{11} & \cdot & & \ldots\ldots & c_{j1} & c_{j0} & & \ldots\ldots & c_{n0} & & & \\
\cdot & \cdot & \cdot & \cdot & \ldots\ldots & \cdot & c_{j1} & \cdot & \ldots\ldots & c_{n1} & c_{n0} & & \\
\cdot & \cdot & \cdot & c_{10} & \ldots\ldots & \cdot & \cdot & \cdot & \ldots\ldots & \cdot & c_{n1} & \cdot & \\
\cdot & \cdot & \cdot & c_{11} & \ldots\ldots & \cdot & \cdot & \cdot & c_{j0} \ldots\ldots & \cdot & \cdot & \cdot & \\
c_{1m_1} & \cdot & \cdot & \cdot & \ldots\ldots & c_{jm_j} & \cdot & \cdot & c_{j1} \ldots\ldots & c_{nm_n} & \cdot & \cdot & c_{n0} \\
 & c_{1m_1} & \cdot & \cdot & \ldots\ldots & & c_{jm_j} & \cdot & \cdot \ldots\ldots & & c_{nm_n} & \cdot & c_{n1} \\
 & & \cdot & \cdot & \ldots\ldots & & & \cdot & \cdot \ldots\ldots & & & \cdot & \cdot \\
 & & & \cdot & \ldots\ldots & & & \cdot & \cdot \ldots\ldots & & & \cdot & \cdot \\
 & & & c_{1m_1} & \ldots\ldots & & & c_{jm_j} & \ldots\ldots & & & c_{nm_n} &
\end{array}\right\| \quad m_1 + m_2 \text{ rows}
\end{array} \tag{7}$$

Therefore if the matrix (7) vanishes, the functions $\rho_0, \rho_1, \ldots\ldots$ are all equal to zero.

But the matrix (7) vanishes whenever the matrix $M_N(F_1 \ldots F_n)$ vanishes. For, corresponding to every determinant Δ of order $m_1 + m_2$ in (7), there is a determinant of order N in $M_N(F_1 \ldots F_n)$ which is equal to $c_{10}^{N-(m_1+m_2)} \cdot \Delta$, and since c_{10} is different from zero, this determinant can not vanish unless Δ does. Therefore if the matrix $M_N(F_1 \ldots F_n)$ vanishes, the functions $\rho_0, \rho_1, \ldots\ldots$ are all equal to zero, and the polynomials F_j have a common factor containing x.

§ 4. *Sub-resultant Matrices. Degree of the Highest Common Factor.*

It has been shown in the preceding section that the vanishing of the resultant matrix $M_N(F_1 \ldots F_n)$ is necessary and sufficient for the existence of a common factor of first or higher degree of the polynomials (4). In the

present section it will be shown that the exact degree of the highest common factor can be stated in terms of the rank of the resultant matrix, the final result being reached in Theorem III. To this end we define a certain sequence of matrices called *sub-resultant matrices* and denoted by

$$M_N^{(i)}(F_1 \ldots . F_n) \quad (i = 1, 2, \ldots, m_n).$$

DEFINITION. *$M_N^{(i)}(F_1 \ldots . F_n)$ is the matrix formed from $M_N(F_1 \ldots . F_n)$ by striking out the first i rows and the last i rows, and the first i columns of each group G_j.* It has therefore $N - 2i$ rows and $\sum_{j=1}^{n} (N - m_j - i)$ columns.

THEOREM II. *The highest common factor of the polynomials* (4) *is of degree k ($\leqq m_n$), if and only if*

$$M_N(F_1 \ldots . F_n) = 0; \quad M_N^{(i)}(F_1 \ldots . F_n) = 0 \quad (i = 1, 2, \ldots, k-1), \tag{8}$$

while

$$M_N^{(k)}(F_1 \ldots . F_n) \neq 0. \tag{9}$$

The proof of this theorem is accomplished by means of the following:

LEMMA III. *If the polynomials F_j have a common factor*

$$D^{(h)}(x) \equiv x^h + \alpha_1 x^{h-1} + \alpha_2 x^{h-2} + \ldots . + \alpha_h,$$

and if

$$F_j \equiv D^{(h)} \cdot F_j^{(h)} \quad (j = 1, 2, \ldots, n), \tag{10}$$

then:

(1) *The matrix $T_\alpha M_N(F_1 \ldots . F_n)$, where α is the sequence $(\alpha_1, \alpha_2, \ldots, \alpha_h)$ of coefficients of $D^{(h)}$, has only zero elements in the last h rows.*

(2) *The sub-resultant matrix $M_N^{(h)}(F_1 \ldots . F_n)$ is equivalent to the resultant matrix $M_{N-2h}(F_1^{(h)} \ldots . F_n^{(h)})$ of the quotient polynomials $F_j^{(h)}$.*

For the proof of the lemma, let any one of the polynomials F_j be denoted temporarily by

$$F \equiv c_0 x^m + c_1 x^{m-1} + \ldots . + c_m,$$

and the corresponding quotient polynomial $F/D^{(h)}$ by

$$F^{(h)} \equiv c_0' x^{m-h} + c_1' x^{m-h-1} + \ldots . + c_{m-h}'.$$

Then the relations existing between the coefficients of F, $D^{(h)}$ and $F^{(h)}$, on account of (10), are

$$c_k' = c_k - \sum_{i=1}^{h} \alpha_i c_{k-i}' \quad (k = 0, 1, \ldots, m), \tag{11}$$

where the notations c_k' $(k > m - h)$ and c_{k-i}' $(i > k)$ are to be interpreted as zero.

From these relations and the definition of the transformation T_a it follows that the matrix

$$\begin{array}{c} \overbrace{\qquad\qquad}^{N-m \text{ columns}} \\ \left\| \begin{array}{ccccc} c_0 & & & & \\ c_1 & c_0 & & & \\ \cdot & c_1 & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & c_0 \\ \cdot & \cdot & \cdot & \cdot & c_1 \\ c_m & \cdot & \cdot & \cdot & \cdot \\ & c_m & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \\ & & & & c_m \end{array} \right\| \end{array} \quad \left. \vphantom{\begin{array}{c} a \\ a \\ a \\ a \\ a \\ a \\ a \\ a \\ a \\ a \\ a \end{array}} \right\} N \text{ rows} \tag{12}$$

formed from the columns of $M_N(F_1 \ldots . F_n)$ which correspond to the polynomial F, is transformed by the transformation T_a into

$$\begin{array}{c} \overbrace{\qquad\qquad}^{N-m \text{ columns}} \\ \left\| \begin{array}{ccccc} c'_0 & & & & \\ c'_1 & c'_0 & & & \\ \cdot & c'_1 & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \\ c'_{m-h} & \cdot & \cdot & \cdot & c'_0 \\ & c'_{m-h} & \cdot & \cdot & c'_1 \\ & & & \cdot & \cdot \\ & & & \cdot & \cdot \\ & & & & c'_{m-h} \\ & & & & \\ & & & & \end{array} \right\| \end{array} \quad \begin{array}{l} \left. \vphantom{\begin{array}{c} a \\ a \\ a \\ a \\ a \\ a \\ a \\ a \\ a \end{array}} \right\} N-h \text{ rows} \\ \left. \vphantom{\begin{array}{c} a \\ a \end{array}} \right\} h \text{ rows} \end{array} \tag{13}$$

in which the last h rows contain only zero elements. Since, then, every matrix formed from a group G_j of columns of $M_N(F_1 \ldots . F_n)$ is transformed by T_a into a matrix which contains only zero elements in the last h rows, the same is true of the matrix $M_N(F_1 \ldots . F_n)$ itself, and this proves the first part of the lemma.

Now, the group of columns in $M_N^{(h)}(F_1 \ldots . F_n)$ which corresponds to the polynomial F is, by definition, obtained from (12) by crossing out the first and last h rows and the first h columns. The matrix formed from this group of columns is transformed by T_a into the matrix obtained by striking out the first and last h rows and the first h columns of (13). But it is evident that this

latter matrix consists of exactly those columns of $M_{N-2h}(F_1^{(h)}\ldots.F_n^{(h)})$ which correspond to the polynomial $F^{(h)}$. From this follows at once

$$T_a M_N^{(h)}(F_1\ldots.F_n) = M_{N-2h}(F_1^{(h)}\ldots.F_n^{(h)}),$$

and the second conclusion of the lemma.

We now prove Theorem II.

(a) The conditions of the theorem are necessary. For, the existence of a common factor $D^{(k)}$ of degree k of the polynomials F_j necessitates the existence of common factors of degrees $1, 2, \ldots, k-1$. If these be denoted respectively by $D^{(i)}$ $(i = 1, 2, \ldots, k-1)$, and if

$$F_j \equiv D^{(i)} F_j^{(i)} \quad (j = 1, 2, \ldots, n),$$

then each of the sets of polynomials $F_j^{(i)}$ (for a fixed value of i) possesses a common factor of first or higher degree. Hence, by Theorem I, each of the resultant matrices $M_{N-2i}(F_1^{(i)}\ldots.F_n^{(i)})$ vanishes; and by Lemma III the same is true of the matrices $M_N^{(i)}(F_1\ldots.F_n)$ $(i = 1, 2, \ldots, k-1)$.

Moreover, $M_N^{(k)}(F_1\ldots.F_n)$ does not vanish. For if it did, the same would be true of the equivalent matrix $M_{N-2k}(F_1^{(k)}\ldots.F_n^{(k)})$, and therefore the polynomials $F_j^{(k)}$ would have a common factor. This would require that the highest common factor of the polynomials F_j be of degree greater than k, which is contrary to our assumption.

(b) The conditions of the theorem are sufficient. For, these conditions being satisfied, the polynomials F_j have, by Theorem I, a common factor of at least the first degree. They have, moreover, a common factor of degree k. For suppose the highest common factor were of degree i $(< k)$. Let it be denoted by $D^{(i)}$, and let

$$F_j = D^{(i)} F_j^{(i)} \quad (j = 1, 2, \ldots, n).$$

Then, by Lemma III, the resultant matrix $M_{N-2i}(F_1^{(i)}\ldots.F_n^{(i)})$ is equivalent to $M_N^{(i)}(F_1\ldots.F_n)$, which vanishes by hypothesis, and therefore the polynomials $F_j^{(i)}$ have a common factor of at least the first degree, which contradicts the supposition that $D^{(i)}$ was the highest common factor of the polynomials F_j. The polynomials F_j have therefore a common factor of degree k.

This factor of degree k is the highest common factor. For if

$$F_j \equiv D^{(k)} F_j^{(k)} \quad (j = 1, 2, \ldots, n),$$

then $M_{N-2k}(F_1^{(k)}\ldots.F_n^{(k)})$ is equivalent to $M_N^{(k)}(F_1\ldots.F_n)$, which does not vanish. Therefore the polynomials $F_j^{(k)}$ have no common factor containing x, and the highest common factor of the polynomials F_j is of degree k.

THEOREM III. *The highest common factor of the polynomials* $F_1, F_2, \ldots, F_n$ *is of degree* k, *where* $N-k$ *is the rank of the resultant matrix* $M_N(F_1 \ldots F_n)$. *The rank can not be smaller than* $N-m_n$, *where* m_n *is the smallest of* $m_1, m_2, \ldots, m_n$.

The last part of the theorem follows at once from the fact that the determinant of order $N-m_n$ formed from the elements common to the first $N-m_n$ rows and the last $N-m_n$ columns of the matrix is equal to $c_{n0}^{N-m_m}$, and is therefore different from zero.

For the proof of the first part of the theorem, it is sufficient, on account of Theorem II, to show that conditions (8) and (9) are equivalent to the conditions that the matrix $M_N(F_1 \ldots F_n)$ be of rank $N-k$.

First, if conditions (8) and (9) be satisfied, then the matrix $M_N(F_1 \ldots F_n)$ is of rank $N-k$.

It can not be of rank greater than $N-k$. For under the assumption of conditions (8), the polynomials F_j have a common factor of degree k, and the matrix $M_N(F_1 \ldots F_n)$ can, by Lemma III, be transformed into a matrix $M'_N(F_1 \ldots F_n)$ of the same rank, and such that all elements of the last k rows are zero. Since, then, $M'_N(F_1 \ldots F_n)$ is obviously of rank not greater than $N-k$, the same is true of $M_N(F_1 \ldots F_n)$.

To prove that the rank of $M_N(F_1 \ldots F_n)$ is exactly $N-k$, it suffices to note that, by condition (9), there is at least one determinant, call it K, of order $N-2k$ in $M_N^{(k)}(F_1 \ldots F_n)$, which is different from zero. Therefore the determinant of order $N-k$ formed by annexing to K those elements of the first k rows of $M_N(F_1 \ldots F_n)$ which lie in the same columns with K (all of which elements are zero), and those elements of the first k columns of $M_N(F_1 \ldots F_n)$ which lie in the first $N-k$ rows, does not vanish, for this determinant is equal to $c_{10}^k K$.

Conversely, if the matrix $M_N(F_1 \ldots F_n)$ is of rank $N-k$, then the conditions (8) and (9) are satisfied.

To prove by the indirect method that (8) is satisfied, suppose for some positive integer h, less than k, there were a determinant H of order $N-2h$ in $M_N^{(h)}(F_1 \ldots F_n)$ which did not vanish. Then the determinant of order $N-h$ formed by annexing to H those elements from the first h rows of $M_N(F_1 \ldots F_n)$ which lie in the same columns with H (which elements are all zero), and those elements of the first h columns of $M_N(F_1 \ldots F_n)$ which lie in the first $N-h$ rows, would equal $c_{10}^h H$ and therefore would not vanish. This would contradict the hypothesis that $M_N(F_1 \ldots F_n)$ is of rank k, and the contradiction proves that (8) must be satisfied.

To prove that (9) is satisfied, suppose that $M_N^{(k)}(F_1 \ldots . F_n)$ vanished. Then the polynomials F_j would have a common factor of degree $k+1$, and the matrix $M_N(F_1 \ldots . F_n)$ could, by Lemma III, be transformed into an equivalent matrix in which all elements of the last $k+1$ rows were zero. Since this latter matrix could obviously not be of rank $N-k$, the matrix $M_N(F_1 \ldots . F_n)$ could not be of that rank, as it is by hypothesis. This completes the proof of the theorem.

§ 5. *Determination of the Highest Common Factor.*

Suppose now the rank of the matrix $M_N(F_1 \ldots . F_n)$ is $N-k$, that is, the polynomials F_j have a common factor of degree k; then, as has been seen, there is a determinant of order $N-2k$ in the sub-resultant matrix $M_N^{(k)}(F_1 \ldots . F_n)$ which does not vanish.* Let this determinant be denoted by Δ_0.

In the $N-2k$ columns of $M_N(F_1 \ldots . F_n)$ which contain the columns of Δ_0, the elements of the first k rows are all zero. The matrix formed from these $N-2k$ columns by striking out the first k rows of zero elements will be denoted by M, and is a matrix of $N-k$ rows and $N-2k$ columns. According to our definition, the determinant formed from the first $N-2k$ rows of M is Δ_0. In general, the determinant whose first $N-2k-1$ rows are the first $N-2k-1$ rows of M, and whose last row is the $(N-2k+i)$-th row of M, will be denoted by Δ_i. There are $k+1$ such determinants, namely, Δ_0, Δ_1, $\ldots .$, Δ_k, and in terms of them we can state

THEOREM IV. *The highest common factor of the polynomials F_j is*

$$\Delta_0 x^k + \Delta_1 x^{k-1} + \ldots . + \Delta_k . \tag{D}$$

Since the matrix $M_N(F_1 \ldots . F_n)$ is the matrix of coefficients of the system of equations (6), it is evident that M is the matrix of coefficients of a system of equations

$$x^{i_p} \cdot F_{j_p} = 0 \quad (p = 1, 2, \ldots . , N-2k). \tag{14}$$

Now, let the co-factors of the elements of the last row of Δ_0 be denoted by $C_1, C_2, \ldots . , C_{N-2k}$; C_p being the co-factor of that element which is a coefficient in the p-th equation of (14). If the left sides of equations (14) be multiplied respectively by the corresponding co-factors C_p, and the results be added, we obtain the identity

$$\sum_{p=1}^{N-2k} C_p x^{i_p} \cdot F_{j_p} \equiv \Delta_0 x^k + \Delta_1 x^{k-1} + \ldots . + \Delta_k \equiv D, \tag{15}$$

the coefficients of the higher powers of x on the left-hand side being zero in

* Of course it is entirely possible that there be several determinants of this character. The important point is that there is one such.

consequence of the well-known theorem that the sum of the products of the elements of one row of a determinant by the co-factors of the corresponding elements of another row is zero.

From the identity (15), it follows that every common factor of the polynomials $F_{j_p}(p = 1, 2, \ldots, N - 2k)$ is a factor of the polynomial D, and therefore the highest common factor of the polynomials $F_j(j = 1, 2, \ldots, n)$, which is known to be of degree k, must be a factor of D. Therefore D must be the highest common factor.

CHAPTER III.

Resultant Determinants.

§ 6. *Definitions and Notations.*

In Chapter II it was shown that a necessary and sufficient condition for the existence of a common factor of first or higher degree of the polynomials (4) is the vanishing of the matrix $M_N(F_1 \ldots . F_n)$. This is not in general a single condition, but many, since it means the vanishing of all the determinants of order N in the matrix $M_N(F_1 \ldots . F_n)$. These determinants are not, however, independent, and from the known result of Meyer that there are $n-1$ rational integral functions of the coefficients whose vanishing constitutes necessary and sufficient conditions for the existence of a common factor, one is led to suspect that there are $n-1$ particular determinants of the matrix whose vanishing is equivalent to the vanishing of the matrix. This seems not to be strictly true, in that it seems impossible to characterize formally and generally* $n-1$ determinants of the matrix whose vanishing is equivalent to the vanishing of the matrix.

However, it will be shown in the present chapter that in every *numerical* case there are $n-1$ determinants of the matrix whose vanishing is equivalent to the vanishing of the matrix. As a rule these determinants have to be constructed successively, the form of each after the first depending upon the ranks of those which precede. However, in certain important cases, the $n-1$ determinants are easily characterized.

In order not to disturb the continuity of the discussion later, we introduce here some definitions and notations.

$M_N(F_r F_s)$. First we note that the columns of the groups G_r and G_s

* Cf. the restrictions upon the coefficients required in Foethke's conditions, *loc. cit.*

(where r and s are any two different numbers of the set $1, 2, \ldots, n$) of the matrix $M_N(F_1 \ldots F_n)$ constitute the matrix $M_N(F_r F_s)$.

$R(F_r F_s)$. The ordinary resultant determinant (as formed by Euler and Sylvester) of the two polynomials F_r and F_s will be denoted by $R(F_r F_s)$. It is of order $m_r + m_s$ and is obtained from $M_N(F_r F_s)$ by striking out the first $N - (m_r + m_s)$ rows, the first $N - (m_r + m_s)$ columns of G_r, and the first $N - (m_r + m_s)$ columns of G_s.

$R^{(i)}(F_r F_s)$. The sub-resultant determinant $R^{(i)}(F_r F_s)$ is obtained from $R(F_r F_s)$ by striking out the first i rows, the last i rows, the first i columns containing coefficients of F_r, and the first i columns containing coefficients of F_s. It is of order $m_r + m_s - 2i$.

$P_N^{(i)}(F_r F_s)$. A determinant of order $N - i$ which is equal to

$$c_{r0}^{N-(m_r+m_s)+i} \cdot R^{(i)}(F_r F_s)$$

can evidently be formed by bordering the determinant $R^{(i)}(F_r F_s)$ on the top by the zero elements which lie above it in the matrix $M_N(F_1 \ldots F_n)$ and at the left by the first $N - (m_r + m_s) + i$ columns of the group G_r. This determinant is denoted by $P_N^{(i)}(F_r F_s)$. It vanishes if and only if $R^{(i)}(F_r F_s)$ vanishes.

$P_N^{(i)}(F_r F_s / F_j)$. The determinant of order N formed by annexing to $P_N^{(i)}(F_r F_s)$ those elements of the last i rows of $M_N(F_r F_s)$ which lie in the same columns with $P_N^{(i)}(F_r F_s)$, and to the matrix so formed annexing the last i columns of the group G_j (where j is any number of the set $1, 2, \ldots, n$, different from r and s), is denoted by $P_N^{(i)}(F_r F_s / F_j)$.

$R^{(i)}(F_r F_s / F_j)$. The determinant $P_N^{(i)}(F_r F_s / F_j)$ just defined is, from its form, equal to a product of a certain power of c_{r0} into another determinant of lower order. This latter determinant is denoted by $R^{(i)}(F_r F_s / F_j)$. Since it plays an important rôle, we give the following more direct method of constructing* it.

To the determinant $R^{(i)}(F_r F_s)$ annex those elements of the last i rows of $M_N(F_r F_s)$ which lie in the same columns with $R^{(i)}(F_r F_s)$. The resulting matrix has $m_r + m_s - i$ rows and $m_r + m_s - 2i$ columns. To the right of this matrix annex i columns consisting of those elements in the last i columns and the last $m_j + i$ rows of the group G_j. If $m_j + i = m_r + m_s - i$, the result is a square matrix of order $m_r + m_s - i$, and the corresponding determinant is $R^{(i)}(F_r F_s / F_j)$. If $m_j + i < m_r + m_s - i$, fill in the upper right-hand corner

* The determinants $R^{(1)}(F_1 F_2 / F_3)$ and $R^{(2)}(F_1 F_2 / F_3)$, where F_1, F_2 and F_3 are cubics, are displayed in Chapter IV, and will serve as illustrations of the constructions here described.

with zeros until a square matrix of order $m_r + m_s - i$ is obtained. The corresponding determinant is $R^{(i)}(F_r F_s/F_j)$. If $m_j + i > m_r + m_s - i$, prefix those elements of G_r which lie in the $m_j + i - (m_r + m_s - i)$ columns immediately preceding $R^{(i)}(F_r F_s)$ and in the last $m_j + i$ rows. By filling in above $R^{(i)}(F_r F_s)$ with zeros, a square matrix of order $m_j + i$ is then obtained, whose corresponding determinant is $R^{(i)}(F_r F_s/F_j)$.

§ 7. *The Case* $n = 2$.

If the number of polynomials considered is two, and if N be taken equal to $m_1 + m_2$, then $M_N(F_1 F_2)$ is the matrix of the resultant determinant $R(F_1 F_2)$, and the sub-resultant matrices $M_N^{(i)}(F_1 F_2)$ are respectively the matrices of the determinants $R^{(i)}(F_1 F_2)$. We have, therefore, as special cases of Theorems II and III, the two well-known theorems:

*The highest common factor of two polynomials F_1 and F_2 is of degree k, if and only if $R(F_1 F_2)$ and $R^{(i)}(F_1 F_2)$ $(i = 1, 2, \ldots, k-1)$ all vanish, while $R^{(k)}(F_1 F_2)$ does not vanish.**

The highest common factor of two polynomials F_1 and F_2 is of degree k, where $N - k$ is the rank of the resultant determinant $R(F_1 F_2)$.†

§ 8. *The Case* $n = 3$.

Regarding three polynomials

$$\left.\begin{aligned} F_1(x) &\equiv a_0 x^\lambda + a_1 x^{\lambda-1} + \ldots + a_\lambda, \\ F_2(x) &\equiv b_0 x^\mu + b_1 x^{\mu-1} + \ldots + b_\mu, \\ F_3(x) &\equiv c_0 x^\nu + c_1 x^{\nu-1} + \ldots + c_\nu, \end{aligned}\right\} \quad (16)$$

we have the following

THEOREM V. *If F_r and F_s are any two of the three polynomials* (16), *and if $R^{(i)}(F_r F_s) = 0$ $(i = 1, 2, \ldots, k-1)$, but $R^{(k)}(F_r F_s) \neq 0$, then necessary and sufficient conditions for the vanishing of the matrix $M_N(F_1 F_2 F_3)$, and therefore for the existence of a common factor of first or higher degree, are*

$$R(F_r F_s) = 0, \quad R^{(k)}(F_r F_s/F_j) = 0 \quad (j \neq r, s). \quad (17)$$

That the conditions are necessary is evident since the determinants $R(F_r F_s)$ and $R^{(k)}(F_r F_s/F_j)$ are, when multiplied by certain powers of a_0, equal to determinants of order N in $M_N(F_1 F_2 F_3)$.

Assuming now that $R(F_r F_s) = 0$, it follows from the hypothesis of the theorem that the highest common factor of the polynomials F_r and F_s is of degree k. Therefore by Lemma III of Chapter II, the matrix $M_N(F_1 F_2 F_3)$ can be transformed by a transformation T_α into an equivalent matrix $M'_N(F_1 F_2 F_3)$

* Cf. Scheibner: "Mathematische Bemerkungen," *Leipz. Ber.*, 40, pp. 1–13.

† Cf. Capelli: *Palermo Rend.*, 23, pp. 130–136.

in which the last k rows of $M'_N(F_r F_s)$ contain only zero elements. By this same transformation the determinant $P_N^{(k)}(F_r F_s)$, which is equal to

$$a_0^{N-(m_r+m_s-k)} \cdot R^{(k)}(F_r F_s)$$

and is therefore different from zero, is changed into a determinant $P_N^{(k)\prime}(F_r F_s)$ which does not vanish. If, for the sake of definiteness in writing, we suppose $r=1$, $s=2$, then $M'_N(F_1 F_2 F_3)$ will be of the form

$$M'_N(F_1 F_2 F_3) = \left\| \begin{array}{ccccccccccccc} \overbrace{\quad}^{N-\lambda \text{ columns}} & & & \overbrace{\quad}^{N-\mu \text{ columns}} & & & \overbrace{\quad}^{N-\nu \text{ columns}} & & & & & & \\ a'_0 & & & b'_0 & & & c'_0 & & & & & & \\ a'_1 & a'_0 & & b'_1 & b'_0 & & c'_1 & c'_0 & & & & & \\ . & a'_1 & . & . & b'_1 & . & . & c'_1 & . & & & & \\ . & . & . . & . & . & . . & . & . & . . & & & & \\ a'_{\lambda-k} & . & . . a'_0 & b'_{\mu-k} & . & . . b'_0 & . & . & . . c'_0 & & & & \\ & a'_{\lambda-k} & . . a'_1 & & b'_{\mu-k} & . . b'_1 & . & . & . . c'_1 & & & & \\ & & . . . & & & . . . & . & . & . . . & & & & \\ & & . . & & & . . & . & . & . . . & & & & \\ & & a'_{\lambda-k} & & & b'_{\mu-k} & . & . & . . . & & & & \\ & & & & & & c'_{N-k} & c'_{N-k-1} & . . c'_{\nu-k+1} & & & & \\ & & & & & & . & . & . . . & & & & \\ & & & & & & . & . & . . . & & & & \\ & & & & & & c'_{N-1} & c'_{N-2} & . . c'_\nu & & & & \end{array} \right\| \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}

Now, the determinant of order N of $M'_N(F_1F_2F_3)$ formed from the $N-k$ columns containing $P_N^{(k)'}(F_r\ F_s)$ and the k columns containing D'_k is equal to $D'_k.P_N^{(k)'}(F_rF_s)$, and therefore its vanishing necessitates the vanishing of D'_k and therefore the vanishing of the matrix $M'_N(F_1F_2F_3)$. But this determinant is exactly the transform of $P_N^{(k)}(F_rF_s/F_j)$ by T_a. Hence, since this transformation does not change the rank of a matrix, the vanishing of $P_N^{(k)}(F_rF_s/F_j)$ necessitates the vanishing of $P_N^{(k)'}(F_rF_s/F_j)$, therefore of the matrix $M'_N(F_1F_2F_3)$ and finally of the matrix $M_N(F_1F_2F_3)$. But by definition, the vanishing of $R^{(k)}(F_rF_s/F_j)$ is equivalent to the vanishing of $P_N^{(k)}(F_rF_s/F_j)$. Therefore the condition $R^{(k)}(F_rF_s/F_j)=0$, together with the assumed condition $R(F_rF_s)=0$, is equivalent to the vanishing of the matrix $M_N(F_1F_2F_3)$.

§ 9. *The Case in which a Sub-resultant Determinant $R^{(1)}(F_rF_s)$ is Different from Zero.*

THEOREM VI. *If F_r and F_s are any two of the polynomials* (4), *and if $R^{(1)}(F_rF_s)\neq 0$, then necessary and sufficient conditions for the vanishing of the matrix $M_N(F_1\ldots.F_n)$, and therefore for the existence of a common factor of first or higher degree of the polynomials* (4), *are*

$$R(F_rF_s)=0,\ R^{(1)}(F_rF_s/F_j)=0\ \ (j=1,2,\ldots.,n,\text{ excepting } r,s). \tag{19}$$

The conditions are necessary, since the determinants in (19) are, when multiplied by certain powers of c_{r0}, equal to determinants of order N in the matrix $M_N(F_1\ldots.F_n)$.

If now the determinant $R(F_rF_s)$ vanishes, then the polynomials F_r and F_s have a linear factor in common, and by means of a transformation T_a, the matrix $M_N(F_1\ldots.F_n)$ can, according to Lemma III of Chapter II, be transformed into an equivalent matrix $M'_N(F_1\ldots.F_n)$ such that the last row of the matrix $M'_N(F_rF_s)$ will consist wholly of zero elements. The groups of columns G_j $(j=1,2,\ldots.,n,$ excepting $r,s)$ will be transformed into

$$\left.\begin{array}{ccccc} c'_{j0} & & & & \\ c'_{j1} & c'_{j0} & & & \\ . & c'_{j1} & . & & \\ . & . & . & . & \\ . & . & . & . & c'_{j0} \\ . & . & . & . & c'_{j1} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ c'_{jN-1} & c'_{jN-2} & . & . & c'_{jm_j} \end{array}\right\} \left(\begin{array}{l} j=1,2,\ldots.,n, \\ \text{excepting } r,s \end{array}\right).$$

Since now by hypothesis $R^{(1)}(F_r F_s)$ does not vanish, the same is true of the determinant $P_N^{(1)}(F_r F_s)$, and therefore of its transform $P_N^{(1)\prime}(F_r F_s)$. This latter determinant is, by definition, of order $N-1$; and by a well-known theorem in the theory of matrices, the matrix $M'_N(F_1 \ldots . F_n)$ vanishes if every determinant of which $P_N^{(1)\prime}(F_r F_s)$ is a first minor vanishes. Certain of these determinants are obviously zero, namely those which are contained altogether in the matrix $M'_N(F_r F_s)$. The others are except for sign equal respectively to

$$c'_{jk} \cdot P_N^{(1)\prime}(F_r F_s) \quad \begin{pmatrix} j = 1, 2, \ldots, n, \text{ excepting } r, s, \\ k = m_j,\ m_{j+1},\ \ldots,\ N-1 \end{pmatrix}$$

and vanish if and only if the quantities c'_{jk} vanish.

But by Lemma II of § 2, all the quantities c'_{jk} vanish if the quantities c'_{jm_j} vanish. Therefore the matrix $M'_N(F_1 \ldots . F_n)$ vanishes if the $n-2$ determinants of that matrix which are equal to

$$c'_{jm_j} P_N^{(1)\prime}(F_r F_s) \quad (j = 1, 2, \ldots, n, \text{ excepting } r, s)$$

all vanish.

The latter determinants are, however, exactly the transforms of the determinants $P_N^{(1)}(F_r F_s/F_j)$ $(j = 1, 2, \ldots, n, \text{ excepting } r, s)$; and therefore the vanishing of the determinants $P_N^{(1)}(F_r F_s/F_j)$, together with the vanishing of $R(F_r F_s)$, constitutes, under the hypothesis of the theorem, sufficient conditions for the vanishing of $M_N(F_1 \ldots . F_n)$. Then since the vanishing of $R^{(1)}(F_r F_s/F_j)$ is from the definition of this determinant equivalent to the vanishing of $P^{(1)}(F_r F_s/F_j)$, it follows at once that conditions (19) are sufficient for the vanishing of $M_N(F_1 \ldots . F_n)$.

Remark. It should be noted that if $R^{(1)}(F_r F_s)$ vanishes, then conditions (19) are by no means sufficient conditions for the vanishing of $M_N(F_1 \ldots . F_n)$. In fact, if $R(F_r F_s) = R^{(1)}(F_r F_s) = 0$, then all of the determinants $R^{(1)}(F_r F_s/F_j)$ vanish whatever be the coefficients of F_j. For under these conditions the polynomials F_r and F_s have a quadratic factor in common, and therefore $M_N(F_1 \ldots . F_n)$ can be transformed into an equivalent matrix $M''_N(F_1 \ldots . F_n)$ such that the matrix $M''_N(F_r F_s)$ has only zero elements in the last *two* rows. Therefore the determinant $P_N^{(1)\prime\prime}(F_r F_s)$ vanishes, and so

$$c''_{jm_j} \cdot P_N^{(1)\prime\prime}(F_r F_s) = 0 \quad (j = 1, 2, \ldots, n, \text{ excepting } r, s). \tag{20}$$

But as before, the transforms of the determinants $P_N^{(1)}(F_r F_s/F_j)$ are exactly equal to the quantities on the left side of (20), and so $P_N^{(1)}(F_r F_s/F_j) = 0$, and therefore $R^{(1)} \cdot (F_r F_s/F_j) = 0$ $(j = 1, 2, \ldots, n, \text{ excepting } r, s)$.

§ 10. *The General Case.*

THEOREM VII. *In any numerical case there are $n-1$ determinants of the matrix $M_N(F_1 \ldots F_n)$ whose simultaneous vanishing is equivalent to the vanishing of the matrix, and therefore constitutes necessary and sufficient conditions for the existence of a common factor of first or higher degree of the polynomials* (4).

This theorem has been proved in § 8 for the case $n = 3$; it will now be proved in general by a complete induction. We assume it true for $n = q$, and prove it true for $n = q + 1$.

By our assumption, there are $q-1$ determinants of the matrix $M_N(F_1 \ldots F_q)$ whose vanishing is equivalent to the vanishing of the matrix. Let these determinants be denoted by

$$R(F_1F_2),\ R^{(k_3)}(F_1F_2/F_3),\ R^{(k_4)}(F_1F_2F_3/F_4),\ \ldots,\ R^{(k_q)}(F_1F_2\ldots F_{q-1}/F_q),$$

and suppose that they all vanish. Then the matrix $M_N(F_1 \ldots F_q)$ has a certain rank $N - k_{q+1}$, where $k_{q+1} > 0$; the polynomials $F_1, F_2, \ldots, F_q$ have a highest common factor of degree k_{q+1}; and there exists a determinant of order $N - 2k_{q+1}$ of the sub-resultant matrix $M_N^{(k_{q+1})}(F_1 \ldots F_q)$ which does not vanish.* Call it $R_N^{(k_{q+1})}(F_1 \ldots F_q)$. There exists, therefore, a determinant of order $N - k_{q+1}$ formed from the first $N - k_{q+1}$ rows of $M_N(F_1 \ldots F_q)$ which is equal to $c_{10}^{k_{q+1}} \cdot R^{(k_{q+1})}(F_1 \ldots F_q)$ and so does not vanish. Call it $P_N^{(k_{q+1})}(F_1 \ldots F_q)$.

Now, since the polynomials $F_1, F_2, \ldots, F_q$ have a common factor of degree k_{q+1}, the matrix $M_N(F_1 \ldots F_{q+1})$ can be changed by a transformation T_a into an equivalent matrix $M'_N(F_1 \ldots F_{q+1})$ such that the last k_{q+1} rows of $M'_N(F_1 \ldots F_q)$ contain only zero elements. By this same transformation the non-vanishing determinant $P_N^{(k_{q+1})}(F_1 \ldots F_q)$ is changed into an equivalent determinant $P_N^{(k_{q+1})'}(F_1 \ldots F_q)$.

For simplicity in writing, let

$$F_{q+1} \equiv c_0 x^m + c_1 x^{m-1} + \ldots + c_m,$$

and for the moment let the number k_{q+1} be denoted simply by k. Then it is evident that each of the determinants of order N of the matrix $M'_N(F_1 \ldots F_{q+1})$

* There may of course be several such determinants. That there is at least one, follows from Theorems II and III.

which are not obviously zero, has as a factor a determinant of order k of the matrix

$$\left\| \begin{matrix} c'_{N-k} & c'_{N-k-1} & \cdot & \cdot & \cdot & c'_{m-k+1} \\ c'_{N-k+1} & c'_{N-k} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c'_{N-1} & c'_{N-2} & \cdot & \cdot & \cdot & c'_{m} \end{matrix} \right\| \begin{matrix} \\ \\ \\ \\ \\ \end{matrix}

CHAPTER IV.

An Illustrative Example. Three Cubics.

Given

$$\left.\begin{aligned} F_1(x) &\equiv a_0 x^3 + a_1 x^2 + a_2 x + a_3, \\ F_2(x) &\equiv b_0 x^3 + b_1 x^2 + b_2 x + b_3, \\ F_3(x) &\equiv c_0 x^3 + c_1 x^2 + c_2 x + c_3. \end{aligned}\right\} \quad (23)$$

The resultant matrix of lowest possible order is

$$M_6(F_1 F_2 F_3) \equiv \left\| \begin{matrix} a_0 & & b_0 & & c_0 & \\ a_1 & a_0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\ a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\ & a_3 & a_2 & & b_3 & b_2 & & c_3 & c_2 \\ & & a_3 & & & b_3 & & & c_3 \end{matrix} \right\|$$

Its vanishing is necessary and sufficient for the existence of a common factor of first or higher degree of F_1, F_2 and F_3.

The two resultant determinants can now be constructed, their form depending, however, upon the values of the sub-resultants of some pair of the polynomials (see Theorem V). For definiteness, we consider the sub-resultants of the pair F_1 and F_2. We have by definition

$$R(F_1 F_2) \equiv \left| \begin{matrix} a_0 & & b_0 & \\ a_1 & a_0 & b_1 & b_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 \\ a_3 & a_2 & a_1 & b_3 & b_2 & b_1 \\ & a_3 & a_2 & & b_3 & b_2 \\ & & a_3 & & & b_3 \end{matrix} \right|$$

$$R^{(1)}(F_1 F_2) \equiv \left| \begin{matrix} a_0 & & b_0 & \\ a_1 & a_0 & b_1 & b_0 \\ a_2 & a_1 & b_2 & b_1 \\ a_3 & a_2 & b_3 & b_2 \end{matrix} \right| \qquad R^{(2)}(F_1 F_2) \equiv \left| \begin{matrix} a_0 & b_0 \\ a_1 & b_1 \end{matrix} \right|$$

There are then three distinct cases to be considered.

Case I: *In which* $R^{(1)}(F_1 F_2) \neq 0$. The two resultant determinants whose vanishing is equivalent to the vanishing of $M_6(F_1 F_2 F_3)$ are

$$R(F_1 F_2), \quad R^{(1)}(F_1 F_2 / F_3) \equiv \left| \begin{matrix} a_0 & & b_0 & & \\ a_1 & a_0 & b_1 & b_0 & c_0 \\ a_2 & a_1 & b_2 & b_1 & c_1 \\ a_3 & a_2 & b_3 & b_2 & c_2 \\ & a_3 & & b_3 & c_3 \end{matrix} \right|$$

Case II: *In which* $R^{(1)}(F_1 F_2) = 0$, $R^{(2)}(F_1 F_2) \neq 0$. The resultant determinants are

$$R(F_1 F_2), \quad R^{(2)}(F_1 F_2/F_3) \equiv \begin{vmatrix} a_0 & & c_0 & \\ a_1 & a_0 & b_0 & c_1 & c_0 \\ a_2 & a_1 & b_1 & c_2 & c_1 \\ a_3 & a_2 & b_2 & c_3 & c_2 \\ & a_3 & b_3 & & c_3 \end{vmatrix}$$

Case III: *In which* $R^{(1)}(F_1 F_2) = R^{(2)}(F_1 F_2) = 0$, $R^{(3)}(F_1 F_2) \neq 0$. The resultant determinants are

$$R(F_1 F_2), \quad R^{(3)}(F_1 F_2/F_3) \equiv \begin{vmatrix} b_0 & & c_0 & \\ b_1 & b_0 & c_1 & c_0 \\ b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\ b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\ & b_3 & b_2 & & c_3 & c_2 \\ & & b_3 & & & c_3 \end{vmatrix}$$

In this case, if $R(F_1 F_2) = 0$, then the polynomials F_1 and F_2 differ only by a constant factor. Also $R^{(3)}(F_1 F_2/F_3)$ is identical with $R(F_1 F_3)$. It is therefore directly evident that the vanishing of $R(F_1 F_2)$ and $R^{(3)}(F_1 F_2/F_3)$ is sufficient for the existence of a common factor of first or higher degree of F_1, F_2 and F_3.

COLUMBIA UNIVERSITY.

*Linear Mixed Equations and their Analytic Solutions.**

By R. D. Carmichael.

The most important early papers on mixed equations are those by Biot† and Poisson.‡ Each of them deals with the practical integration of special equations and the application of the results to the solution of geometrical problems. The principal contents of both papers, together with references to the previous literature, are to be found in Lacroix's *Traité du Calcul.* §

Questions of an analytic nature have been further treated by Cesaro, Lecornu, Brajtzew and others. ||

Several applications of mixed equations in the treatment of geometrical problems are to be found in memoirs by Puiseux and Combescure.¶

From this brief bibliography it appears that the general questions as to the existence and the nature of solutions of mixed equations have not been treated. It is the object of this paper to consider these questions for the system of mixed equations,

$$D_i f_i(x) = \sum_{j=1}^{n} \psi_{ij}(x) f_j(x), \qquad i = 1 \ldots . n,$$

where

$$\psi_{ij}(x) = \psi'_{ij} x^{-2} + \psi''_{ij} x^{-3} + \ldots . , \qquad |x| \geqq R;$$

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* Read before the American Mathematical Society, October 29, 1910.

† Biot, "Sur les Équations aux Différences Mêlées," *Mémoires de l'Institut de France, Savans étrangers,* Vol. I (1806), pp. 296–327.

‡ Poisson, "Sur les Équations aux Différences Mêlées," *Journal de l'École Polytechnique,* Cahier XIII (1806), pp. 126–147.

§ Third edition, Vol. III (1819), pp. xix, 575–600.

|| Cesaro, "Sur une Équation aux Différences Mêlées," *Nouvelles Annales de Mathématiques,* Ser. 3, Vol. IV (1885), pp. 36–40; Lecornu, "Sur Certaines Équations aux Différences Mêlées," *Bulletin de la Société Mathématique de France,* Vol. XXVII (1899), pp. 153–160; Brajtzew, "On the Question of the Integration of Linear Mixed Equations by the aid of Definite Integrals" (Russian), *Moscow Mathematical Collection,* Vol. XXII (1901), pp. 275–284. See also the following papers: Gregory, *Cambridge Mathematical Journal,* Vol. I (1839), p. 54; Walton, *Quarterly Journal,* Vol. X (1870), pp. 248–253; Oltramare, *Association Français, Compte rendu* XX (1891), pp. 66–72; Oltramare, *Ibid.,* XXIV (1895), pp. 175–186; Lémeray, *Edinburgh Mathematical Society Proceedings* (1898), pp. 13–44.

¶ Puiseux, "Sur les Développées et les Développantes des Courbes Planes," *Liouville's Journal,* Vol. IX (1844), pp. 377–399; Combescure, "Sur Quelques Questions qui Dépendent des Différences Finies ou Mêlées," *Annales de l'École Normale Supérieure,* Ser. 2, Vol. III (1874), pp. 305–362.

the symbols D_i, $i=1, \ldots, k$, denote differentiation with respect to x, and the symbols D_i, $i=k+1, \ldots, n$, are identical with the symbol Δ of the difference calculus. It will be noticed that such a system is analogous in form to a system of differential equations of the first order having infinity for an ordinary point.

In Section I two formal expansions, involving arbitrary constants and arbitrary periodic functions are obtained, both of which formally satisfy the given system of mixed equations.

In Section II we demonstrate the convergence of the formal expansions throughout properly determined regions and obtain fundamental existence theorems in two forms, and thus exhibit a remarkable duality in the solutions of mixed equations. A method of applying these results to a single equation of order higher than the first is indicated in a brief remark.

Section III contains an application of the preceding theorems to a system of difference equations. Two fundamental systems of solutions are obtained.

In Section IV we point out an essential difference between mixed equations and either differential or difference equations. This difference is due to the remarkable manner in which the arbitrary elements enter into the solution of the mixed equations.

I.

Let D_i, $i=1 \ldots n$, denote a set of n operations, the first k of them being each identical with D and the remaining ones with Δ, where k is some one of the numbers $0, 1, 2, \ldots, n$, and where D is the symbol of differentiation and Δ is the symbol of differences, that is, $\Delta f(x)=f(x+1)-f(x)$, and form the system of equations

$$D_i f_i(x)=\sum_{j=1}^{n} \psi_{ij}(x) f_j(x), \qquad i=1 \ldots n, \tag{1}$$

where

$$\psi_{ij}(x)=\psi_{ij}'' x^{-2}+\psi_{ij}''' x^{-3}+\ldots, \qquad |x| \geqq R. \tag{2}$$

In applying the method of successive approximation to the problem of finding a solution of (1) we write the auxiliary equations

$$\begin{aligned} &D_i f_i^{(1)}(x)=0, \\ &D_i f_i^{(2)}(x)=\sum_{j=1}^{n} \psi_{ij}(x) f_j^{(1)}(x), \\ &D_i f_i^{(3)}(x)=\sum_{j=1}^{n} \psi_{ij}(x) f_j^{(2)}(x), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \qquad i=1 \ldots n, \tag{3}$$

and by means of them define a sequence of approximation functions which lead to a set of formal expansions satisfying (1).

Each of the above auxiliary systems (3), after the first, is of the form

$$D_i g_i(x) = \eta_i(x), \qquad i = 1 \ldots . n. \tag{4}$$

Considering first those equations of this system in which D_i denotes Δ we have evidently two particular formal solutions

$$g_i(x) = -\sum_{\nu=0}^{\infty} \eta_i(x+\nu), \tag{5}$$

$$i = k+1 \ldots . n.$$

$$g_i(x) = \sum_{\nu=1}^{\infty} \eta_i(x-\nu), \tag{6}$$

The other equations of system (4), namely those in which D_i denotes D, have each two formal solutions analogous respectively to (5) and to (6). We may conveniently write these in the form

$$g_i(x) = -\int_x^{+\infty} \eta_i(x)\,dx, \tag{7}$$

$$i = 1 \ldots . k,$$

$$g_i(x) = -\int_x^{-\infty} \eta_i(x)\,dx, \tag{8}$$

where in the first case the path of integration extends from x to infinity in a direction parallel to the positive real axis, and in the second case it extends from x to infinity parallel to the negative real axis. Equations (5) and (7) may be taken together as constituting a single formal solution of (4), and this solution we write in the form

$$g_i(x) = S_{xi}^{+}(\eta_i), \qquad i = 1 \ldots . n, \tag{9}$$

where $S_{xi}^{+}(\eta_i)$, for a particular i, is identical with that second member of (5) or of (7) which has the same subscript i. Likewise from (6) and (8) we have a second formal solution of (4), and this we write in the form

$$g_i(x) = S_{xi}^{-}(\eta_i), \qquad i = 1 \ldots . n, \tag{10}$$

where $S_{xi}^{-}(\eta_i)$, for a particular i, is identical with that second member of (6) or of (8) which has the same subscript i. The solution of the first system in the set (3) we shall denote by $t_i(x)$, $i = 1 \ldots . n$. For the equations in which D_i is D it is evident that $t_i(x)$ is a constant; for the other equations it is a periodic function of period 1.

We are now in position to obtain successively a set of particular formal solutions of systems (3); we employ in the first instance the formal solutions (9) of (4). When any particular system of (3) is solved, the set of values so found is substituted in the following system and this in turn is then solved. In each case those particular solutions are selected which will be convenient later on; from each system after the first these solutions are obtained by

operating on the equations throughout by S_{xi}^{+} and then adding $t_i(x)$ to the resulting second member. Thus we have

$$f_i^{(1)}(x) = t_i(x),$$

$$f_i^{(2)}(x) = t_i(x) + S_{xi}^{+}\left(\sum_{j=1}^{n} \psi_{ij} t_j\right),$$

$$f_i^{(3)}(x) = t_i(x) + S_{xi}^{+}\left(\sum_{j=1}^{n} \psi_{ij} t_j\right) + S_{xi}^{+}\left\{\sum_{j=1}^{n} \psi_{ij} S_{xj}^{+}\left(\sum_{k=1}^{n} \psi_{jk} t_k\right)\right\}, \qquad i = 1 \ldots . n.$$

. . . . . .

If we denote by $f_i^{+}(x)$ the formal expansion to which the sequence $f_i^{(1)}(x)$, $f_i^{(2)}(x)$, $f_i^{(3)}(x)$, ... leads, we have

$$f_i^{+}(x) = t_i(x) + S_{xi}^{+}\left(\sum_{j=1}^{n} \psi_{ij} t_j\right) + S_{xi}^{+}\left\{\sum_{j=1}^{n} \psi_{ij} S_{xj}^{+}\left(\sum_{k=1}^{n} \psi_{jk} t_k\right)\right\}$$
$$+ S_{xi}^{+}\left[\sum_{j=1}^{n} \psi_{ij} S_{xj}^{+}\left\{\sum_{k=1}^{n} \psi_{jk} S_{xk}^{+}\left(\sum_{l=1}^{n} \psi_{kl} t_l\right)\right\}\right] + \ldots . , \qquad i = 1 \ldots . n. \quad (11)$$

It is easy to verify that $f_i(x) = f_i^{+}(x)$, $i = 1 \ldots . n$, formally satisfies (1). To this end let us perform on the n equations (11), in order, the operations denoted by D_i, $i = 1 \ldots . n$, making use of the facts that $D_i S_{xi}^{+}(\eta) \equiv \eta$ and $D_i t_i(x) \equiv 0$ and distributing the operations over the terms of the series in the second members; we have:

$$D_i f_i^{+}(x) = 0 + \sum_{j=1}^{n} \psi_{ij} t_j + \sum_{j=1}^{n} \psi_{ij} S_{xj}^{+}\left(\sum_{k=1}^{n} \psi_{jk} t_k\right) + \ldots .$$
$$= \sum_{j=1}^{n} \psi_{ij}\left\{t_j(x) + S_{xj}^{+}\left(\sum_{k=1}^{n} \psi_{jk} t_k\right) + \ldots .\right\}$$
$$= \sum_{j=1}^{n} \psi_{ij}(x) f_j^{+}(x), \qquad i = 1 \ldots . n.$$

We have thus shown that the functions (11) formally satisfy equations (1).

If now we start from the formal solution (10) instead of (9) and carry out an argument entirely analogous to the preceding one, we shall obtain another formal solution $f_i(x) = f_i^{-}(x)$, $i = 1 \ldots . n$, representable as follows:

$$f_i^{-}(x) = t_i(x) + S_{xi}^{-}\left(\sum_{j=1}^{n} \psi_{ij} t_j\right) + S_{xi}^{-}\left\{\sum_{j=1}^{n} \psi_{ij} S_{xj}^{-}\left(\sum_{k=1}^{n} \psi_{jk} t_k\right)\right\}$$
$$+ S_{xi}^{-}\left[\sum_{j=1}^{n} \psi_{ij} S_{xj}^{-}\left\{\sum_{k=1}^{n} \psi_{jk} S_{xk}^{-}\left(\sum_{l=1}^{n} \psi_{kl} t_l\right)\right\}\right] + \ldots . , \qquad i = 1 \ldots . n. \quad (12)$$

II.

In this section we determine the nature of the convergence and certain properties of the functions formally defined by the expansions already obtained. From the form of $\psi_{ij}(x)$ given in (2) it follows that there exists a constant M such that

$$|\psi_{ij}(x)| < \frac{M}{|x|^2}, \text{ for } |x| \geq R. \quad (13)$$

We assume that the functions $t_i(x)$, $i = 1 \ldots n$, when not constant, are periodic functions of period 1 *analytic* throughout the finite plane. Let $x = u + v\sqrt{-1}$, u and v being real. We shall prove first that the expansions (11) are uniformly convergent series of analytic functions throughout a region in which

$$u \geqq R, \quad u \geqq Mn + 1 + \varepsilon, \quad \varepsilon = \text{any positive constant.} \tag{14}$$

Let us consider the series

$$\bar{S}_{xi}^{+}\left(\sum_{j=1}^{n} \psi_{ij} t_j\right) + \bar{S}_{xi}^{+}\left\{\sum_{j=1}^{n} \psi_{ij} \bar{S}_{xj}^{+}\left(\sum_{k=1}^{n} \psi_{jk} t_k\right)\right\} + \ldots, \tag{15}$$

where $\bar{S}_{xi}^{+}$ is defined by $\bar{S}_{xi}^{+}(\eta) \equiv |S_{xi}^{+}(|\eta|)|$. For fixed v and varying u let A_{vi} be the maximum value of $|t_i(x)|$. Let A_v be the greatest of the quantities A_{vi}, $i = 1 \ldots n$. Using this notation and employing inequalities (13) and (14) it is clear that we may write

$$\bar{S}_{xi}^{+}\left(\sum_{j=1}^{n} \psi_{ij} t_j\right) < Mn A_v \bar{S}_{xi}^{+}\left(\frac{1}{x^2}\right). \tag{16}$$

Let us consider the case in which S_{xi}^{+} is an inverse of Δ. Then we have

$$\bar{S}_{xi}^{+}\left(\frac{1}{x^2}\right) = \frac{1}{|x|^2} + \frac{1}{|x+1|^2} + \frac{1}{|x+2|^2} + \ldots,$$

$$\therefore \bar{S}_{xi}^{+}\left(\frac{1}{x^2}\right) \leqq \frac{1}{u^2} + \frac{1}{(u+1)^2} + \frac{1}{(u+2)^2} + \ldots.$$

But the sum of the last series is less than the value of the integral

$$\int_{u-1}^{\infty} \frac{dz}{z^2} = \frac{1}{u-1}.$$

Hence

$$\bar{S}_{xi}^{+}\left(\sum_{j=1}^{n} \psi_{ij} t_j\right) < \frac{Mn A_v}{u-1}, \text{ when } S_{xi}^{+} \text{ is an inverse of } \Delta. \tag{17}$$

In the other case, namely that in which S_{xi}^{+} is an inverse of D, we have

$$\bar{S}_{xi}^{+}\left(\frac{1}{x^2}\right) \leqq \int_{u}^{\infty} \frac{M}{u^2} du = \frac{M}{u}.$$

Hence

$$\bar{S}_{xi}^{+}\left(\sum_{j=1}^{n} \psi_{ij} t_j\right) < \frac{Mn A_v}{u}, \text{ when } S_{xi}^{+} \text{ is an inverse of } D. \tag{18}$$

Combining (17) and (18) we have

$$\bar{S}_{xi}^{+}\left(\sum_{j=1}^{n} \psi_{ij} t_j\right) < \frac{Mn A_v}{u-1}, \qquad i = 1 \ldots n;$$

and therefore

$$|S_{xi}^{+}\left(\sum_{j=1}^{n} \psi_{ij} t_j\right)| < \frac{Mn A_v}{u-1}, \qquad i = 1 \ldots n.$$

Furthermore, since at any finite point x, for which relations (14) are satisfied, each of the n^2+n functions $\psi_{ij}(x)$, $t_j(x)$ is analytic, it is evident from the preceding argument that $S_{xi}^{+}(\psi_{ij}t_j)$ is analytic throughout the portion of the finite plane defined by (14).

Let us now consider the second term in (17). In view of the preceding discussion it is easy to obtain step by step the results indicated in the following relations:

$$\begin{aligned}\bar{S}_{xi}^{+}\{\sum_{j=1}^{n}\psi_{ij}\bar{S}_{xj}^{+}(\sum_{k=1}^{n}\psi_{jk}t_k)\} &\leqq Mn\,A_v\,\bar{S}_{xi}^{+}\Big\{\sum_{j=1}^{n}|\psi_{ij}|\bar{S}_{xj}^{+}\Big(\frac{1}{x^2}\Big)\Big\}\\ &< Mn\,A_v\,\bar{S}_{xi}^{+}\Big\{\sum_{j=1}^{n}|\psi_{ij}|\,\frac{1}{u-1}\Big\}\\ &< \frac{Mn\,A_v}{u-1}\bar{S}_{xi}^{+}\{\sum_{j=1}^{n}|\psi_{ij}|\}\\ &< \frac{M^2n^2A_v}{(u-1)^2}.\end{aligned}$$

One also proves readily that $S_{xi}^{+}\{\sum_{j=1}^{n}\psi_{ij}S_{xj}^{+}(\sum_{k=1}^{n}\psi_{jk}t_k)\}$ is an analytic function of x in the region defined by (14), and that its absolute value is less than $M^2n^2A_v/(u-1)^2$. By a continuation of the process it may be shown that every term of the series in (11) is analytic and that the series itself is term by term less in absolute value than

$$A_v+\frac{Mn}{u-1}A_v+\Big(\frac{Mn}{u-1}\Big)^2A_v+\Big(\frac{Mn}{u-1}\Big)^3A_v+\ldots. \tag{19}$$

But for u satisfying (14) this last series is uniformly convergent. The same is therefore true of the series in (11), and hence the latter converges to a function which is analytic throughout the portion of the finite plane defined by (14). It is easy to see then that (11) affords a solution of (1) in the region defined.

Starting from (15) and (16) and working in a similar manner, we may prove the convergence of the n series (11) in a different region of the plane. We suppose in this case that $|v|\geqq R$. We have

$$\bar{S}_{xi}^{+}\Big(\frac{1}{x^2}\Big)=\sum_{r=1}^{\infty}\frac{1}{(u+r)^2+v^2},$$

where S_{xi}^{+} is an inverse of Δ. But it is evident that

$$\sum_{r=1}^{\infty}\frac{1}{(u+r)^2+v^2}<\sum_{r=-\infty}^{\infty}\frac{1}{(u+r)^2+v^2}<2\Big\{\int_0^{\infty}\frac{dz}{z^2+v^2}+\frac{1}{v^2}\Big\}.$$

Hence

$$\bar{S}_{xi}^{+}\Big(\frac{1}{x^2}\Big)<\frac{\pi}{|v|}+\frac{2}{v^2}.$$

Comparing with (16) we have

$$\bar{S}^{+}_{xi}\left(\sum_{j=1}^{n}\psi_{ij}t_j\right) < Mn\,A_v\left(\frac{\pi}{|v|}+\frac{2}{v^2}\right), \tag{20}$$

where S^{+}_{xi} is an inverse of Δ. Now when S^{+}_{xi} is an inverse of D it is easy to show that

$$\bar{S}^{+}_{xi}\left(\frac{1}{x^2}\right) < 2\int_0^{\infty}\frac{du}{u^2+v^2};$$

whence it follows that (20) holds also for these values of i. It is clear that one may continue the process just as in the preceding argument and thus show that each series (11) is term by term less in absolute value than the series

$$A_v + Mn\left(\frac{\pi}{|v|}+\frac{2}{v^2}\right)A_v + M^2n^2\left(\frac{\pi}{|v|}+\frac{2}{v^2}\right)^2 A_v + \ldots\ldots \tag{21}$$

Furthermore, for x in a region in which

$$|v| \geqq R,\ \ Mn\left(\frac{\pi}{|v|}+\frac{2}{v^2}\right) \leqq 1-\varepsilon,\ \ \varepsilon \text{ any positive constant}, \tag{22}$$

it is easy to carry out an argument similar to that used in the preceding case, and to show that in such a region every term of the series in (11) is an analytic function, that the series as a whole converges uniformly (in any closed part of the region) and that therefore it represents a function of x which is analytic throughout the whole region.

It is obvious that the two regions thus found overlap, but that each one contains a part which is not in the other. The form of the total region, which is composed of these two, is easily made out from inequalities (14) and (22).

From the form of this region and the statements connected with (19) and (21) it follows that when x approaches infinity along any straight line parallel to the positive real axis we must have

$$\lim\{f_i^{+}(x) - t_i(x)\} = 0.$$

The principal results thus obtained are stated in the following theorem:

I. *The system of mixed equations*

$$D_i f_i(x) = \sum_{j=1}^{n}\psi_{ij}(x)f_j(x), \qquad i = 1\ldots n,$$

where

$$\psi_{ij}(x) = \psi''_{ij}x^{-2} + \psi'''_{ij}x^{-3} + \ldots., \qquad |x| \geqq R,$$

has a solution

$$f_i(x) = f_i^{+}(x), \qquad i = 1\ldots n,$$

where

$$f_i^{+}(x) = t_i(x) + S^{+}_{xi}\left(\sum_{j=1}^{n}\psi_{ij}t_j\right) + S^{+}_{xi}\left\{\sum_{j=1}^{n}\psi_{ij}S^{+}_{xj}\left(\sum_{k=1}^{n}\psi_{jk}t_k\right)\right\}$$
$$+ S^{+}_{xi}\left[\sum_{j=1}^{n}\psi_{ij}S^{+}_{xj}\left\{\sum_{k=1}^{n}\psi_{jk}S^{+}_{xk}\left(\sum_{l=1}^{n}\psi_{kl}t_l\right)\right\}\right] + \ldots., \qquad i = 1\ldots n,$$

where $t_i(x)$ is a constant or a periodic function of x of period 1, analytic throughout the finite plane, according as $D_i \equiv D$ or $D_i \equiv \Delta$. Each function of this solution is analytic throughout that portion of the finite plane for which either of the following sets of relations is true:

$$u \geqq R, \quad u \geqq Mn + 1 + \varepsilon, \tag{23}$$

$$|v| \geqq R, \quad Mn\left(\frac{\pi}{|v|} + \frac{2}{v^2}\right) \leqq 1 - \varepsilon,$$

where M is a properly chosen positive constant, ε is any positive constant whatever, and u and v are real, being defined by $x = u + v\sqrt{-1}$. Further, if x approaches infinity along any straight line parallel to the positive real axis then

$$\lim \{f_i^+(x) - t_i(x)\} = 0. \tag{24}$$

It is clear that the formal expansions (12) may be treated in a manner entirely analogous to that just employed in the investigation of the properties of (11) and that corresponding results will be found which are in every respect parallel to those in Theorem I.

II. *With the following verbal alterations I is changed into the new theorem which is thus obtained: Replace f^+ by f^-, S_{xi}^+ by S_{xi}^- and instead of* (23) *write*

$$u \leqq -R, \quad u \leqq -(Mn + 1 + \varepsilon).$$

We have thus established the remarkable fact that there are two classes of solutions of (1) which are equally simple, the first of them being related to the right side of the plane as the second is to the left, and *vice versa.*

REMARK 1. It is clear that if the periodic functions $t_i(x), i = k+1 \ldots n$, are not assumed to be analytic throughout the finite plane, the general convergence proof can be carried out in a manner analogous to the foregoing; that the solutions so obtained are analytic in the same region as before except for singularities at the singularities of $t_i(x), i = k+1 \ldots n$, and that the limit for x approaching infinity along a line parallel to the positive real axis in general exists and has the same form as in the preceding cases.

REMARK 2. The preceding results are readily applied to a single mixed equation of order higher than the first. It will be sufficient to illustrate this remark by means of an example. For instance, if (1) has the following special form

$$g'(x) = \psi_{11}(x)g(x) + \psi_{12}(x)f(x),$$
$$\Delta f(x) = \psi_{21}(x)g(x) + \psi_{22}(x)f(x),$$

and if we differentiate the second of these equations with respect to x and

eliminate $g(x)$ and $g'(x)$ from the two given equations and the one thus obtained, we have

$$\Delta f'(x) - \psi_{22}(x) f'(x) - \left\{\psi_{11}(x) + \frac{\psi'_{21}(x)}{\psi_{21}(x)}\right\}\Delta f(x)$$
$$+\left\{\psi_{22}(x)\frac{\psi'_{21}(x)}{\psi_{21}(x)} - \psi'_{22}(x) - \psi_{12}(x)\psi_{21}(x) + \psi_{11}(x)\psi_{22}(x)\right\} f(x) = 0,$$

an equation of the type studied by Poisson. Hence for any equation

$$\Delta f'(x) + a f'(x) + b\Delta f(x) + c f(x) = 0,$$

which may be written in the preceding form, theorems I and II assert the existence of integrals having specified properties. Whether a given equation is reducible to this form is easily determined in any special case. Also, suitable values of the ψ's for effecting this reduction, in case it is possible, may be readily obtained.

III.

So far no essential use has been made of the fact that (1) is a *mixed* system; and therefore the results which we have obtained are true if every D_i is identical with D or if every D_i is identical with Δ. Hence our theorems are true of a system either of differential equations or of difference equations. If one seeks to apply them to differential equations it turns out that they are not suitable for deriving the existence theorem in a form of greatest simplicity; this is owing to the particular path of integration which has been chosen, there being in this case of course an infinity of paths from which choice can be made.

But with the difference equations the matter is essentially otherwise; the *path of summation* (to use a term analogous to *the path of integration*) for the sum which gives an inverse of Δ must have infinity for one of its extremities, must pass through the point x and be parallel to the axis of reals. This gives only two essentially different paths of summation; namely, the two extending from x to infinity along the two directions of this line. But, for the case in which (1) is a system of difference equations these two are entirely sufficient, as we shall see, to lead to the existence theorem in a form possessing the desired simplicity.*

* Compare the analogous existence theorem for a different system of equations treated in my thesis. See also a paper by G. D. Birkhoff, *Transactions of the American Mathematical Society*, Vol. XII (1911), pp. 243–284. The theorem here derived is a special case of Birkhoff's results. It may also be readily obtained as a corollary to the treatment in my thesis. It is nevertheless interesting to have a proof of it by the relatively simple method of the present paper, especially since it comes out as a corollary of the theorems in Section II.

If every D_i is identical with Δ, so that (1) is a system of difference equations, we may write this system in the form

$$f_i(x+1)=\sum_{j=1}^{n}\phi_{ij}(x)f_j(x),\quad \phi_{ij}(x)=\delta_{ij}+\psi_{ij}(x),\qquad i,j=1\ldots.n, \tag{25}$$

where δ_{ij} equals 0 or 1 according as i is or is not equal to j.

From Theorem II it follows that there exists a line l perpendicular to the negative axis of reals and such that the solutions referred to in Theorem II are all analytic throughout the part of the finite plane which lies to the left of l. If we know $f_i^-(x_0)$, $i=1\ldots.n$, then (25) gives $f_i^-(x_0+1)$, $i=1\ldots.n$; this known, $f_i^-(x_0+2)$, $i=1\ldots.n$, is found; and so on. Hence equation (25) may be used to extend the solution $f_i^-(x)$ of Theorem II across the plane to the right; it is evident that this process leads to the determination of $f_i^-(x)$, $i=1\ldots.n$, throughout the finite plane and that the singularities of these functions occur at the points congruent on the right to the singularities of $\psi_{ij}(x)$, $i,j=1\ldots.n$. (A point A is said to be *congruent on the right* to a point B if $A-B$ is a positive integer; it is *congruent on the left* if $A-B$ is a negative integer.)

If (25) is solved for $f_i(x)$, $i=1\ldots.n$, in terms of $f_i(x+1)$, $i=1\ldots.n$, and x is replaced by $x-1$, the system may be written in the form

$$f_i(x-1)=\sum_{j=1}^{n}\bar{\psi}_{ij}(x)f_j(x),\qquad i=1\ldots.n.$$

If one starts from this equation and employs Theorem I, it is easy to show that the functions $f_i^+(x)$, $i=1\ldots.n$, are analytic throughout the finite plane except at points congruent on the left to the singularities of $\bar{\psi}_{ij}(x)$, $i,j=1\ldots.n$.

For the present case, namely that in which (1) is a system of difference equations, the periodic functions $t_i(x)$, $i=1\ldots.n$, may be removed from under the signs S_{xi}^+, $i=1\ldots.n$; that is,

$$S_{xi}^+(\eta t_j)=t_jS_{xi}^+(\eta).$$

From this it follows that $f_i^+(x)$ may be written in the form:

$$\begin{aligned}f_i^+(x)=\sum_{j=1}^{n}t_j(x)\Big[&\delta_{ij}+S_{xi}^+(\psi_{ij})+S_{xi}^+\{\sum_{a=1}^{n}\psi_{ia}S_{xa}^+(\psi_{aj})\}\\&+S_{xi}^+[\sum_{a=1}^{n}\psi_{ia}S_{xa}^+\{\sum_{\beta=1}^{n}\psi_{a\beta}S_{x\beta}^+(\psi_{\beta j})\}]+\ldots.\Big],\\&i=1\ldots.n;\quad \delta_{ij}=0 \text{ if } i\neq j;\quad \delta_{ij}=1 \text{ if } i=j.\end{aligned} \tag{26}$$

As particular solutions of (1) we may therefore choose

$$f_{1j}^{+}(x), f_{2j}^{+}(x), \ldots, f_{nj}^{+}(x), \qquad j=1\ldots n,$$

where

$$f_{ij}^{+}(x)=\delta_{ij}+S_{xi}^{+}(\psi_{ij})+S_{xi}^{+}\{\sum_{a=1}^{n}\psi_{ia}S_{xa}^{+}(\psi_{aj})\}$$
$$+S_{xi}^{+}[\sum_{a=1}^{n}\psi_{ia}S_{xa}^{+}\{\sum_{\beta=1}^{n}\psi_{a\beta}S_{x\beta}^{+}(\psi_{\beta j})\}]+\ldots., \qquad i,j=1\ldots n. \quad (27)$$

Thus $f_{ij}^{+}(x)$, j fixed and $i=1\ldots n$, is a special case of the preceding solution $f_i^{+}(x)$.

Moreover, $f_{ij}^{+}(x)$, $i,j=1\ldots n$, constitute a fundamental system of solutions of (1); for they satisfy the necessary and sufficient condition of independence, namely that the determinant $|f_{ij}^{+}|\not\equiv 0$. To prove this we have only to notice that as x approaches infinity along a line parallel to the positive real axis $f_{ij}^{+}(x)$ approaches δ_{ij}; and hence, that the determinant $|f_{ij}^{+}|$ approaches 1. From the independence of these solutions it follows that the general solution of the system of difference equations is

$$f_i(x)=\sum_{j=1}^{n}\omega_j(x)f_{ij}^{+}(x), \qquad i=1\ldots n,$$

where $\omega_j(x)$, $j=1\ldots n$, are arbitrary periodic functions of x of period 1.

In a similar manner one arrives at a second fundamental system of solutions

$$f_{1j}^{-}(x), f_{2j}^{-}(x), \ldots, f_{nj}^{-}(x), \qquad j=1\ldots n,$$

where

$$f_{ij}^{-}(x)=\delta_{ij}+S_{xi}^{-}(\psi_{ij})+S_{xi}^{-}\{\sum_{a=1}^{n}\psi_{ia}S_{xa}^{-}(\psi_{aj})\}+\ldots., \qquad i,j=1\ldots n, \quad (28)$$

$f_{ij}^{-}(x)$, j fixed and $i=1\ldots n$, being a special case of $f_i^{-}(x)$, $i=1\ldots n$.

In case we have a solution which belongs to either of these fundamental solutions, the quantity A_v entering into (19) and (21) is equal to 1. Hence, if x approaches infinity along any ray from the origin, the negative real axis excepted, we have

$$\lim f_{ij}^{+}(x)=\delta_{ij},$$

with a like result for functions of the other fundamental system.

The principal results of this section may be constructed into the following theorem:

The system of difference equations

$$f_i(x+1)=\sum_{j=1}^{n}\phi_{ij}(x)f_j(x) \quad or \quad f_i(x-1)=\sum_{j=1}^{n}\bar{\psi}_{ij}(x)f_j(x), \qquad i=1\ldots n,$$

where

$$\phi_{ij}(x)=\delta_{ij}+\psi_{ij}(x),\ \psi_{ij}(x)=\psi_{ij}''x^{-2}+\psi_{ij}'''x^{-3}+\ldots., \qquad |x|\geqq R,$$

has two equally simple fundamental systems of solutions $f_{ij}^{+}(x)$, $f_{ij}^{-}(x)$, $i, j = 1 \ldots n$, *defined by means of equations* (27) *and* (28) *respectively; each function of the first solution is analytic throughout the finite plane except at points congruent on the right to the singularities of* $\bar{\psi}_{ij}(x)$, $i, j = 1 \ldots n$, *and each function of the second is analytic throughout the finite plane except at points congruent on the left to singularities of* $\psi_{ij}(x)$, $i, j = 1 \ldots n$.

If x *approaches infinity along any ray from the origin, the negative real axis excepted, then*

$$\lim f_{ij}^{+}(x) = \delta_{ij};$$

if along any ray except the positive real axis, then

$$\lim f_{ij}^{-}(x) = \delta_{ij}.$$

IV.

In the preceding section we have seen that when (1) is a system of difference equations the general solution (11) may be written in the form (26); and that this form has the property of being linear in the arbitrary periodic functions $t_i(x)$, $i = 1 \ldots n$. It is on account of the possibility of writing (11) in this form that we have been able to express the general solution of (25) in terms of a fundamental system consisting of n properly chosen particular solutions.

A reference to the discussion of this matter in Section III will bring out the fact that it depends essentially on the equation

$$S_{xi}^{+}(\eta t_j) = t_j S_{xi}^{+}(\eta),$$

where S_{xi}^{+} is an inverse of Δ. But when S_{xi}^{+} is a definite integral and $t_j(x)$ is a periodic function (not a constant) the corresponding equality does not exist. It follows, therefore, that the method which was effective in the case of difference equations for obtaining a fundamental system of solutions will not apply to the case of a mixed system of equations, and, in fact, simple examples are sufficient to indicate that the relations among the infinity of solutions of such systems are probably not expressible in a simple way in terms of a finite number of them singled out as fundamental. In connection with this remark one may compare the paper by Lecornu, to which reference has already been made.

PRINCETON UNIVERSITY, *December*, 1910.

*On the Theory of Linear Difference Equations.**

By R. D. Carmichael.

Under the above general heading I present here a discussion of each of the following topics:

I. On the Finite Integral of a Function.

II. Entire Functions Defined by Linear Homogeneous Equations with Rational Coefficients.

III. Applications of the Preceding Results to Non-homogeneous Equations.

The three parts of the paper are closely connected by having the common purpose of investigating the character of functions defined by linear difference equations, especial emphasis being put on the possibility of obtaining fundamental systems of solutions each function of which is an entire function. Where such a fundamental system does not exist, solutions are obtained having as simple sets of singularities as possible.

Each part of the paper has its own introduction, containing references to the pertinent literature and giving a brief statement of the results obtained in that part.

I. On the Finite Integral of a Function.

Introduction.

Guichard† has proved that if $G(x)$ is any entire function there exists another entire function $u(x)$ satisfying the relation

$$u(x+1)-u(x)=G(x).$$

Such a function $u(x)$ he obtains in the form of an integral. Appell‡ and Hurwitz§ have given other proofs of the same theorem. Hurwitz,‖ by an elegant method, has proved the additional theorem that if $\phi(x)$ is any function of x which is meromorphic throughout the finite plane there exists a function

* Read before the American Mathematical Society, Chicago, April 1912.

† *Annales Scientifiques de l'École Normale Supérieure*, (3) 4 (1887), pp. 361–380.

‡ (Liouville) *Journal de Mathématiques* (4) 7 (1891), pp. 157–219; especially pp. 159–176.

§ *Acta Mathematica* 20 (1897), pp. 285–312.

‖ *Loc. cit.*, pp. 308–312.

$F(x)$ which is also meromorphic throughout the finite plane and satisfies the relation

$$F(x+1)-F(x)=\phi(x).$$

The primary object of the present part of this paper is to prove Guichard's important theorem by direct means. Accordingly, in § 1 a power series for $u(x)$ is assumed and direct substitution is made in the first equation above. There results an infinite system of equations for the determination of the coefficients in the series. This system is solved by what is believed to be a method of considerable value,* the solution obtained being such that the power series converges throughout the finite plane.

In § 2 I determine properties of the finite integral of a function which is analytic outside of a given circle, except possibly at infinity.

In §§ 3 and 4 I treat the same problem for functions which are analytic within a circle and within a circular ring respectively.

§ 1. *The Finite Integral of an Entire Function.*

Let $G(x)$ be any entire function and let it be expanded in the form

$$G(x)=a_0+a_1x+\frac{a_2}{2!}x^2+\frac{a_3}{3!}x^3+\dots.$$

We shall show that there exists an entire function $u(x)$ satisfying the relation

$$u(x+1)-u(x)=G(x). \tag{1}$$

Let us write

$$u(x)=u_0+u_1x+\frac{u_2}{2!}x^2+\frac{u_3}{3!}x^3+\dots,$$

and substitute in (1). If we equate the coefficients of like powers of x we have the following infinite system of equations in $u_1, u_2, \dots,$

$$\sum_{j=1}^{\infty}\frac{1}{j!}u_{i+j}=a_i, \qquad i=0,1,2,\dots. \tag{2}$$

It is clear that u_0 is arbitrary.

It is natural to expect that a solution of system (2) is linear in the a's. Accordingly, let us seek a solution in the form

$$u_k=\sum_{l=0}^{\infty}\alpha_{kl}a_l, \qquad k=1,2,\dots. \tag{3}$$

If we substitute the value of u_k from (3) into (2) we have

$$\sum_{j=1}^{\infty}\frac{1}{j!}\sum_{l=0}^{\infty}\alpha_{i+j,l}a_l=a_i, \qquad i=0,1,2,\dots.$$

*It is my purpose in a future paper to develop the general consequences of this method of dealing with non-homogeneous infinite systems of equations.

We shall determine the α's so that each of these equations is a formal identity in the a's. A necessary and sufficient condition for this is that the α's satisfy the doubly infinite system of equations

$$\sum_{j=1}^{\infty} \frac{1}{j!} \alpha_{i+j,\,l} = \delta_{il}, \qquad \begin{cases} i = 0, 1, 2, \ldots., \\ l = 0, 1, 2, \ldots., \end{cases} \tag{4}$$

where δ_{il} is equal to 1 or 0 according as i is or is not equal to l.

For any fixed value of l we have the singly infinite system of equations

$$\sum_{j=1}^{\infty} \frac{1}{j!} \alpha_{i+j,\,l} = \delta_{il}, \qquad i = 0, 1, 2, \ldots. \tag{5}$$

for determining those α's which have the given second subscript l. The $(l+1)$th equation of the set has the second member 1; all the other equations of the set have the second member 0.

We shall now obtain a particular solution $\alpha_{kl} = \bar{\alpha}_{kl}$ of (5) in which $\bar{\alpha}_{kl} = 0$ when $k > l+1$. For this case (5) reduces to the finite system (the order of equations being inverted)

$$\begin{aligned} \bar{\alpha}_{l+1,\,l} &= 1, \\ \bar{\alpha}_{l,\,l} + \frac{1}{2!}\bar{\alpha}_{l+1,\,l} &= 0, \\ \bar{\alpha}_{l-1,\,l} + \frac{1}{2!}\bar{\alpha}_{l,\,l} + \frac{1}{3!}\bar{\alpha}_{l+1,\,l} &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, & \\ \bar{\alpha}_{1l} + \frac{1}{2!}\bar{\alpha}_{2l} + \ldots. + \frac{1}{(l+1)!}\bar{\alpha}_{l+1,\,l} &= 0. \end{aligned} \tag{6}$$

This system may readily be solved by determinants, but a solution more convenient for our purposes may be obtained in the form of a contour integral. If we write

$$\frac{\xi}{e^{\xi} - 1} = c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + \ldots., \tag{7}$$

expand the denominator in powers of ξ, clear of fractions and in the resulting equation equate to zero the coefficients of the different powers of ξ, we have for determining the c's the following equations:

$$\begin{aligned} c_0 &= 1, \\ c_1 + \frac{c_0}{2!} &= 0, \\ c_2 + \frac{c_1}{2!} + \frac{c_0}{3!} &= 0, \\ \ldots\ldots\ldots\ldots & \end{aligned}$$

Comparing this system with (6) we see that $\bar{\alpha}_{l+1-k,\,l} = c_k$. But from (7), by means of the Cauchy theorem on residues, we have

$$\int_C \frac{d\xi}{(e^\xi - 1)\xi^k} = 2\pi\sqrt{-1}\, c_k,$$

where the contour of integration C incloses the point zero and no other singularity of the integrand. Therefore

$$\bar{\alpha}_{l+1-k,\,l} = \frac{1}{2\pi\sqrt{-1}} \int_C \frac{d\xi}{(e^\xi - 1)\xi^k}, \quad k = 0, 1, \ldots, l; \quad \bar{\alpha}_{kl} = 0, \quad k > l+1.$$

Clearly this result may be written in the simpler form

$$\bar{\alpha}_{kl} = \frac{1}{2\pi\sqrt{-1}} \int_C \frac{d\xi}{(e^\xi - 1)\xi^{l+1-k}}, \qquad k = 1, 2, 3. \ldots\ldots \tag{8}$$

If in (8) we let l run over all the numbers 0, 1, 2, ..., we obtain a complete solution of (4). Substituting from (8) into (3) we have

$$u_k = \frac{1}{2\pi\sqrt{-1}} \sum_{s=0}^{\infty} a_s \int_C \frac{d\xi}{(e^\xi - 1)\xi^{s+1-k}}. \tag{9}$$

This result affords a *formal* solution of system (2). That in general it is divergent and hence does not afford an actual solution is easily shown. For we have

$$\frac{\xi}{e^\xi - 1} = \frac{1}{2\pi\sqrt{-1}} \sum_{s=0}^{\infty} \xi^s \int_C \frac{d\xi}{(e^\xi - 1)\xi^s},$$

as one sees readily from the value of c_k obtained above. Now the radius of convergence of this series is not greater than 2π, since $\xi/(e^\xi - 1)$ has a pole at $\xi = 2\pi\sqrt{-1}$. Hence if $a_t = m^t$, as in the case when $G(x) = e^{mx}$, where $m > 2\pi$, series (9) diverges at least for $k = 1$.

This (divergent) formal solution of (2) has been obtained by taking a particular solution of (4). It will now be shown that (4) has an infinitude of solutions, and that if an appropriately chosen particular solution be selected from this totality it will lead to an actual solution of (2).

If we write $\alpha_{kl} = \bar{\alpha}_{kl} + \beta_{kl}$, substitute in (4) and make use of the fact that $\bar{\alpha}_{kl}$ is a solution of (4), it is clear that we have

$$\sum_{j=1}^{\infty} \frac{1}{j!} \beta_{i+j,\,l} = 0, \qquad \begin{cases} i = 0, 1, 2, \ldots., \\ l = 0, 1, 2, \ldots.. \end{cases} \tag{10}$$

Thus for each value of l we have to solve a system of equations of the form

$$\sum_{j=1}^{\infty} \frac{1}{j!} \beta_{i+j} = 0, \qquad i = 0, 1, 2, \ldots.. \tag{11}$$

If $\beta(x)$ is any periodic function of period 1 which is analytic at zero, and we write

$$\beta(x) = \beta_0 + \beta_1 x + \frac{\beta_2}{2!} x^2 + \frac{\beta_3}{3!} x^3 + \ldots,$$

substitute in the equation $\beta(x+1) - \beta(x) = 0$ and equate to zero the coefficients of like powers of x we have system (11). Therefore, for every periodic function of period 1, and analytic at zero, there exists a corresponding formal solution of (11); and this solution is furnished by the coefficients $\beta_1, \beta_2, \ldots$. in the expansion of $\beta(x)$ above. If this expansion converges for $x = 1$ the formal solution becomes an actual solution. This result will be used in constructing an appropriate solution of (10).

From the function $ce^{2n\pi\sqrt{-1}x}$, where c is any constant and n is any positive or negative integer, we have a solution $\beta_k = c(2n\pi\sqrt{-1})^k$ of (11). Now the residue of the function

$$\frac{1}{(e^\xi - 1)\xi^{l-k+1}},$$

at the point $2n\pi\sqrt{-1}$, is $(2n\pi\sqrt{-1})^{k-l-1}$. Hence, if $c = (2n\pi\sqrt{-1})^{-l-1}$, we have β_k equal to the residue in consideration. That is, we have

$$\beta_k = \frac{1}{2\pi\sqrt{-1}} \int \frac{d\xi}{(e^\xi - 1)\xi^{l-k+1}},$$

where the contour of integration incloses the point $2n\pi\sqrt{-1}$ and no other singularity of the integrand.

Now the sum of any finite number of solutions of system (11) is also a solution; hence it is clear that a set of values satisfying (10) is

$$\beta_{kl} = \frac{1}{2\pi\sqrt{-1}} \Sigma \int \frac{d\xi}{(e^\xi - 1)\xi^{l-k+1}}, \qquad \begin{cases} k = 1, 2, \ldots, \\ l = 0, 1, 2, \ldots, \end{cases}$$

where for a fixed l the sum of $2l$ integrals is to be taken, one around each of the $2l$ contours inclosing separately each of the $2l$ points $2n\pi\sqrt{-1}$, $n = \pm 1, \pm 2, \ldots, \pm l$ and no other singularity of the integrand. Combining this value of β_{kl} with that of $\bar{\alpha}_{kl}$ in (8) we have for α_{kl} the value

$$\alpha_{kl} = \frac{1}{2\pi\sqrt{-1}} \int_{C_l} \frac{d\xi}{(e^\xi - 1)\xi^{l-k+1}}, \qquad \begin{cases} k = 1, 2, \ldots, \\ l = 0, 1, 2, \ldots, \end{cases}$$

where the contour of integration C_l incloses the points $2n\pi\sqrt{-1}$, $n = 0$, $\pm 1, \pm 2, \ldots, \pm l$ and no other singularity of the integrand.

Substituting this value of a_{kl} in (3) we have

$$u_k = \frac{1}{2\pi\sqrt{-1}} \sum_{s=0}^{\infty} a_s \int_{C_s} \frac{d\xi}{(e^{\xi}-1)\xi^{s-k+1}}, \qquad k = 1, 2, \ldots . \tag{12}$$

In order to show that this is a valid solution of (2) we must first prove that the series is convergent for every value of k.

Denote by I_{sk} the integral

$$I_{sk} = \int_{C_s} \frac{d\xi}{(e^{\xi}-1)\xi^{s-k+1}}. \tag{13}$$

Choose as the contour of integration the square formed by four lines each at a distance $(2s+1)\pi$ from the origin, two of them being parallel to the axis of reals and the other two being parallel to the axis of imaginaries. Making the transformation $\xi = (2s+1)\pi w$, we have

$$I_{sk} = \frac{1}{(2s+1)^{s-k}\pi^{s-k}} \int_S \frac{dw}{(e^{(2s+1)\pi w}-1)w^{s-k+1}},$$

where the contour S is the square formed by four lines each at a distance unity from the origin, two of them being parallel to the axis of reals and the other two being parallel to the axis of imaginaries. Along the contour of integration $1/|e^{(2s+1)\pi w}-1|$ remains finite; let us say that it is always less than A. Along the path of integration $|w|$ is equal to or greater than 1. Hence

$$|I_{sk}| < \frac{A}{(2s+1)^{s-k}\pi^{s-k}} \int_S |dw| = \frac{8A}{(2s+1)^{s-k}\pi^{s-k}}. \tag{14}$$

From this result it follows that the series in (12) is convergent for every k provided that the same is true of the series

$$\sum_{s=0}^{\infty} \frac{a_s}{(2s+1)^{s-k}\pi^{s-k}}.$$

That this series is convergent follows at once by comparison with the convergent series

$$\sum_{s=0}^{\infty} \frac{a_s}{s!} \frac{1}{\pi^{s-k}},$$

the convergence of the latter being a consequence of the fact that $G(x)$ is entire.

Substituting these values for u_k in the expansion of $u(x)$ and choosing a convenient value for the arbitrary constant u_0, we have the following result:

$$u(x) = \frac{1}{2\pi\sqrt{-1}} \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{s=0}^{\infty} a_s \int_{C_s} \frac{d\xi}{(e^{\xi}-1)\xi^{s-k+1}}. \tag{15}$$

We shall next determine the circle of convergence of this power series.

Let R denote any circle of radius R about the point zero. We shall show that (15) is convergent in this circle R. Form the set of functions

$$u_s(x)=\frac{1}{2\pi\sqrt{-1}}\sum_{k=0}^{\infty}\frac{x^k}{k!}a_s\int_{C_s}\frac{d\xi}{(e^{\xi}-1)\xi^{s-k+1}},\quad s=0,\,1,\,2,\,\ldots\ldots \tag{16}$$

In view of the inequality (14) it is easy to see that each of these series is convergent in the circle R. One has only to compare the sth series with the series

$$\sum_{k=0}^{\infty}\frac{R^k}{k!}a_s\,\frac{8A}{(2s+1)^{s-k}\pi^{s-k}},$$

the convergence of the latter being obvious.

Next we shall show that the series

$$u_0(x)+u_1(x)+u_2(x)+\ldots. \tag{17}$$

is uniformly convergent throughout R. By a theorem of Weierstrass it will then follow that the sum is analytic in R and may be written in the form of the second member of (15). From (14) and (16) we have

$$|u_s(x)|<\alpha_s,\quad |x|\leqq R,$$

where

$$\alpha_s=\frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{R^k}{k!}|a_s|\,\frac{8A}{(2s+1)^{s-k}\pi^{s-k}}.$$

One shows readily that

$$\frac{\alpha_{s+1}}{\alpha_s}=\frac{|a_{s+1}|}{|a_s|}\,\frac{(2s+1)^s}{(2s+3)^{s+1}}\,\frac{1}{\pi}\,\frac{\sum_{k=0}^{\infty}\frac{R^k}{k!}(2s+3)^k\pi^k}{\sum_{k=0}^{\infty}\frac{R^k}{k!}(2s+1)^k\pi^k}$$

$$=\frac{|a_{s+1}|}{|a_s|}\,\frac{(2s+1)^s}{(2s+3)^{s+1}}\,\frac{1}{\pi}\,e^{2R\pi}.$$

Hence

$$\frac{\alpha_{s+1}}{\alpha_s}<\frac{|a_{s+1}|}{|a_s|}\,\frac{1}{2s+3}\,\frac{1}{\pi}\,e^{2R\pi}<\frac{|a_{s+1}|}{|a_s|}\,\frac{1}{s+1}\,\frac{e^{2R\pi}}{2\pi}.$$

Since $G(x)$ is an entire function the series

$$\sum_{s=0}^{\infty}|a_s|\,\frac{1}{s!}\,\bar{R}^s$$

is convergent for every value of $\bar{R}$. Hence the ratio of two consecutive terms,

$$\frac{|a_{s+1}|}{|a_s|}\,\frac{1}{s+1}\,\bar{R},$$

is ultimately less than or equal to one. If $\bar{R} > e^{2R\pi}/2\pi$ this ratio is greater than the ratio α_{s+1}/α_s; and therefore the series

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \dots.$$

is convergent, since α_{s+1}/α_s ultimately becomes and remains less than 1. But $|u_s(x)| < \alpha_s$, $|x| \leqq R$; and therefore series (16) is uniformly convergent throughout R. Hence its sum is analytic throughout R and can be written in the form (15). But R is any circle about the origin. Hence (15) converges throughout the entire plane; that is, $u(x)$ is an entire function.

It is easy to show directly that $u(x)$ as defined in (15) actually satisfies relation (1). We have thus proved the following theorem:

I. *If $G(x)$ is an entire function expansible in the form*

$$G(x) = a_0 + a_1 x + \frac{a_2}{2!} x^2 + \frac{a_3}{3!} x^3 + \dots,$$

then there exist entire functions $u(x)$ satisfying the difference equation

$$u(x+1) - u(x) = G(x).$$

One such entire function is

$$u(x) = \frac{1}{2\pi\sqrt{-1}} \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{s=0}^{\infty} a_s \int_{C_s} \frac{d\xi}{(e^\xi - 1)\xi^{s-k+1}},$$

where the contour of integration C_s incloses the points $2n\pi\sqrt{-1}$, $n = 0, \pm 1, \pm 2, \dots, \pm s$, and no other singularity of the integrand.

Evidently, the general solution of the difference equation is obtained by adding to the above solution any arbitrary periodic function of period 1. If the solution is restricted to be entire this periodic function must be entire.

The value of the integral I_{sk}, defined by equation (13), takes a very simple form when $s - k$ is even. For, then, the residues of the integrand at the two points $2n\pi\sqrt{-1}$ and $-2n\pi\sqrt{-1}$, $n \neq 0$, being respectively $(2n\pi\sqrt{-1})^{k-s-1}$ and $(-2n\pi\sqrt{-1})^{k-s-1}$, are such that the part of the integral which comes from these two residues is zero. Hence

$$I_{sk} = \int_{C_0} \frac{d\xi}{(e^\xi - 1)\xi^{s-k+1}}, \text{ when } s - k \text{ is even.}$$

That is, I_{sk} is in this case $2\pi\sqrt{-1}$ times the coefficient of ξ^{s-k+1} in the development of $\xi/(e^\xi - 1)$ in powers of ξ. Hence, when $s - k = 0$ we have $I_{sk} = -\pi\sqrt{-1}$. It is easy to see that I_{sk} is equal to zero when $s - k$ is even and not zero. To prove this it is sufficient to notice that

$$\frac{\xi}{e^\xi - 1} + \tfrac{1}{2}\xi \text{ or } \tfrac{1}{2}\frac{\xi(e^\xi + 1)}{e^\xi - 1}$$

is an even function; for then the coefficients of odd powers of ξ (greater than the first) in the development of $\xi/(e^{\xi}-1)$ are all zero.

Making use of these results as to the value of I_{sk} we have at once the following theorem:

II. *The function $u(x)$ of Theorem I may be written in the form*

$$u(x) = -\tfrac{1}{2}G(x) + \frac{1}{2\pi\sqrt{-1}} \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{s=0}^{\infty}{}' a_s \int_{C_s} \frac{d\xi}{(e^{\xi}-1)\xi^{s-k+1}},$$

where the accent on the second summation sign indicates that for each value of k only those values of s are to be taken for which $s-k$ is odd.

COROLLARY 1. *If $G(x)$ is an even function* (so that a_s is zero when s is odd), *then the equation*

$$u(x+1) - u(x) = G(x)$$

has a solution which is the sum of $-\frac{1}{2}G(x)$ and an odd function which is entire.

COROLLARY 2. *If $G(x)$ is an odd function* (so that a_s is zero when s is even), *then the equation*

$$u(x+1) - u(x) = G(x)$$

has a solution which is the sum of $-\frac{1}{2}G(x)$ and an even function which is entire.

§ 2. *The Finite Integral of a Function which is Analytic Outside of a Circle, except at Infinity.*

Let us consider the equation

$$u(x) = G(x) + \frac{c}{x-a} + \phi(x) \equiv X(x), \tag{18}$$

where $G(x)$ is an entire function and $\phi(x)$ is analytic outside of a circle of finite radius R about the point a as a center, the development of $\phi(x)$ in negative powers of $x-a$ containing no constant term and no term in $(x-a)^{-1}$. It is evident that a solution $u(x)$ of (18) may be written in the form

$$u(x) = u_1(x) + u_2(x) + u_3(x),$$

where $u_1(x)$, $u_2(x)$, $u_3(x)$ are solutions respectively of the equations

$$\left.\begin{aligned} u_1(x+1) - u_1(x) &= G(x), \\ u_2(x+1) - u_2(x) &= \frac{c}{x-a}, \\ u_3(x+1) - u_3(x) &= \phi(x). \end{aligned}\right\} \tag{19}$$

A solution $u_1(x)$ of the first equation (19), analytic throughout the finite plane, was found in the preceding section. For the second and third equations we have the following two formal solutions:

$$u_2(x) = -\frac{1}{x-a} + \sum_{i=1}^{\infty}\left(\frac{1}{i} - \frac{1}{x-a+i}\right), \quad u_2(x) = \sum_{i=1}^{\infty}\left(\frac{1}{x-a-i} + \frac{1}{i}\right),$$

$$u_3(x) = -\sum_{i=0}^{\infty}\phi(x+i), \qquad u_3(x) = \sum_{i=1}^{\infty}\phi(x-i).$$

Let us consider the first pair of formal solutions for $u_2(x)$ and $u_3(x)$. It is easy to see that there exists a line parallel to the axis of imaginaries to the right of which each of the series is uniformly convergent. Hence to the right of this line $u_2(x)$ and $u_3(x)$, as defined by these series, are analytic except at infinity. From this result and the fact that $u_1(x)$ is analytic throughout the finite plane it follows that $u(x)$ is analytic to the right of the same line. Now, by means of the equation

$$u(x-1) = u(x) - X(x-1),$$

the value of $u(x)$ is known at $x_0 - 1$ when it is known at x_0. In this way the solution obtained above may be defined throughout any finite region of the plane. Thus we see that $u(x)$ is analytic throughout the finite plane except at the singularities of $X(x)$ in the finite plane and the points congruent to them on the left. (A point $\alpha - n$ is said to be congruent to α on the left if n is a positive integer; similarly, we say that $\alpha + n$ is congruent to α on the right if n is a positive integer.)

In a similar way we may treat the second pair of formal solutions for $u_2(x)$ and $u_3(x)$. Here we consider first the part of the plane which is to the left of a suitably chosen line parallel to the axis of imaginaries. It is clear that the solution $u(x)$ of (18), which we obtain in this case, is analytic throughout the finite plane except at the points congruent on the right to the singularities of $X(x)$ in the finite plane.

Thus we have the following theorem:

III. *Let $X(x)$ be any function of x which is analytic outside of a circle about the center a of finite radius R, except possibly at infinity. Then there exists a function $u(x)$ satisfying the relation*

$$u(x+1) - u(x) = X(x)$$

and having either one of the following two properties:

1. *It is analytic throughout the finite plane except at the singularities of $X(x)$ in the finite plane and the points congruent to them on the left.*

2. *It is analytic throughout the finite plane except at points congruent on the right to the singularities of $X(x)$ in the finite plane.*

§3. *The Finite Integral of a Function which is Analytic within a Circle.*

Let $\psi(x)$ be any function which is analytic within a circle R of radius R about the point a as a center. A method due to Weber * gives us in this case an immediate solution of the problem of finding a function $u(x)$ which satisfies the equation

$$u(x+1)-u(x)=\psi(x)$$

and is analytic in any circle within the circle R. Let $\bar{R}$ be any circle about a as a center and having a radius less than that of the circle R. Denote by $(R\bar{R})$ the circular ring between these two circles. On a line through a and parallel to the axis of imaginaries take two points A and B, lying within the circular ring $(R\bar{R})$, the imaginary coefficient of A being less than that of a and the imaginary coefficient of B being greater than that of a. Let x be any point in the circle $\bar{R}$ and denote by l the line through the points $\ldots . x-1$, x, $x+1$, $\ldots .$

Now form the integral

$$u(x)=\int_C \frac{\psi(z)\,dz}{1-e^{2\pi i(x-z)}}, \qquad i=\sqrt{-1}, \tag{20}$$

where the path of integration C extends from A to B, lies entirely within the circle R, passes through a point P lying on the line l between $x-1$ and x and has no other point in common with the line l. Then, on account of the periodicity of the integrand considered as a function of x, it is clear that

$$u(x+1)=\int_{\bar{C}} \frac{\psi(z)\,dz}{1-e^{2\pi i(x-z)}},$$

where the path of integration is as in the preceding case except that the point P is now between x and $x+1$. The difference $u(x+1)-u(x)$ is expressible as a contour integral taken along the curve made up of $\bar{C}$ taken in a positive sense and C taken in a negative sense, the integrand being the same as above. Hence, by Cauchy's theorem, $u(x+1)-u(x)=\psi(x)$. But $u(x)$, as defined by (20), is evidently an analytic function of x at every point x in the circle R. Hence we have the following theorem:

IV. *If $\psi(x)$ is analytic within a circle of radius R about the point a and $\bar{R}$ is any circle about a with radius less than R, then there exists a function $u(x)$ which is analytic throughout $\bar{R}$ and satisfies the relation*

$$u(x+1)-u(x)=\psi(x).$$

Such a function is defined in equation (20).

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* *Acta Mathematica*, Vol. XXVII (1903), p. 226.

§ 4. *The Finite Integral of a Function which is Analytic in a Circular Ring.*

If $\phi(x)$ is any function which is analytic within a circular ring we may deal with the problem of solving the equation

$$u(x+1)-u(x)=\phi(x)$$

in a manner similar to that employed in the preceding section. We suppose that $\phi(x)$ is analytic within the circular ring bounded by two circles each having the center a, the radii being r and R, where $r+1<R$. Take any two quantities $\bar{r}$ and $\bar{R}$ such that

$$r<\bar{r}, \quad \bar{r}+1<\bar{R}<R,$$

and form circles about a with radii $\bar{r}$ and $\bar{R}$. Denote by r, $\bar{r}$, $\bar{R}$, R the circles having the radii r, $\bar{r}$, $\bar{R}$, R respectively. Take points A and B lying within the circular ring bounded by $\bar{R}$ and R and lying on a line through a parallel to the axis of imaginaries, A being below and B above a. Let us define a region H in the following manner: Let the line AB cut $\bar{r}$ and $\bar{R}$ in the points L, l, m, M, these points being named in the order of increasing imaginary coefficients. From l and m draw tangents to $\bar{r}$ proceeding to the left and cutting $\bar{R}$ in the points T_1 and T_2 respectively. The region H has the following boundary: The arc T_1LMT_2 of the circle $\bar{R}$, the tangent T_2m, the right-hand semi-circle ml, the tangent lT_1. Let x be a point in the region H and let λ be the line through the points $\ldots. x-1, x, x+1, \ldots.$ Form the function

$$u(x)=\int_C \frac{\phi(z)\,dz}{1-e^{2\pi i(x-z)}}, \tag{21}$$

where the path of integration C extends from A to B, lies wholly in the ring bounded by r and R, cuts the line λ in a point P between $x-1$ and x, and has no other point in common with λ.

It is easy to show, by the method employed in the preceding section, that the function $u(x)$, as thus defined, satisfies the relation $u(x+1)-u(x)=\phi(x)$. It is evidently analytic at every point of the region H.

Now

$$u(x-1)=u(x)-\phi(x-1).$$

By means of this equation the value of $u(x)$ at a point x_0-1 can be found when its value at x_0 is known. In this way we may determine the function $u(x)$ in that part of the circular ring between r and R in which it is not already defined. It is thus clear that $u(x)$ is analytic throughout this circular

ring except (possibly) at the points congruent on the left to the singularities of $\phi(x)$ in r.

The right and left half-planes, as divided by the line AB, enter unsymmetrically into the above discussion. It is clear that we may make a similar discussion of the problem in which the two half-planes exchange rôles.

Thus we have the following two-fold theorem:

V. *If $\phi(x)$ is any function which is analytic within a circular ring bounded by two circles about a as center and having radii r and R where $r+1<R$, and if any other concentric circular ring is formed with radii $\bar{r}$ and $\bar{R}$, where $r<\bar{r}$, $\bar{r}+1<\bar{R}<R$, then there exists a function $u(x)$ satisfying the relation*

$$u(x+1)-u(x)=\phi(x)$$

and having either of the following properties:

1. *It is analytic throughout the ring bounded by the circles $\bar{r}$ and $\bar{R}$ except at points congruent on the left to the singularities of $\phi(x)$ which lie in the circle r.*

2. *It is analytic throughout the ring bounded by the circles $\bar{r}$ and $\bar{R}$ except at points congruent on the right to the singularities of $\phi(x)$ which lie in the circle r.*

II. Entire Functions Defined by Linear Homogeneous Equations with Rational Coefficients.

Let us consider a single difference equation of the nth order

$$R_0(x)H(x+n)+R_1(x)H(x+n-1)+\ldots.+R_n(x)H(x)=\phi(x),$$

where $R_0(x), R_1(x), \ldots., R_n(x)$ are rational functions. Let α be the least integer such that $x^{-i\alpha}R_i(x)/R_0(x)$ is analytic at infinity for every value of i, $i=1, 2, \ldots., n$; and let R_i be the constant term in the expansion of $x^{-i\alpha}R_i(x)/R_0(x)$ in powers of x^{-1}. We assume that the constants R_i are such that the roots $\rho_1, \rho_2, \ldots., \rho_n$ of the characteristic equation

$$\rho^n+R_1\rho^{n-1}+R_2\rho^{n-2}+\ldots.+R_{n-1}\rho+R_n=0$$

are in absolute value different from each other and from zero. An equation having these properties, whether homogeneous or non-homogeneous, is here said to be of *simple character*. In what follows we confine attention for the most part to equations of simple character.

If the equation (whether of simple character or not) is homogeneous and the coefficients are constants it is easy to determine by elementary means a fundamental system of solutions, each function of which is entire. Guichard* has gone further and has shown that if the coefficients are constants and the independent term $\phi(x)$ is any entire function, then there exists a particular solution of the equation which is an entire function. In this case the general solution can be expressed in the form

$$H(x) = E(x) + p_1(x)E_1(x) + \ldots + p_n(x)E_n(x),$$

where $E(x), E_1(x), \ldots, E_n(x)$ are entire functions and $p_1(x), \ldots, p_n(x)$ are arbitrary periodic functions of period unity.

The question naturally arises as to whether corresponding theorems are true for equations in which the coefficients are no longer constants, but are restricted to be rational functions. Such is the subject of investigation in this and the following section of the present paper.

The theorem on page 133 of my dissertation,† taken in connection with the remarks which follow, leads at once to a theorem which we shall need; it may be stated thus:

If

$$P_0(x)H(x+n) + P_1(x)H(x+n-1) + \ldots + P_n(x)H(x) = 0$$

is a homogeneous difference equation of simple character in which the coefficients $P_0(x), P_1(x), \ldots, P_n(x)$ are polynomials in x, then

1) *There exists a fundamental system of solutions $H_1(x), \ldots, H_n(x)$ each function of which is analytic throughout the finite plane except at the zeros of $P_n(x)$ and the points congruent to them on the left. If x approaches infinity in the positive direction along any line parallel to the real axis, we have*

$$\lim H_j(x)\{\Gamma(x)\}^{-\alpha}a_j^{-x}x^{-m_j} = 1, \qquad j = 1, \ldots, n,$$

where α, a_j, m_j are appropriately determined constants;

2) *There exists a fundamental system of solutions $\overline{H}_1(x), \ldots, \overline{H}_n(x)$ each function of which is analytic throughout the finite plane except at the zeros of $P_0(x-n)$ and the points congruent to them on the right. If x approaches infinity in the negative direction along any line parallel to the real axis, we have*

$$\lim \overline{H}_j(x)\{\overline{\Gamma}(x)\}^{-\beta}b_j^{-x}x^{-n_j} = 1, \qquad j = 1, \ldots, n,$$

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* *Annales de l'École Normale Supérieure* (3), 4 (1887), pp. 368–370.

† *Transactions of the American Mathematical Society*, Vol. XII (1911), pp. 99–134.

where β, b_j, n_j *are appropriately determined constants and*

$$\bar{\Gamma}(x) = (1 - e^{2\pi i x})\Gamma(x), \qquad i = \sqrt{-1}.$$

From either of these fundamental systems of solutions of (1) *we may obtain* (at once and in an obvious way) *a fundamental system which consists of entire functions;* for this purpose, it is sufficient to multiply each of the given solutions by an appropriate periodic function of period 1 so that the zeros of the latter will balance off the poles of the former, thus giving a product function which has no singularity in the finite plane.

Thus we see that all functions defined by linear homogeneous difference equations of simple character can be expressed in terms of entire functions and functions which are periodic of period 1. That the entire functions defined by an equation are sometimes not the most interesting ones which the equation may define is evidenced by the importance of the gamma function. It has been found of much more use than any entire function which satisfies the same difference equation as the gamma function itself.

That non-homogeneous equations, even when the coefficients and the independent term are all polynomials, do not always possess a solution which is an entire function will be shown in part III by means of an example.

III. Applications of the Preceding Results to Non-homogeneous Equations.

The results of the preceding parts of the paper will now be applied to an investigation of properties of the solutions of non-homogeneous equations whose coefficients are polynomials in x but whose independent term is a meromorphic function or an entire function of x. The results, other than the negative one afforded by the following example, are stated in Theorems I and II.

That non-homogeneous difference equations in which the coefficients and the independent term are polynomials do not always possess solutions which are entire functions is shown by the equation

$$xg(x+1) - (x+1)g(x) = x,$$

the general solution of which may be written in the form

$$g(x) = x\left\{p(x) + \sum_{i=1}^{\infty}\left(\frac{1}{x+1-i} + \frac{1}{i}\right)\right\},$$

where $p(x)$ is an arbitrary periodic function of x of period 1. It is obvious that $g(x)$ has singularities in the finite plane for every choice of $p(x)$. But if $p(x)$ is any periodic function which has in the finite plane no singularity other

than poles, then the corresponding $g(x)$ is analytic throughout the finite plane except for poles.

This example suggests the following theorem, which will now be proved:

I. *If*

$$P_0(x)H(x+n)+P_1(x)H(x+n-1)+\ldots.+P_n(x)H(x)=\phi(x) \qquad (1)$$

is a difference equation of simple character in which the coefficients are polynomials in x, while $\phi(x)$ is analytic throughout the finite plane except for poles, then its general solution $H(x)$ may be written in the form

$$H(x)=M(x)+p_1(x)E_1(x)+\ldots.+p_n(x)E_n(x),$$

where $E_1(x), \ldots., E_n(x)$ are entire functions, $M(x)$ is analytic throughout the finite plane except for poles, and $p_1(x), \ldots., p_n(x)$ are arbitrary periodic functions of period 1.

In section II we have shown that the corresponding homogeneous equation (that is, the equation obtained from (1) by writing zero for $\phi(x)$) has a fundamental system of solutions each function of which is entire. We let $E_1(x), \ldots., E_n(x)$ be the functions of this solution. It is now obvious that our theorem will have been demonstrated when we have proved the existence of a single function $M(x)$ satisfying the non-homogeneous equation (1) and possessing the required character of being analytic throughout the finite plane except for poles. We shall now prove the existence of such a function. Our method is the classic one of variation of parameters.

We seek a particular solution $\overline{H}(x)$ of (1) in the form

$$\overline{H}(x)=\alpha_1(x)E_1(x)+\alpha_2(x)E_2(x)+\ldots.+\alpha_n(x)E_n(x). \qquad (2)$$

In order that $\overline{H}(x)$ shall be a solution of (1) it is sufficient that the functions $\alpha_i(x)$ satisfy the set of equations

$$\begin{aligned}
&E_1(x+1)\Delta\alpha_1(x)+\ldots.+E_n(x+1)\Delta\alpha_n(x)=0,\\
&E_1(x+2)\Delta\alpha_1(x)+\ldots.+E_n(x+2)\Delta\alpha_n(x)=0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&E_1(x+n-1)\Delta\alpha_1(x)+\ldots.+E_n(x+n-1)\Delta\alpha_n(x)=0,\\
&E_1(x+n)\Delta\alpha_1(x)+\ldots.+E_n(x+n)\Delta\alpha_n(x)=\phi(x).
\end{aligned} \qquad (3)$$

This may readily be shown by the method usually employed in finding a solution by means of variation of parameters. It may easily be verified directly; for, from (2) and the first equation in (3) one has

$$\overline{H}(x+1)=\alpha_1(x)E_1(x+1)+\ldots.+\alpha_n(x)E_n(x+1).$$

By means of this equation and the second one in (3) one obtains the result

$$\overline{H}(x+2) = \alpha_1(x)E_1(x+2) + \ldots + \alpha_n(x)E_n(x+2).$$

The process may evidently be continued till $\overline{H}(x+n)$ is found. Then a direct substitution in (1) will show that that equation is satisfied.

The determinant of (3) is not identically zero; for this is the condition that $E_1(x), \ldots, E_n(x)$ form a fundamental system of solutions of the homogeneous equation corresponding to (1). Hence (3) can be solved for $\Delta\alpha_1(x), \ldots, \Delta\alpha_n(x)$. Thus we have

$$\Delta\alpha_i(x) = M_i(x), \qquad i = 1, \ldots, n, \tag{4}$$

where the functions $M_i(x)$, $i = 1, \ldots, n$, are obviously analytic throughout the finite plane except for poles. From Hurwitz' theorem, quoted above (p. 163), it follows then that there exist functions $\alpha_i(x)$, $i = 1, \ldots, n$, having the same property. If these functions $\alpha_i(x)$ are put in (2) we have a function $\overline{H}(x)$ satisfying (1) and having the property of being analytic throughout the finite plane except for poles. We let $\overline{H}(x)$ be the function $M(x)$ of the theorem. Thus the proof of the theorem is completed.

We shall now turn our attention to a more restricted form of equation (1). We suppose that $\phi(x)$ is an entire function and that $P_n(x) \equiv 1$. We write

$$P_0(x)H(x+n) + \ldots + P_{n-1}(x)H(x+1) + H(x) = \phi(x). \tag{1'}$$

Then according to the theorem quoted in part II above, there exists a fundamental system of solutions,

$$H_1(x), \quad H_2(x), \ldots, \quad H_n(x),$$

of the corresponding homogeneous equation each function of which is analytic throughout the finite plane; and these functions verify the relations

$$\lim H_j(x)\{\Gamma(x)\}^{-\alpha}\rho_j^{-x}x^{-m_j} = 1, \qquad j = 1, \ldots, n, \tag{5}$$

where the limit is taken for x approaching infinity in the positive sense along any line parallel to the real axis. Here the constants ρ_j, $j = 1, \ldots, n$, are the roots of the associated algebraic equation and m_j, $j = 1, \ldots, n$, are properly determined constants.

We seek now a solution $H^1(x)$ of (1') of the form

$$H^1(x) = \alpha_1(x)H_1(x) + \ldots + \alpha_n(x)H_n(x).$$

Corresponding to equations (3) we have now the system

$$\begin{aligned} &H_1(x+1)\Delta\alpha_1(x)+\ldots.+H_n(x+1)\Delta\alpha_n(x)=0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ &H_1(x+n-1)\Delta\alpha_1(x)+\ldots.+H_n(x+n-1)\Delta\alpha_n(x)=0,\\ &H_1(x+n)\Delta\alpha_1(x)+\ldots.+H_n(x+n)\Delta\alpha_n(x)=\phi(x). \end{aligned} \tag{6}$$

The determinant of this system is $D(x+1)$ where

$$D(x)=\begin{vmatrix} H_1(x) & H_2(x) & \ldots.H_n(x)\\ H_1(x+1) & H_2(x+1) & \ldots.H_n(x+1)\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ H_1(x+n-1) & H_2(x+n-1) & \ldots.H_n(x+n-1) \end{vmatrix}.$$

If we employ the notation $\lim \psi(x)$ in the same sense as in (5), then in view of (5) and the definition of $D(x)$ it is easy to prove that

$$\lim D(x)\{\Gamma(x)\Gamma(x+1)\ldots. \Gamma(x+n-1)\}^{-a}(\rho_1\rho_2\ldots.\rho_n)^{-x}x^{-m_1-m_2\ldots.-m_n}=D, \tag{7}$$

where

$$D=\begin{vmatrix} 1 & 1 & \ldots.1\\ \rho_1^{-1} & \rho_2^{-1} & \ldots.\rho_n^{-1}\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ \rho_1^{-n+1} & \rho_2^{-n+1} & \ldots.\rho_n^{-n+1} \end{vmatrix},$$

whence D is a constant different from zero, since the quantities $\rho_1, \rho_2, \ldots., \rho_n$ are all distinct.

By Stirling's formula for the gamma function we have

$$\lim \Gamma(x+r)\,(x+r)^{-x-r+\frac{1}{2}}e^{x+r}=\sqrt{2\pi},$$

where r is any integer. Hence we have readily

$$\lim \Gamma(x+r)x^{-x}e^{x}x^{-r+\frac{1}{2}}=\sqrt{2\pi}.$$

Taking this result in connection with (7) it is easy to see that there exist constants $a, m, c, c\neq 0$, such that

$$\lim D(x)x^{nax}a^{-x}x^{-m}=c. \tag{8}$$

Now $D(x)$ satisfies a difference equation of the first order which may be found in the following manner: In the determinant defining $D(x)$ replace x by $x+1$ and then substitute for $H_i(x+n)$ its value

$$H_i(x+n)=-\frac{1}{P_0(x)}[P_1(x)H_i(x+n-1)+\ldots.+H_i(x)]$$

gotten by means of the homogeneous equation corresponding to (1). One thus obtains the result

$$D(x+1) = (-1)^n \frac{D(x)}{P_0(x)}.$$

If we make the transformation

$$\overline{D}(x) = D(x) x^{nax} a^{-x} x^{-m}$$

we have readily the result that

$$\overline{D}(x+1) = \{1 + \psi(x)\}\overline{D}(x), \qquad \lim \overline{D}(x) = c, \tag{9}$$

where $\psi(x)$ is analytic at infinity and is expansible in the form

$$\psi(x) = \psi'' x^{-2} + \psi''' x^{-3} + \ldots, \qquad |x| \geqq R,$$

R being an appropriately chosen constant. That $\overline{D}(x)$ is uniquely defined by the fact that it satisfies relations (9) is easily shown. For, if $\overline{D}_1(x)$ is any function satisfying the given difference equation then the most general solution of that equation is $p(x)\overline{D}_1(x)$, where $p(x)$ has the period 1. The existence and value of the limits $\lim \overline{D}_1(x) = c$, $\lim p(x)\overline{D}_1(x) = c$ require that $p(x)$ shall equal 1. Hence $\overline{D}(x)$ is uniquely defined by relations (9).

In view of the general theorem in § 1 of my dissertation, p. 108, it is now clear that $\overline{D}(x)$ is such that there exists a line parallel to the axis of imaginaries to the right of which $\overline{D}(x)$ has no zero. Then, from the relation between $\overline{D}(x)$ and $D(x)$ it follows that there exists a line l to the right of which $D(x)$ has no zero.

From these results we conclude that system (1′) has a solution in the form

$$\Delta\alpha_i(x) = M_i(x), \qquad i = 1, \ldots, n, \tag{10}$$

where the functions $M_i(x)$ have no singularities to the right of the line l. A theorem of Hurwitz* then states that there exist functions $\alpha_i(x)$ satisfying equations (10) and having the property that there exists a line λ, parallel to the axis of imaginaries, to the right of which no one of these functions has a singularity. Hence, if we take such values of $\alpha_i(x)$, the function $H^1(x)$ has no singularities to the left of the line λ. Then, making use of the fact that $H^1(x)$ satisfies (1′), we conclude readily that $H^1(x)$ has no singularity in the finite plane. Hence the general solution of (1′) may be written in the form

$$H(x) = H^1(x) + p_1(x)H_1(x) + \ldots + p_n(x)H_n(x),$$

where $p_1(x), \ldots, p_n(x)$ are arbitrary periodic functions of period 1 and the functions $H^1(x), H_1(x), \ldots, H_n(x)$ have no singularities in the finite plane.

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* *Acta Mathematica* Vol. XX (1897), pp. 308-310.

In a similar manner it may be shown that if we modify (1) by taking $P_0(x)$ instead of $P_n(x)$ equal to 1 and making $\phi(x)$ entire as before, we obtain the same general result.

Thus we have the following theorem:

II. *If*

$$P_0(x)H(x+n)+P_1(x)H(x+n-1)+\ldots.+P_n(x)H(x)=\phi(x)$$

is a difference equation of simple character in which $\phi(x)$ *is an entire function and the coefficients* $P_i(x)$, $i=0, 1, \ldots., n$, *are polynomials in* x, *and if either* $P_0(x)\equiv 1$ *or* $P_n(x)\equiv 1$, *then the general solution* $H(x)$ *may be written in the form*

$$H(x)=E(x)+p_1(x)E_1(x)+\ldots.+p_n(x)E_n(x),$$

where $E(x)$, $E_1(x)$, $\ldots.$, $E_n(x)$ *are entire functions and* $p_1(x)$, $\ldots.$, $p_n(x)$ *are arbitrary periodic functions of period* 1.

INDIANA UNIVERSITY, *April*, 1912.

On the Product of Two Quadro-Quadric Space-Transformations.

By Hilda P. Hudson.

Although the analogy of the plane theory calls attention to the composition and decomposition of space-transformations, yet the subject has received very slight attention except from Herr S. Kantor,* who has dealt fully with the general case of composition of quadro-quadric transformations. As so often in this theory, the more specialized cases are in some ways more interesting, and worth considering in detail.

The object of this paper is to show that the transformation compounded of two quadro-quadric transformations belongs to one of the following types:

(1) A quarto-quartic, whose fundamental system in either space consists of a double and a simple point, a nodal conic and a simple quartic.
(2) A cubo-quartic, with, in the first space, a double and a simple point, a simple cubic and conic.
(3) A cubo-cubic, with a double point, a simple quartic and conic.
(4) A cubo-cubic, with a simple cubic, conic and straight line.
(5) The general cubo-cubic with a nodal line.
(6) The general quadro-quartic.
(7) The general quadro-cubic.
(8) The general quadro-quadric.
(9) The general linear transformation, including identity.

In the quadro-quadric transformation, the homaloidal family in one space, corresponding to planes in the other, consists of quadrics passing through a fixed conic c and a fixed point p. If the plane of c is taken as $w = 0$, and the coordinates of p are $(0, 0, 0, 1)$, the equations of transformation are those given below (A). There are two distinct ways of specializing this transformation. In the first place, c may break up into a line-pair $a\,b$; equations (A) still apply, and the quadratic function f which occurs in them falls into linear

* Kantor, American Journal of Mathematics, Vol. XVII (1895), p. 219; Vol. XIX (1897), p. 1; *Acta Math.*, Vol. XXI (1896), p. 1. See also Noether, *Math. Ann.*, Vol. III (1871), p. 547; Del Pezzo, *Rendic. R. Acc. Napoli* (3), Vol. II (1896), p. 288; Beloch, *Ann. di Mat.* (3), Vol. XVI (1909), p. 64; Tummarello, *Rendic. R. Acc. Napoli* (3), Vol. XVII (1911), p. 439.

factors. In the second place, the fundamental point p may lie on c; then equations (A) cease to be applicable. In this case the quadrics all touch a fixed plane, taken as $z=0$, different from that of c, at the fixed point p of c. If also c breaks up, the quadrics touch the plane $w=0$ at the point of intersection of $a\,b$, so that p can not lie at this point. In all cases the fundamental systems in the two spaces are of the same nature.

I. Consider then the transformation between the first space and an intermediate space, given by the equations

$$\left.\begin{aligned} x' : y' : z' : w' &= x\,w : y\,w : z\,w : f_2(x, y, z), \\ x : y : z : w &= x'\,w' : y'\,w' : z'\,w' : f_2(x', y', z'), \end{aligned}\right\} \quad (A)$$

with fundamental systems

$$\begin{aligned} (c)\ 0 = w &= f, & (p)\ 0 = x &= y = z; \\ (c')\ 0 = w' &= f', & (p')\ 0 = x' &= y' = z'; \end{aligned}$$

and a second transformation between the intermediate and final spaces, given by

$$X : Y : Z : W = X'\,W' : Y'\,W' : Z'\,W' : F_2(X', Y', Z'), \text{ etc.},$$

with fundamental systems C', P'; C, P; where $x'\,y'\,z'\,w'$, $X'\,Y'\,Z'\,W'$ refer to different frames of reference, both in the intermediate space, and are connected by either of the equivalent sets of relations

$$\begin{aligned} X' &= a_1 x' + b_1 y' + c_1 z' + d_1 w', & x' &= A_1 X' + B_1 Y' + C_1 Z' + D_1 W', \\ Y' &= a_2 x' + b_2 y' + c_2 z' + d_2 w', & y' &= A_2 X' + B_2 Y' + C_2 Z' + D_2 W', \\ Z' &= a_3 x' + b_3 y' + c_3 z' + d_3 w', & z' &= A_3 X' + B_3 Y' + C_3 Z' + D_3 W', \\ W' &= a_4 x' + b_4 y' + c_4 z' + d_4 w', & w' &= A_4 X' + B_4 Y' + C_4 Z' + D_4 W'. \end{aligned}$$

Compounding these, we have the quarto-quartic transformation No. 1, between the first and final spaces, given by

$$\begin{aligned} X : Y : Z : W &= \{(a_1 x + b_1 y + c_1 z)\,w + d_1 f\}\,\{(a_4 x + b_4 y + c_4 z)\,w + d_4 f\} \\ &\quad : \{(a_2 x + b_2 y + c_2 z)\,w + d_2 f\}\,\{(a_4 x + b_4 y + c_4 z)\,w + d_4 f\} \\ &\quad : \{(a_3 x + b_3 y + c_3 z)\,w + d_3 f\}\,\{(a_4 x + b_4 y + c_4 z)\,w + d_4 f\} \\ &\quad : F_2[\{(a_1 x + b_1 y + c_1 z)\,w + d_1 f\}, \{(a_2 x + b_2 y + c_2 z)\,w + d_2 f\}, \\ &\qquad\qquad \{(a_3 x + b_3 y + c_3 z)\,w + d_3 f\}], \end{aligned}$$

with a fundamental system h in the first space consisting of: a double point p; a simple point q corresponding to P', given by

$$\frac{1}{d_1}(a_1 x + b_1 y + c_1 z) = \frac{1}{d_2}(a_2 x + b_2 y + c_2 z) = \frac{1}{d_3}(a_3 x + b_3 y + c_3 z) = -\frac{f}{w};$$

a nodal conic c; and a simple quartic c_4 corresponding to C', having two branches through p and meeting c four times.

The results for the reverse transformation are similar.

Either or both of the component transformations can be specialized in the ways mentioned above; any combination of these gives a particular case of No. 1, with a corresponding specialization of h.

Specialization also occurs when the two fundamental systems in the intermediate space are specially related. If c', C' intersect in k points ($k = 1, 2$ or 4), then c_4 breaks up into k generators of the cone (p, c) and a proper curve c_{4-k}. Any other relation of incidence gives a different transformation of lower degree, and not a particular case of No. 1. These transformations of lower degree occur when the quartic homaloidal family (s_4) in the first space breaks up into a fixed part and a family of lower degree. The fixed part corresponds to part of the system c', p', which must lie on all the quadrics of the second family in the intermediate space (corresponding to planes in the final space), and must therefore coincide with part of the system C', P'. Besides (I, above) the arrangements which give no reduction of degree, which have been sufficiently discussed, we have to consider the following cases and their combinations:

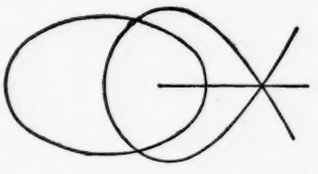
II. Component transformations not specialized.
- (i) p' lies on C'.
- (ii) $p' \equiv P'$.
- (iii) $c' \equiv C'$.

III. Components specialized; c', C' break up with a common line $b' \equiv B'$.
- (i) No further specialization.
- (ii) p' lies on A'.
- (iii) $p' \equiv P'$.

IV. Components specialized; p' lies on c'.

II. *Components not specialized.* (i) *p' lies on C'.* In the first space, s_4 breaks up into the plane of c and a cubic surface with a node at p and containing c, q; instead of c_4 we have a plane cubic curve with a node at p. In the final space, S_4 does not break up, but instead of Q (representing p') it contains a generator of the cone (P, C). We have the cubo-quartic transformation No. 2.

If also P' lies on c', in the first space q is replaced by a generator of the cone (p, c); the transformation is cubo-cubic No. 4. The node of s_3 at p is now a necessary consequence of the passage of the surface through this generator and the cubic curve. This is a particular case of the bilinear transformation in which the fundamental sextic breaks up as in the diagram, which shows the actual intersections of the system h.



This includes the reciprocal transformation in which the sextic breaks up into the six edges of a tetrahedron. Analytically, the transformation

$$X : Y : Z : W = \frac{1}{x} : \frac{1}{y} : \frac{1}{z} : \frac{1}{w}$$

can be compounded of

$$x' : y' : z' : w' = xw : yw : zw : yz \text{ and } X : Y : Z : W = y'z' : z'x' : x'y' : x'w'.$$

Here, in the intermediate space, both conics break up, and each fundamental point lies at the point of intersection of the other line-pair.

(ii) $p' \equiv P'$. In the first space, s_4 breaks up into the plane of c and a cubic surface with a node at p and containing c, c_4; the simple point q disappears. The reverse is similar, and we have the cubo-cubic transformation No. 3.

(iii) $c' \equiv C'$. Here s_4 breaks up into the cone (p, c) and a quadric containing c, q; we have a general quadro-quadric transformation No. 8. If also $p' \equiv P'$, then s_4 breaks up into the cone (p, c), the plane of c and a variable plane; and we have the general linear transformation No. 9.

III. $c' \equiv a' + b'$, $C' \equiv A' + b'$. The cone (p, c) breaks up into two planes $p\,a$, $p\,b$ corresponding to a', b'; then s_4 breaks up into the plane $p\,b$ and a cubic surface s_3 with a as a nodal line and containing b, p, q.

(i) *No further specialization.* To A' corresponds a pair of straight lines, viz., to the point $A'\,b'$, a generator lying on the plane $p\,b$ and not on s_3, and to the rest of A', a straight line lying on s_3, meeting a and this generator. We have the general cubo-cubic transformation No. 5, with a fundamental nodal line, two simple lines meeting it and two points.

If also the point $a'\,b'$ coincides with the point $A'\,b'$, the two simple lines meet on the nodal line; if the line-pairs are coplanar, one of the points lies on one of the simple lines; *i. e.*, at this point s_3 touches a fixed plane.

(ii) *p' lies on A'.* Then s_3 breaks up into the plane of c and a quadric s_2 containing a, p, q. Since A' is a generator of the cone (p', c'), there corresponds to it a single point of b through which s_2 passes. In the final space, Q is replaced by a generator, so that P is not an isolated point. We have the general quadro-cubic transformation No. 7.

If also P' lies on a', the transformation is quadro-quadric.

(iii) $p' \equiv P'$. Here s_3 breaks up into the plane of c and a quadric; the transformation is quadro-quadric.

IV. *p' lies on c'.* Then the cone (p', c') breaks up into the plane $w' = 0$ of the conic and another plane $z' = 0$ which touches the first quadric family in

the intermediate space at p'. A surface containing c' has the degree of its corresponding surface diminished by two, as if it did not contain p', unless it touches z', when the degree is diminished by three.

The only new case occurs when the quadrics of the second family in the intermediate space touch z' at p'. This can happen in two ways:

(i) P' lies on C' and coincides with p', the planes Z', z' also coinciding; this leads to a quadro-quadric transformation of the same type as the components.

(ii) $C' \equiv A' + B'$, and the line-pair lie in the plane z' and intersect at p'. Then, in the first space, s_4 breaks up into the plane of c counted twice and a quadric touching z at p and containing q and two points corresponding to A', B'. In the final space, S_4 has P as a double point and A, B as nodal lines; to c' there correspond a double line joining P to the point (A, B) and a conic through P meeting A, B. We have the general quadro-quartic transformation No. 6.

Analytically, the transformation

$$X : Y : Z : W = yz \quad : zx \quad : xy \quad : w(px+qy+rz),$$
$$x : y : z : w = YZF : ZXF : XYF : XYZW, \qquad F \equiv pYZ + qZX + rXY,$$

can be compounded of

$$x : y : z : w = x'p' : y'p' : z'p' : x'w', \qquad p' \equiv px' + qy' + rz',$$
$$x' : y' : z' : w' = YZ : ZX : XY : XW.$$

As an example of the possible extension of the theory, consider all the cubic transformations with a fundamental nodal line a. There are four such: a cubo-quintic with six fundamental points besides the nodal line, a cubo-quartic with a generator and four points, a cubo-cubic with two generators and two points, and a cubo-quadric with three generators. To the cubic homaloidal family of the first of these apply a quadro-quadric transformation whose fundamental system consists of p (one of the six points), a and another line b meeting a and passing through q (another of the six points). The family is transformed into another cubic homaloidal family with b' as a nodal line, and, corresponding to q, a simple line meeting b' and absorbing the simple point p', and four other simple points. This is the homaloidal family of the second of the transformations. If we now repeat the process, we again lose two points and gain a line, and the family becomes the homaloidal family of the cubo-cubic transformation. Another repetition leads to the last homaloidal family with three generators, and as there are no fundamental points left, we can not continue the process; but at this or any stage after the first, we can use the nodal

line, a generator and any other common point as the fundamental system of a quadro-quadric transformation which transforms the family into a family of quadrics.

Since all quadric transformations are compounded of quadro-quadric ones, it follows that if the homaloidal family of a given transformation can be converted into a quadric family by a series of quadric transformations, then the given transformation is compounded of quadro-quadric transformations only.

With reference to the list of cubic transformations given in a previous paper,* and the statements there made as to their composition, it is clear that a cubic family can be converted into a quadric family if it belongs to one of the following species; viz., the fundamental system contains either:

(i) a nodal line (Nos. 1, 5, 11, 27 of that list), or

(ii) a conic and a node not lying on it (Nos. 21, 30, 36, 38, 57), or

(iii) a conic touching the fixed plane at a binode (Nos. 16, 20, 22, 33, 37, 45, 49, 58).

In Nos. 46, 61, using two rays and a point of contact as the fundamental system of a quadro-quadric transformation (and in No. 55, two rays and a simple point), we convert the cubic families into families of species (iii). A similar treatment of No. 48 leads to species (ii). In No. 54, using p, C_1 and a line through q meeting C_1, we obtain species (i).

In No. 43, apply the cubo-quadric transformation in which the three generators are C_1 and the two rays, and we obtain species (i); No. 31 can be treated in the same way, if two of the generators coincide in the ray of contact. The same treatment applied to No. 41 leads to No. 50; for the latter, apply a quadro-quadric transformation using pq, pr and the point of contact, and we obtain species (ii).

In the same way, if the fundamental system contains a conic and any other point, or a node and a straight line not passing through it, the family can be transformed into another cubic family belonging to the same or a different transformation.

NEWNHAM COLLEGE, CAMBRIDGE, ENG., *April*, 1911.

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* AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIV (1911), p. 207.

*On Some Topological Properties of Plane Curves and a Theorem of Möbius.**

By Solomon Lefschetz.

1. In this paper an application is made of the transformation by inversion to the topology of plane curves. Of the results given many are new, and when they are not, the proofs are. A representative sketch is given for a certain type of closed branches, and is then applied to a proof and generalization of the theorem of Möbius† concerning the inflexions of odd branches, proof based upon entirely different premises. We refer to his paper and to others of C. A. Scott‡ for different applications of the transformation by inversion to the study of plane curves.

2. Let C be a plane branch of curve with a certain number of real points at infinity, and C' its inverse with respect to a pole O exterior to C. As to C', which will have in O a point of multiplicity k and all its points at finite distance, we will assume that it is a *continuous, closed, rectifiable*§ branch having only a finite number of multiple points.

If, then, we fix a positive direction on the arc at an arbitrary point of C', such a direction is defined at any other point on any element of arc through it. The number of directions of advancement from any point is even, since to a positive direction must correspond a negative one.

Since an arbitrary circle divides the plane into two separate regions, it follows that it meets C' in an even number of points, for starting from any point of C' we can come back to it after having described the whole branch and must therefore have passed from one region to the other an even number of times. *A circle infinitely near O meets C' in the neighborhood of O, in a number of points the parity of which is independent of the circle.* For, if $O\alpha$ and $O\alpha'$ are two opposite directions of advancement on C' at O, and D is a line

* Presented to the American Mathematical Society, September 11, 1911.

† "On the Higher Singularities of a Plane Curve," American Journal of Mathematics, Vol. XIV (1892), p. 301.

‡ "The Nature and Effect of Singularities of a Plane Curve," *Ibid.*, Vol. XV (1893) p. 221.

§ For a definition of these words see Camille Jordan, *Traité d'Analyse*, Vol. I, second edition, p. 90.

through O not tangent to them, then either $O\alpha$ and $O\alpha'$ are on the same side of D (cuspidal arc), or they are separated by it. If now M is a point of intersection of C and Γ near O on $O\alpha$, then, in the first case, to M there corresponds a point M' on $O\alpha'$, and if M disappears, which can happen only if it crosses O, M' also will; while in the second case, if M disappears, M' will appear, and this shows that the number of intersections varies only by a multiple of 2. It follows that the intersections of Γ and C', which are not in the neighborhood of O, have a parity independent of Γ. Passing to the limit, it is found that *the number of points other than O, where the circles through this point meet C', differs only by multiples of* 2. These propositions still hold if we replace Γ everywhere by any closed branch without multiple points. If these variable intersections are in odd (even) number, C is said to be an *odd* (*even*) *branch*, and the inversion gives this: *An odd* (*even*) *branch is met by any line of its plane in an odd* (*even*) *number of points.* The maximum of this number will define the *order* of the branch.

3. In what follows we will further assume that the branches considered are non-cuspidal, and also that the curvature is continuous, and its first derivative everywhere definite. *In this case C is of even class.* For, when a line OD revolves around a point O, its intersections with C will change, and that by two units, only when OD touches C; but to every appearance of intersection points, there must correspond a disappearance, which shows that the number of contacts is even. *If C is closed, it has an even number of inflexions.* For, when a point M describes the whole branch, the curvature at M, with respect to O, changes an even number of times; but such changes occur only at inflexions, or when OM is tangent in M; and since the class is even, the same holds for the number of inflexions. A closed branch met by any straight line in at most two points is called an *oval.* It can evidently have no multiple points or inflexions, and conversely, *a closed branch without multiple points or inflexions is an oval.* For, if it were met by a straight line in four points, considerations of continuity would show that it has tangents which meet in two more points besides their contact, and then by considering these tangents, it would be found that it has at least one bitangent, hence at least two inflexions.*

4. Let AB be an arc of curve on which the curvature is increasing from A to B (Fig. 1), while α, β are the contacts with the evolute, of the normals at A and B. We have: Chord $\alpha\beta <$ arc $\alpha\beta = \alpha A - \beta B$, which shows that the circle of curvature at B is interior to the one at A.

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* H. G. Zeuthen, "Sur les courbes du 4e ordre," *Math. Ann.*, Vol. VII, p. 412.

A point A at which the curvature is maximum (Fig. 2) or minimum (Fig. 3) will be called a *vertex.** To a point P, on one side of A on the arc, corresponds Q on the other side, such that the curvature is the same at both points. If $\pi, \alpha, \varkappa$ correspond to P, A, Q on the evolute, and if AP is sufficiently small, we have $O = \pi P - \varkappa Q < \pi\varkappa < \pi P + \varkappa Q$, which shows that *circles of curva-*

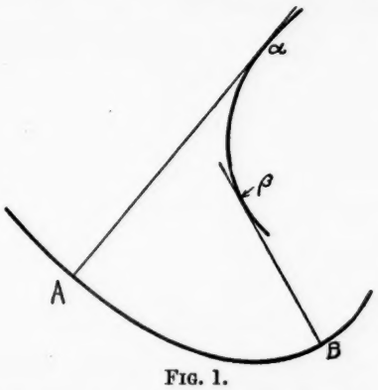


FIG. 1.

ture at points separated by A in the neighborhood of A have two real intersections. Since $\alpha P - \alpha Q$ is very small, αPQ differs but little from an isosceles triangle, $\pi\varkappa$ is nearly parallel to PQ, and therefore αA, normal at A, differs but little from the perpendicular bisector of $\pi\varkappa$, which is the common chord of the osculating circles at P and Q; hence these two circles have a common

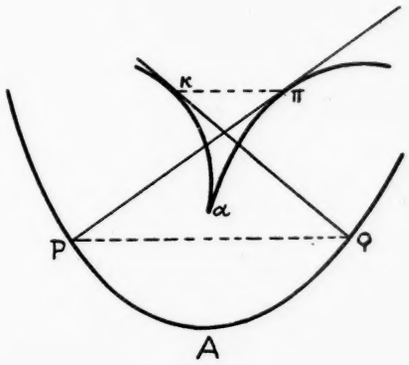


FIG. 2.

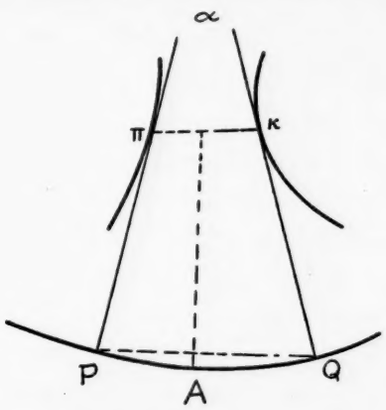


FIG. 3.

point in the neighborhood of A, in the direction of the normal, *that is not on the arc PAQ itself.* At a vertex, the osculating circle is *hyperosculating.* This can be seen from the fact that at such a point the circle is on the same side of the arc on both sides of the contact. We will not go further into this question.

* For an analytic and more rigorous treatment, see: Lilienthal, "Vorlesungen über differentiale Geometrie," Vol. I, p. 56.

5. Since the curvature vanishes at an inflexion, it follows from Rolles' theorem that it has at least one maximum, and the arc at least one vertex, more generally an odd number of vertices between two inflexions. Hence, if the curvature is constantly increasing from A to B (Fig. 1), there is at most one inflexion on AB. If there is none, it follows from § 4 that the osculating circle at A is exterior to the one at B, and that no circle osculating the arc can go through a point interior or exterior to both circles, while one, and one only, passes through any point in the remaining region. If there is one inflexion on the arc, it follows again from § 4 that no circle of curvature passes through a point interior to either circle, and one, and one only, through any point exterior to both. In both cases there are two regions, such that through any point of the one there goes a circle osculating the arc, while none goes through a point of the other.

If the inverse of a closed branch C be taken with respect to a pole M exterior to it, the circles osculating C and going through M become inflexions; hence *their number is even*, and since it changes only when we cross a hyperosculating circle, the proposition is still true if M is on C. Hence *the circles osculating C at a point different from a point M and going through M are in odd number*, and this still holds if M is a point of odd multiplicity. Since C has an even number of inflexions, it follows, from the beginning of this section, that it has also an *even number of vertices.*

6. *A branch $EA'AE'$ with two asymptotes D, D' which it meets in A, A', and with no multiple points, has at least two inflexions* (Fig. 4). For, by reasoning as in § 4, we can find on the arcs $A'E$ and AE' two points F, F' at which the branch has a given curvature arbitrarily small. If we then draw the semi-ellipse Δ hyperosculating the circles of curvatvre Γ, Γ' at F, F' of the given branch, when the curvature at these two points is small enough, we obtain a closed branch, having no multiple points, but met by some straight lines (for example, the base of the semi-ellipse Δ) in four points at least; hence (§ 3) this closed branch is not an oval, and therefore has at least two inflexions, which, since they can not be on the arc $\Gamma\Delta\Gamma'$, are necessarily on the arc $FA'AF'$.

If the branch meets only one of its asymptotes, it has at least one inflexion (Fig. 5). For, by a projection conserving the line at infinity, we may add the condition that the branch is orthogonal to the asymptote which it meets at their intersection A. If we then take the symmetric of the arc AE with respect to D, we can, by a reasoning similar to the preceding, show that there are at least two inflexions on the compound arc, EAE_1, or one at least on the arc AE.

Let us now consider a *loop*, L (Fig. 6), that is an arc of curve having both ends coincident in O. By inverting with respect to O, we obtain a branch

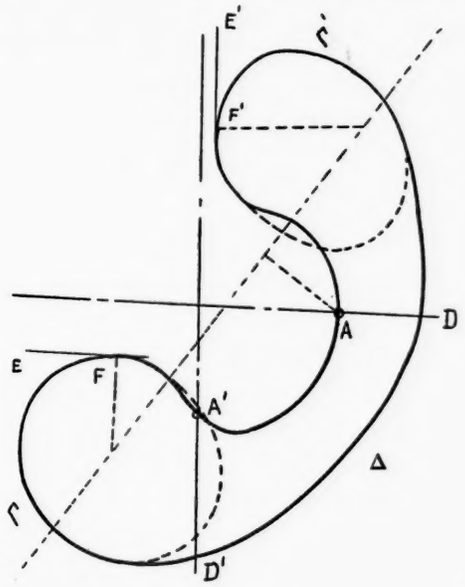


FIG. 4.

such as has just been considered; hence *there are at least one or two circles osculating the loop and going through O, according as one or both circles of curvature at O meet the loop elsewhere.*

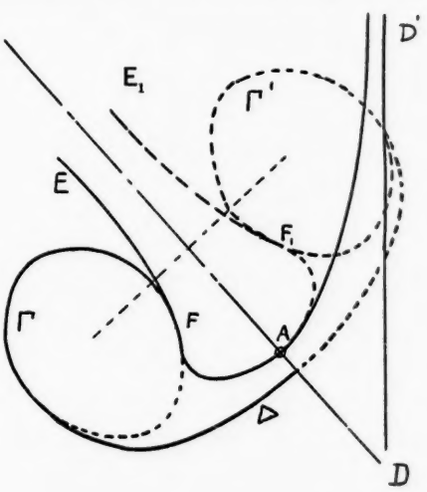


FIG. 5.

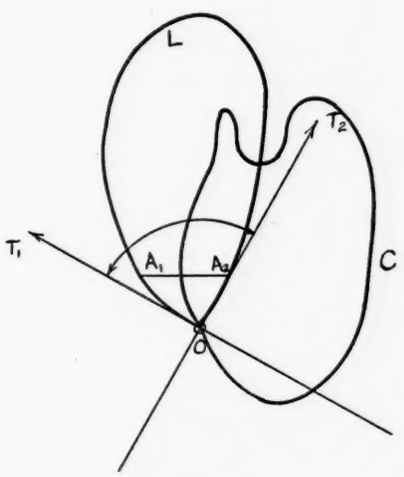


FIG. 6.

Let OT_1 and OT_2 be the semi-tangents to L at O — we will suppose them distinct, although this will not materially affect our conclusions — and let A_1, A_2 be two points in the directions OT_1, OT_2. The arc A_1LA_2 forms, together

with the segment $A_1 A_2$, a closed branch without multiple points, and it will be met by any other closed branch C in an even number of points. In particular, *if C goes through O, it will meet L in an odd (even) number of points, if it has in O an odd (even) number of directions of advancement in the angle $T_1 O T_2$.*

7. Given, as in § 2, a branch C, inverse of a closed branch, we can always cut it by a straight line, meeting it in at least two points, passing through none of its multiple points or inflexions, and not touching C. Projecting this line into the line at infinity, a branch C_1 is obtained which will be supposed to have no multiple points, and the inverse C_1' of which will therefore have only one multiple point of order $k \geq 2$ at the pole O, while the excess of the order

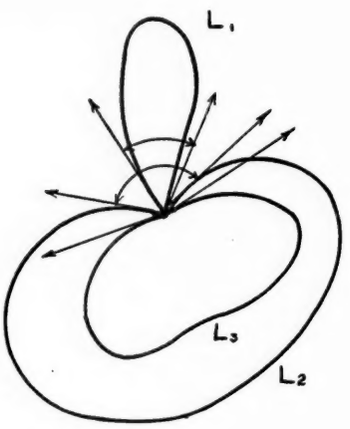


FIG. 7.

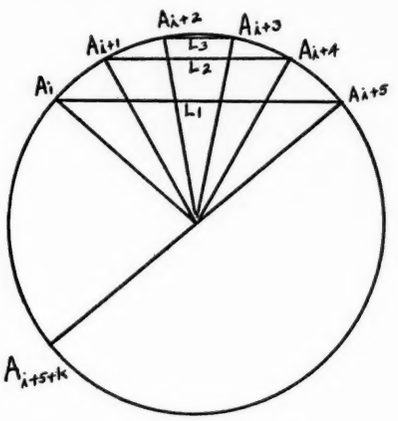
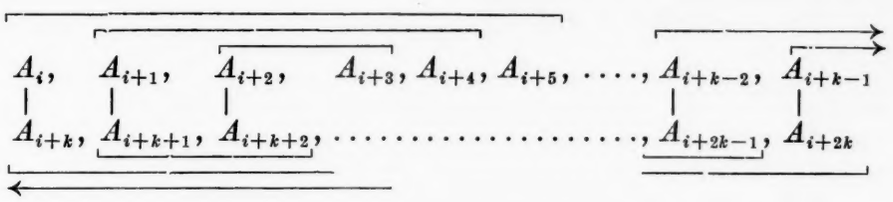


FIG. 8.

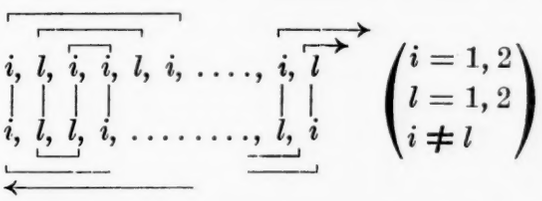
of C on k is even. C_1' is composed of a system of k loops having a common origin O. The loop L_1 will be said to be interior to the loop L_2 (Fig. 7), whenever the angle less than π formed by the directions of advancement on L_1 is interior to the corresponding angle for L_2. O being the only multiple point of C_1', the directions of advancement on one loop can not be separated by those on another; hence two loops are either interior or exterior to each other. A system of loops such as $L_1, L_2, \ldots, L_n$, where L_i is interior to L_{i+1} $(i \leq n-1)$, will be called a *nest* of loops. Counting from L_n each loop will have a certain rank, and the two directions of advancement on L_i will receive the index 1 or 2, when L_i is of odd or even rank.

8. Let us join the center P of a circle to the vertices $A_1, \ldots, A_{2k}$ of an inscribed regular polygon of $2k$ sides (Fig. 8). The radii will represent, in the order in which they are met, the $2k$ directions of advancement from O. Thus $PA_i, PA_{i+1}, \ldots, PA_{i+5}$ will represent the directions of advancement on

the loops of the nest (L_1, L_2, L_3) in Fig. 7. The loop L_1 will be represented by the triangle OA_iA_{i+5} or by the chord A_iA_{i+5}, etc. We thus obtain a diagram of C_1'. Since the latter is a closed branch, the starred polygon $A_iA_{i+5+k}\ldots$. must be closed. Starting from A_i, we may write on a horizontal line the first k vertices in the order $A_i, A_{i+1}, A_{i+2}, \ldots$, then the vertices $A_{i+k+1}, A_{i+k+2}, \ldots$. on the line below. Indicating then how these vertices are joined, we obtain for C_1' a notation or sketch as follows:



where $A_i = A_j$ if $i \equiv j \pmod{2k}$. More conveniently we may merely write the indices of the corresponding direction of advancement as:



$$\begin{pmatrix} i = 1, 2 \\ l = 1, 2 \\ i \neq l \end{pmatrix}$$

and in both schemes the vertical bars represent the diametral sides of the polygon, while the other sides are represented by the horizontal lines. Since the directions of advancement from O on one loop are never separated by those on another, the chords $A_i\,A_j$ can meet inside of the circle P.

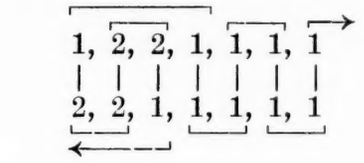
9. Given $A_1, \ldots, A_{2k}$, a starred polygon representing a closed branch C_1' can be obtained from these points in a finite number of ways only. The two properties, given in § 9 will serve to eliminate the polygons corresponding to impossible cases. This will best be shown on a concrete example, and so *let us suppose* $k = 7$. If we represent each combination by its nests and their number of loops, in their order, we may have the following cases:

a) (1, 1, 1, 1, 1, 1, 1), d) (2, 1, 2, 1, 1), g) (3, 2, 1, 1), j) (3, 3, 1),
b) (2, 1, 1, 1, 1, 1), e) (2, 2, 2, 1), h) (3, 1, 2, 1),
c) (2, 2, 1, 1, 1), f) (3, 1, 1, 1, 1), i) (3, 2, 2),

obtained by taking the partitions of 7 in a sum of numbers none greater than 3. The combinations a), d), g), j) alone give a closed polygon, and therefore we will consider only these.

a) *Seven loops* (1, 1, 1, 1, 1, 1, 1). This case is not interesting.

d) *Two nests of two loops and three of one* (2, 1, 2, 1, 1). Fig. 9 gives the corresponding circle diagram, and the sketch is:



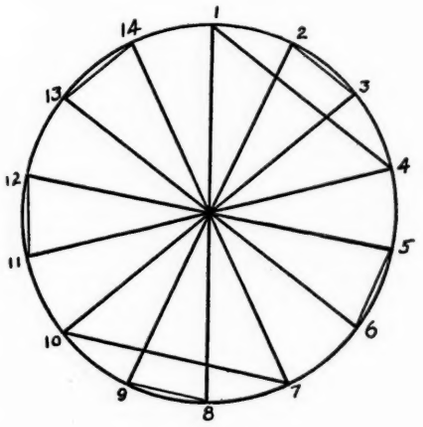
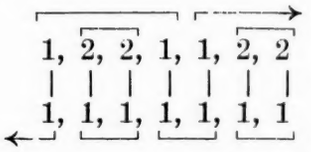


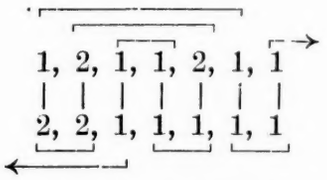
FIG. 9.

It is interesting to notice that the system c) (2, 2, 1, 1, 1) is composite although formed of the same nests as d). In fact its notation is:

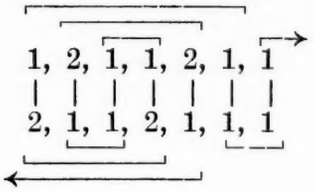


which, as can be seen, does not give a closed polygon.

g) *One nest of three loops, one of two and two of one loop* (3, 2, 1, 1). The notation is:

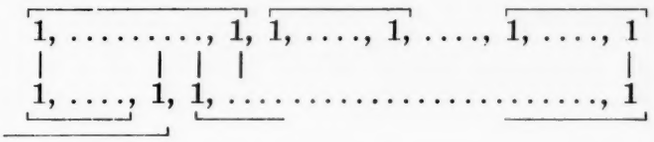


j) *Two nests of three loops and one loop* (3, 3, 1). The notation is:



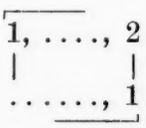
10. We will designate by (i, l) a vertical joining the index i to the index l, in the sketch, independently of the lines on which these indices are found. We may start our sketch from an index corresponding to a loop interior to none. *There are then at least two more couples of consecutive indices* (1) *such as*$\ldots, 1, 1, \ldots$ *than couples of consecutive indices* (2). For all the indices form a certain number of partial systems, each corresponding to a nest of C_1', and both extremities of such a system present a couple $(1, 1)$, while a couple $(2, 2)$ can only be present in the middle of one, and as this can not happen for all middle couples (for then k would be even), our assertion is proved. The dispositions of our sketch can be reduced to two types, according to the relative indices of A_k and A_{k+1}, for both can not have the index 2, as then k would be even. This will be shown when considering case (2).

(1)

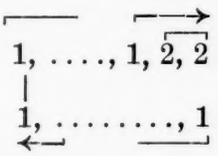


Two verticals $(1, 1)$ are apparent here.

(2)



Each partial system contains an even number of indices, and since k is odd, the same must be true of the number of indices of the partial groups at the end of the first and beginning of the second line; hence we can only have the following disposition for the unwritten indices:



One line $(1, 1)$ is apparent here. We say that there is at least one more. For if not, among the unwritten indices to every couple $(1, 1)$ on one of the lines, there would correspond a couple $(2, 2)$ on the other, so that the couples $(1,1)$ and $(2,2)$ would be in equal number, and we have seen that this can not be if k is odd. Let p, q be the number of bars $(1,1)$ and $(1,2)$ in the sketch. The index (1) will occur $2p + q$ times, and as this number must be even, the same holds for q. Hence the number of bars (i,i) is odd; but it has been found

not to be inferior to 2; hence it is at least equal to 3, or: *When k is odd, there are at least three bars (i, i) in the sketch, and their number is always odd.*

11. A circle of curvature Γ of C_1' at O will be osculating an arc which belongs to two loops A, B. Γ meets C_1' (Fig. 10) in $k+2$ points coincident with O, hence also in an odd number of points different from O. The k tangents at O being all distinct, it follows from § 6 that the loops interior to A and B are met by Γ in an even number of points different from O, while those exterior to either A or B are met by the circle in an odd number of points. Hence if the ranks of A and B are of the same (opposite) parity, these exterior loops meet Γ in an even (odd) number of points, and the loops A, B meet it in a total odd (even) number of points. Hence if the ranks of A, B are of the same parity, one of the loops A, B, say A, meets Γ in an odd number of points,

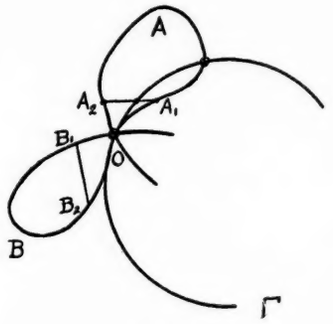


FIG. 10.

and from § 6 we conclude that there is at least one circle osculating A and passing through O. If the circles of curvature for both arcs of A through O meet A, it follows from § 6 that there are at least two circles osculating A and going through O. But if the ranks of A, B are of the same parity, there is a couple (i, i) corresponding to it in the notation for C_1', and conversely. From § 10 it follows then that *if C_1' has an odd number of loops there are at least three, and always an odd number, of circles osculating C_1' at other points than O, and going through O.* By considering C_1 we obtain Möbius' theorem: *An odd branch without multiple points has at least three inflexions.*

12. It will often happen that a minimum greater than 3 can be found for the number of inflexions. Thus, referring to § 9, we can see, in the sketches for C_1' with seven loops, that in all cases there are at least five lines (i, i), except for the C_1' represented by $(3, 3, 1)$. Hence there is only one class of odd branches of the seventh order for which no minimum greater than 3 can be found for the inflexions.

The following is a more interesting application: *A spiraloid of order not inferior to 9 has at least five inflexions.* For a definition of the spiraloid we refer to Fig. 11. To prove our assertion, it is sufficient to cut the branch

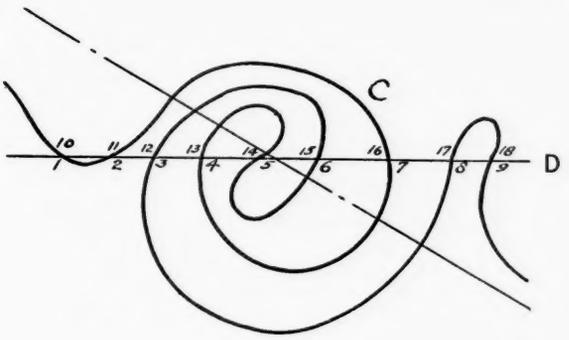
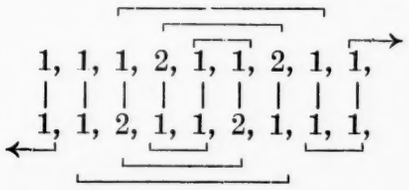


FIG. 11.

by a line D meeting it in at least nine points, to project it into the line at infinity, then to take the inverse C_1' and write its notation, which is found to be the following:



and contains effectively five lines (i, i). This sketch can be easily deduced from C, but we will not enter into any details as to this.

13. From the preceding considerations, minima can in several cases be obtained for the number of vertices of a branch. We have seen in § 5 that no more than one circle of curvature osculating an arc can pass through any point, if there are no vertices on the arc. Hence, from § 5 it follows that *a closed branch with a single* $(2k+1)$*-uple point with* $2k+1$ *distinct tangents has at least* $2k+4$ *vertices*, since there are at least as many osculating circles through O. This supposes $k \geq 1$. If $k = 0$ the inverse of the branch with respect to O is an odd branch with one point at infinity, and at least three inflexions, say I, J, K. By reasoning as in § 5, we find that there is at least one vertex in each of the four intervals (∞I), (IJ), (JK), $(K \infty)$, and since inversion in general changes vertices into vertices, the above proposition is also true if $k = 0$. Hence the generalization of a well-known property of ellipses: *An oval has at least four vertices.* In the general case, we obtain by

inversion the following proposition: *An odd branch having* $2k+1$ *distinct points at infinity has at least* $2k+4$ *vertices.* For example, a cubic with only one real asymptote has at least eight vertices: four on the oval and four on the odd branch.

It is well to point out, however, that the oval of a cubic can have six vertices, as, for example:

$$xyz = \lambda(x+y+z+\varepsilon x)^3,$$

where λ, ε are sufficiently small and $x+y+z=0$ is the line at infinity which has twelve vertices, six on each branch. A bipartite cubic differing very little from three straight lines has also six vertices on the oval. The cubic $y^2 = 2x(x^2-1)$, with an inflexional tangent at infinity, has six vertices on the oval and three on the odd branch, while one differing very little from it, but with a contact at infinity, may have six vertices on the oval and four on the odd branch. These few examples lay the proper emphasis on the care to be taken in applying the propositions of this section.

Finally, we may remark that we have nowhere supposed our curves to be algebraic, but merely to possess certain definite graphical properties.

WORCESTER, *August* 16, 1911.

On a Flat Spread-Sphere Geometry in Odd-dimensional Space.

By J. Eiesland.

§ 1. *Introduction.*

It is a well-known fact that Lie's line-sphere geometry has no analogue in n-dimensional space when n is greater than 3; in fact, in a space $S_n (n > 3)$ there can be no one-to-one correspondence between the ∞^{n+1} spheres and the $\infty^{2(n-1)}$ straight lines; moreover, a contact-transformation which transforms spheres into lines must also transform spheres that touch into lines that intersect. But there is only one necessary and sufficient condition that two spheres shall touch, while the number of conditions that two lines shall intersect is $n-2$. Only if $n=3$ will therefore a line-sphere geometry be possible.

A generalization of Lie's line-sphere geometry becomes possible if, instead of using the straight line as space-element, we introduce a flat spread (or hyperplane) of $\frac{n-1}{2}$ dimensions. In fact, since to spheres that touch must correspond intersecting flats, the condition for such intersection must be unique. Now if two flats of $\frac{n-1}{2}$ dimensions meet in n-space, the condition for intersection is in general unique; moreover, $\frac{n-1}{2}$ must be an integer. Hence, *a flat spread-sphere geometry is only possible in a space of an odd number of dimensions.* It is also necessary that the number of essential parameters of the flat shall be equal to $n+1$, which is the number of parameters of a sphere.

§ 2. *The Minimal Cone.*

A minimal flat spread of $n-1$ dimensions in S_n (n even) may be defined as one whose direction cosines $A_1, A_2, \ldots, A_n$ satisfy the condition

$$A_1^2 + A_2^2 + \ldots + A_n^2 = 0. \qquad (1)$$

If we put

$$A_1=\alpha_1+\beta_1,\quad A_2=i(\alpha_1-\beta_1),\quad A_3=\alpha_2+\beta_2,\quad \ldots,$$
$$A_{n-1}=1-\sum_1^{\frac{n-2}{2}}\alpha_i\beta_i,\quad A_n=i(1+\sum_1^{\frac{n-2}{2}}\alpha_i\beta_i), \tag{2}$$

the minimal flat spread, or simply *flat*, may be written

$$(\alpha_1+\beta_1)X_1+i(\alpha_1-\beta_1)X_2+\ldots.+(1-\Sigma\alpha_i\beta_i)X_{n-1}$$
$$+i(1+\Sigma\alpha_i\beta_i)X_n+T=0, \tag{3}$$

where α_i, β_i, T are the parameters of the flat.

If we put $T=F(\alpha_1,\beta_1,\alpha_2,\beta_2,\ldots.,\alpha_{\frac{n-2}{2}},\beta_{\frac{n-2}{2}})$ and differentiate (3) with respect to α_i and β_i in succession, we obtain the system

$$\left.\begin{aligned}
&(\alpha_1+\beta_1)X_1+i(\alpha_1-\beta_1)X_2+\ldots.\\
&\qquad+(1-\Sigma\alpha_i\beta_i)X_{n-1}+i(1+\Sigma\alpha_i\beta_i)X_n+F=0,\\
&X_1+iX_2-\beta_1X_{n-1}+i\beta_1X_n+F'_{\alpha_1}=0,\\
&X_1-iX_2-\alpha_1X_{n-1}+i\alpha_1X_n+F'_{\beta_1}=0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&X_{n-3}+iX_{n-2}-\beta_{\frac{n-2}{2}}X_{n-1}+i\beta_{\frac{n-2}{2}}X_n+F'_{\alpha_{\frac{n-2}{2}}}=0,\\
&X_{n-3}-iX_{n-2}-\alpha_{\frac{n-2}{2}}X_{n-1}+i\alpha_{\frac{n-2}{2}}X_n+F'_{\beta_{\frac{n-2}{2}}}=0.
\end{aligned}\right\} \tag{4}$$

Multiplying the second of these equations by α_1, the third by β_1, and so on, we get by adding, and subtracting the sum from the first equation,

$$(1+\Sigma\alpha_i\beta_i)X_{n-1}+i(1-\Sigma\alpha_i\beta_i)X_n+F-\alpha_1F'_{\alpha_1}-\beta_1F'_{\beta_1}-\ldots.-F'_{\beta_{\frac{n-2}{2}}},$$

which solved for X_{n-1} gives

$$X_{n-1}=i\frac{(\Sigma\alpha_i\beta_i-1)}{\Sigma\alpha_i\beta_i+1}X_n-\frac{F-\Sigma\alpha_iF'_{\alpha_i}-\Sigma\beta_iF'_{\beta_i}}{\Sigma\alpha_i\beta_i+1}.$$

Substituting this value in (4) we obtain the following system:

$$\left.\begin{aligned}
X_1&=-\frac{i(\alpha_1+\beta_1)}{\Sigma\alpha_i\beta_i+1}X_n+\frac{(\alpha_1+\beta_1)(\Sigma\alpha_iF'_{\alpha_i}+\Sigma\beta_iF'_{\beta_i}-F)-(1+\Sigma\alpha_i\beta_i)(F'_{\alpha_i}+F'_{\beta_i})}{2(\Sigma\alpha_i\beta_i+1)},\\
X_2&=\frac{\alpha_1-\beta_1}{\Sigma\alpha_i\beta_i+1}X_n+i\frac{(\alpha_1-\beta_1)(\Sigma\alpha_iF'_{\alpha_i}+\Sigma\beta_iF'_{\beta_i}-F)+(1+\Sigma\alpha_i\beta_i)(F'_{\alpha_i}-F'_{\beta_i})}{2(\Sigma\alpha_i\beta_i+1)},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
X_{n-2}&=\frac{\alpha_{\frac{n-2}{2}}-\beta_{\frac{n-2}{2}}}{\Sigma\alpha_i\beta_i+1}X_n+i\frac{(\alpha_{\frac{n-2}{2}}-\beta_{\frac{n-2}{2}})(\Sigma\alpha_iF'_{\alpha_i}+\Sigma\beta_iF'_{\beta_i}-F)+(1+\Sigma\alpha_i\beta_i)(F'_{\alpha_{\frac{n-2}{2}}}-F'_{\beta_{\frac{n-2}{2}}})}{2(\Sigma\alpha_i\beta_i+1)},\\
X_{n-1}&=\frac{i(\Sigma\alpha_i\beta_i-1)}{\Sigma\alpha_i\beta_i+1}X_n+\frac{\Sigma\alpha_iF'_{\alpha_i}+\Sigma\beta_iF'_{\beta_i}-F}{\Sigma\alpha_i\beta_i+1},
\end{aligned}\right\} \tag{5}$$

which is equivalent to (4). If we consider α_i, β_i, F, F'_{α_i}, F'_{β_i} as the coordinates of a minimal line in S_n, we have here the ∞^{2n-3} minimal line of that space. We also observe that through any fixed point there will pass ∞^{n-2} such lines.

If we put

$$F = a\Sigma\alpha_i\beta_i + \Sigma b_i\alpha_i + \Sigma c_i\beta_i + d, \tag{6}$$

the system (5) reduces to the following:

$$\left.\begin{aligned}
&X_1 + \frac{b_1 + c_1}{2} = \frac{i(\alpha_1 + \beta_1)}{1 + \Sigma\alpha_i\beta_i}\left(X_n + \frac{i(a+d)}{2}\right),\\
&X_2 - \frac{i(b_1 - c_1)}{2} = \frac{\alpha_1 - \beta_1}{1 + \Sigma\alpha_i\beta}\left(X_n + \frac{i(a+d)}{2}\right),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&X_{n-2} - \frac{i\left(\frac{b_{n-2}}{2} - \frac{c_{n-2}}{2}\right)}{2} = \frac{\frac{\alpha_{n-2}}{2} - \frac{\beta_{n-2}}{2}}{1 + \Sigma\alpha_i\beta_i}\left(X_n + \frac{i(a+d)}{2}\right),\\
&X_{n-1} - \frac{a-d}{2} = \frac{i(\Sigma\alpha_i\beta_i - 1)}{1 + \Sigma\alpha_i\beta_i}\left(X_n + \frac{i(a+d)}{2}\right),
\end{aligned}\right\} \tag{7}$$

which is a minimal cone having its vertex at the point

$$\left.\begin{aligned}
&\xi_1 = -\frac{b_1 + c_1}{2}, \quad \xi_2 = \frac{i(b_1 - c_1)}{2}, \quad \ldots, \quad \xi_{n-3} = -\frac{\frac{b_{n-2}}{2} + \frac{c_{n-2}}{2}}{2},\\
&\xi_{n-2} = \frac{i\left(\frac{b_{n-2}}{2} - \frac{c_{n-2}}{2}\right)}{2}, \quad \xi_{n-1} = \frac{a-d}{2}, \quad \xi_n = -\frac{i(a+d)}{2};
\end{aligned}\right\} \tag{8}$$

in fact, eliminating α_i, β_i from (7), we have

$$(X_1 - \xi_1)^2 + (X_2 - \xi_2)^2 + \ldots + (X_n - \xi_n)^2 = 0. \tag{9}$$

We shall now consider α, β_i, F, F'_{α_i}, F'_{β_i} as coordinates of the surface-elements in an $(n-1)$-dimensional space which will be denoted by $\bar{S}_{n-1}$. (Since n is even $\bar{S}_{n-1}$ will be an odd-dimensional space.) It follows then from the above development that *to all the* ∞^n *hyperboloids* (6) *in* $\bar{S}_{n-1}$ *there correspond the* ∞^n *minimal cones in* S_n. *To the* ∞^{n-2} *points of the hyperboloid correspond the* ∞^{n-2} *minimal lines passing through the point* ξ_i; *the addition to* F *in* (5) *of the function* $a\Sigma\alpha_i\beta_i + \Sigma b_i\alpha_i + \Sigma c_i\beta_i + d$ *has the effect of translating the minimal hypercomplex* (5) *to the point* ξ_i. *Finally, the system* (5) *establishes a one-to-one correspondence between the* ∞^{2n-3} *minimal lines in* S_n *and the* ∞^{2n-3} *surface-elements* F, α_i, β_i, F_{α_i}, F'_{β_i} *in* $\bar{S}_{n-1}$.*

(5)

*This theorem is a generalization of a corresponding one for $n=4$ in a paper by the author published in *Trans. Amer. Math. Soc.*, Vol. XII, pp. 403–428. The same is true of the theorem on p. 207.

Consider now the isotropic hypercomplex (5). Since we may interpret the quantities $\frac{dX_i}{dX_n}$, $(i = 1, 2, \ldots, n-1)$, as coordinates of the flat space at infinity, we see that the lines of the complex meet the sphere at infinity in a point determined by the values of α_i and β_i. We may therefore consider the quantities

$$\frac{i(\alpha_1+\beta_1)}{\Sigma\alpha_i\beta_i+1}, \frac{\alpha_1-\beta_1}{\Sigma\alpha_i\beta_i+1}, \ldots, \frac{i(\Sigma\alpha_i\beta_i-1)}{\Sigma(\alpha_i\beta_i+1}$$

as coordinates of points on this sphere.

§ 3. *The Focal Surface.*

Let the quantities α_{i0}, β_{i0} determine a generator D of the complex (5). *How many minimal developables of two dimensions can we pass through D?*

To answer this question we proceed as follows: It is evident that we must find a curve determined by the relations

$$\alpha_i = \phi_i(\alpha_{\frac{n-2}{2}}), \quad \beta_k = \psi_k(\alpha_{\frac{n-2}{2}}), \qquad \begin{matrix} i = 1, 2, \ldots, \frac{n-4}{2}, \\ k = 1, 2, \ldots, \frac{n-2}{2}, \end{matrix}$$

such that the generators are tangent to it, and moreover such that

$$\alpha_{i0} = \phi_i(\alpha_{\frac{n-2}{2},0}), \quad \beta_{k0} = \psi_k(\alpha_{\frac{n-2}{2},0}).$$

This curve will then be the edge of regression of the minimal developable M_2. The conditions that two consecutive generators of (5) shall intersect are

$$\left.\begin{aligned} &X_1(d\alpha_1 + d\beta_1) + iX_2(d\alpha_1 - d\beta_1) + \ldots . \\ &\quad - X_{n-1}\Sigma(\alpha_i d\beta_i + \beta_i d\alpha_i) + iX_n\Sigma(\alpha_i d\beta_i + \beta_i d\alpha_i) + dF = 0, \\ &(-X_{n-1} + iX_n)d\beta_1 + dF'_{\alpha_1} = 0, \\ &(-X_{n-1} + iX_n)d\alpha_1 + dF'_{\beta_1} = 0, \\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots, \\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots, \\ &(-X_{n-1} + iX_n)d\alpha_{\frac{n-2}{2}} + dF'_{\beta_{\frac{n-2}{2}}} = 0. \end{aligned}\right\} \quad (10)$$

Eliminating $X_1, X_2, \ldots, X_n$ from (4) and (10) we have

$$\left.\begin{aligned} &\text{a)} \quad dF - F'_{\alpha_1}d\alpha_1 - F'_{\beta_1}d\beta_1 - \ldots . - F'_{\alpha_{\frac{n-2}{2}}}d\alpha_{\frac{n-2}{2}} - F'_{\beta_{\frac{n-2}{2}}}d\beta_{\frac{n-2}{2}} = 0, \\ &\text{b)} \quad \frac{d\alpha_i}{d\alpha_{\frac{n-2}{2}}} = \frac{dF'_{\beta_i}}{dF'_{\beta_{\frac{n-2}{2}}}}, \quad \frac{d\beta_k}{d\alpha_{\frac{n-2}{2}}} = \frac{dF'_{\alpha_k}}{dF'_{\beta_{\frac{n-2}{2}}}}, \qquad \begin{matrix} i = 1, 2, \ldots, \frac{n-4}{2}, \\ k = 1, 2, \ldots, \frac{n-2}{2}. \end{matrix} \end{aligned}\right\} \quad (11)$$

The first relation is satisfied *à priori*, since we suppose that the function F is finite and continuous, and possesses finite and continuous derivatives in the region α_{i0}, β_{k0}. The remaining $n-3$ relations need special consideration. We shall adopt the following notation, putting

$$a_{ki}^{(11)} = a_{ik}^{(11)} = \frac{\partial^2 F}{\partial\alpha_i \partial\alpha_k}, \quad a_{ik}^{(22)} = a_{ki}^{(22)} = \frac{\partial^2 F}{\partial\beta_i \partial\beta_k},$$
$$a_{ik}^{(12)} = a_{ki}^{(21)} = \frac{\partial^2 F}{\partial\alpha_i \partial\beta_k}, \quad a_{ik}^{(21)} = a_{ki}^{(12)} = \frac{\partial^2 F}{\partial\beta_i \partial\alpha_k}.$$

We now write the system (11)b as follows:

$$\left.\begin{aligned}
\frac{d\beta_k}{d\alpha_{\frac{n-2}{2}}} &= \frac{a_{1k}^{(11)}d\alpha_1 + a_{2k}^{(11)}d\alpha_2 + \ldots + a_{\frac{n-2}{2}}^{(11)}d\alpha_{\frac{n-2}{2}} + a_{1k}^{(21)}d\beta_1 + \ldots + a_{\frac{n-2}{2}\,k}^{(21)}d\beta_{\frac{n-2}{2}}}{a_{1\frac{n-2}{2}}^{(12)}d\alpha_1 + a_{2\frac{n-2}{2}}^{(12)}d\alpha_2 + \ldots + a_{\frac{n-2}{2}\frac{n-2}{2}}^{(12)}d\alpha_{\frac{n-2}{2}} + a_{1\frac{n-2}{2}}^{(22)}d\beta_1 + \ldots + a_{\frac{n-2}{2}\frac{n-2}{2}}^{(22)}d\beta_{\frac{n-2}{2}}}, \\
\frac{d\alpha_i}{d\alpha_{\frac{n-2}{2}}} &= \frac{a_{1i}^{(12)}d\alpha_1 + a_{2i}^{(12)}d\alpha_2 + \ldots + a_{\frac{n-2}{2}\,i}^{(12)}d\alpha_{\frac{n-2}{2}} + a_{11}^{(22)}d\beta_1 + \ldots + a_{\frac{n-2}{2}\,i}^{(22)}d\beta_{\frac{n-2}{2}}}{a_{1\frac{n-2}{2}}^{(12)}d\alpha_1 + a_{2\frac{n-2}{2}}^{(12)}d\alpha_2 + \ldots + a_{\frac{n-2}{2}\frac{n-2}{2}}^{(12)}d\alpha_{\frac{n-2}{2}} + a_{1\frac{n-2}{2}}^{(22)}d\beta_1 + \ldots + a_{\frac{n-2}{2}\frac{n-2}{2}}^{(22)}d\beta_{\frac{n-2}{2}}}.
\end{aligned}\right\} (12)$$

Introducing a factor of proportionality ρ in this system, we may write it as follows:

$$\left.\begin{aligned}
&(a_{kk}^{(21)} - \rho)d\beta_k + a_{1k}^{(11)}d\alpha_1 + \ldots + a_{\frac{n-2}{2}\,k}^{(11)}d\alpha_{\frac{n-2}{2}} + a_{1k}^{(21)}d\beta_1 + \ldots \\
&\qquad + a_{k-1\,k}^{(21)}d\beta_{k-1} + a_{k+1\,k}^{(21)}d\beta_{k+1} + \ldots = 0, \\
&(a_{ii}^{(12)} - \rho)d\alpha_i + a_{1i}^{(12)}d\alpha_1 + \ldots + a_{i-1}^{(12)}d\alpha_{i-1} + a_{i+1}^{(12)}d\alpha_{i+1} + \ldots \\
&\qquad + a_{11}^{(22)}d\beta_1 + \ldots = 0, \\
&(a_{1\frac{n-2}{2}}^{(12)}d\alpha_1 + a_{2\frac{n-2}{2}}^{(12)}d\alpha_2 + \ldots + a_{1\frac{n-2}{2}}^{(22)}d\beta_1 + \ldots + (a_{\frac{n-2}{2}\frac{n-2}{2}}^{(22)} - \rho)d\beta_{\frac{n-2}{2}} = 0.
\end{aligned}\right\} (12')$$

Eliminating $d\alpha_i$ and $d\beta$ from this system we have the following equation of the $(n-2)$-nd degree in ρ:

$$\left|\begin{array}{llllllll}
a_{11}^{(21)} - \rho & a_{21}^{(21)} & \ldots & a_{11}^{(11)} & a_{21}^{(11)} & \ldots & a_{\frac{n-2}{2}1}^{(11)} \\
a_{12}^{(21)} & a_{22}^{(21)} - \rho & \ldots & a_{12}^{(11)} & a_{22}^{(11)} & \ldots & a_{\frac{n-2}{2}2}^{(11)} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
a_{1\frac{n-2}{2}}^{(21)} & a_{2\frac{n-2}{2}}^{(21)} & \ldots & a_{\frac{n-2}{2}\frac{n-2}{2}}^{(21)} - \rho & a_{1\frac{n-2}{2}}^{(11)} & a_{2\frac{n-2}{2}}^{(11)} & \ldots & a_{\frac{n-2}{2}\frac{n-2}{2}}^{(11)} \\
a_{11}^{(22)} & a_{21}^{(22)} & \ldots & a_{11}^{(12)} - \rho & a_{21}^{(12)} & \ldots & a_{\frac{n-2}{2}1}^{(11)} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
a_{1\frac{n-2}{2}}^{(22)} & a_{2\frac{n-2}{2}}^{(22)} & \ldots & a_{1\frac{n-2}{2}}^{(12)} & a_{2\frac{n-2}{2}}^{(12)} & \ldots & a_{\frac{n-2}{2}\frac{n-2}{2}}^{(12)} - \rho
\end{array}\right| = 0. \quad (13)$$

If we call the roots $\rho_1, \rho_2, \ldots, \rho_{n-2}$, we see at once that

$$\sum_1^{n-2} \rho_i = 2\Sigma \frac{\partial^2 F}{\partial \alpha_i \partial \beta_i}. \tag{14}$$

If we now introduce a root of (13) into (12′), we obtain a linear system of $n-3$ differential equations in $n-2$ variables; we have therefore by solving

$$\left.\begin{aligned} \frac{d\beta_k}{\partial \alpha_{\frac{n-2}{2}}} &= f_k(\alpha_1, \beta_1, \ldots, \alpha_{\frac{n-2}{2}}, \beta_{\frac{n-2}{2}}), \\ \frac{d\alpha_i}{\partial \alpha_{\frac{n-2}{2}}} &= \phi_i(\alpha_1, \beta_1, \ldots, \alpha_{\frac{n-2}{2}}, \beta_{\frac{n-2}{2}}). \end{aligned}\right\} \tag{15}$$

Since we have assumed F and its derivatives finite and continuous, at least for a certain region in the neighborhood of $\alpha_{i0}, \beta_{k0}, \alpha_{\frac{n-2}{2}0}$, the functions f_k and ϕ_k will in general possess the same property. By Cauchy's existence-theorem * there will exist a system of integrals

$$\alpha_i = \phi_i(\alpha_{\frac{n-2}{2}}), \quad \beta_k = \psi_k(\alpha_{\frac{n-2}{2}}) \tag{16}$$

which satisfies (15), and such that for $\alpha_i = a_{i0}$, $\beta_k = \beta_{k0}$ the following relations are satisfied:

$$\alpha_{i0} = \phi_i(\alpha_{\frac{n-2}{2}0}), \quad \beta_{k0} = \psi_k(\alpha_{\frac{n-2}{2}0}). \tag{17}$$

Since ρ has in general $n-2$ distinct values, *there exist $n-2$ minimal developables passing through an arbitrarily chosen generator D of the complex* (5).

These are obtained by solving an equation of the $(n-2)$-nd degree and integrating a system of $n-3$ linear differential equations in $n-2$ variables. Substituting the values of α_i and β_k from (16) in (5) we obtain the required minimal developables, one for each root of (15). It may happen that for a special value of F equation (13) has multiple roots; in this case the number of developables will be reduced by a number equal to the sum of the multiplicity numbers minus the number of multiple roots.

We may now obtain the focal surface of the complex. From the last of equations (10) we get

$$\begin{aligned} (-X_{n-1} + iX_n)\, d\alpha_{\frac{n-2}{2}} + \frac{\partial^2 F}{\partial \alpha_1 \partial \beta_{\frac{n-2}{2}}} d\alpha_1 + \ldots + \frac{\partial^2 F}{\partial \alpha_{\frac{n-2}{2}} \partial \beta_{\frac{n-2}{2}}} d\alpha_{\frac{n-2}{2}} + \ldots \\ + \frac{\partial^2 F}{\partial \beta_1 \partial \beta_{\frac{n-2}{2}}} d\beta_1 + \ldots + \frac{\partial^2 F}{\partial^2 \beta_{\frac{n-2}{2}}} d\beta_{\frac{n-2}{2}} = 0, \end{aligned}$$

* See Picard, *Traité D'Analyse*, Chapter XI.

which, if we take account of the last of equations (12′), may be written

$$-X_{n-1}+iX_n+\rho_i=0, \tag{18}$$

where ρ_i is any one of the roots of (13). If now we add (18) to the system (4) and solve for $X_1, X_2, \ldots, X_n$, we shall obtain the focal surface which, since ρ has $n-2$ values, possesses in general $n-2$ focal sheets. The equation of the focal surface is:

$$\left.\begin{aligned}
X_1 &= \frac{1}{2}[(\alpha_1+\beta_1)\rho - F'_{\alpha_1} - F'_{\beta_1}],\\
X_2 &= \frac{i}{2}[(\alpha_1-\beta_1)\rho + F'_{\alpha_1} - F'_{\beta_1}],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
X_{n-1} &= \frac{1}{2}[(1-\Sigma\alpha_i\beta_i)\rho - F + \Sigma\alpha_i F'_{\alpha_i} + \Sigma\beta_i F'_{\beta_i}],\\
X_n &= \frac{i}{2}[(1+\Sigma\alpha_i\beta_i)\rho + F - \Sigma\alpha_i F'_{\alpha_i} - \Sigma\beta_i F'_{\beta_i}].
\end{aligned}\right\} \tag{19}$$

We have thus proved the

THEOREM: *The isotropic complex C_{n-2} in S_n has a focal surface* (19) *which is the envelope of ∞^{n-3} minimal developables M_2 obtained from the complex by choosing any one of the ∞^{n-2} generators and passing through it $n-2$ developables. This focal surface has $n-2$ sheets. The edges of regression on the surface are minimal curves satisfying the system of differential equations*

$$\left.\begin{aligned}
\frac{d\beta_k}{d\alpha_{\frac{n-2}{2}}} &= f_k(\alpha_1, \beta_1, \ldots, \alpha_{\frac{n-2}{2}}, \beta_{\frac{n-2}{2}}), \quad k=1,2,\ldots,\frac{n-2}{2},\\
\frac{d\alpha_i}{d\alpha_{\frac{n-2}{2}}} &= \phi_i(\alpha_1, \beta_1, \ldots, \alpha_{\frac{n-2}{2}}, \beta_{\frac{n-2}{2}}), \quad i=1,2,\ldots,\frac{n-4}{2}.
\end{aligned}\right\} \tag{20}$$

If we calculate the linear element of the surface (19) we find

$$\overline{d\sigma_k}^2 = \sum_1^{n-1} dX_i^2 = \sum_1^{\frac{n-2}{2}} (\rho_k d\beta - dF'_{\alpha_i})(\rho_k d\alpha_i - dF'_{\beta_i}), \tag{21}$$

where the subscript k refers to the focal sheet S_k. If $n=4$, $k=2$ and $d\sigma^2$ is a perfect square, as was proved in a former paper "On Minimal Lines and Congruences in 4-Dimensional Space," *Trans. Amer. Math. Soc.*, Vol. XII, p. 407. The case $n=4$ seems therefore to be exceptional.

§ 4. *The Surface of Reference.*

If we cut the complex C_{n-2} by the space $X_n=0$ we obtain an $(n-2)$-dimensional surface in a new space which we shall denote by S_{n-1}, viz.:

$$\left.\begin{aligned}
X_1 &= \frac{(\alpha_1+\beta_1)(\Sigma\alpha_i F'_{\alpha_i}+\Sigma\beta_i F'_{\beta_i}-F)-(1+\Sigma\alpha_i\beta_i)(F'_{\alpha_1}+F'_{\beta_1})}{2(\Sigma\alpha_i\beta_i+1)},\\
X_2 &= i\,\frac{(\alpha_1-\beta_1)(\Sigma\alpha_i F'_{\alpha_i}+\Sigma\beta_i F'_{\beta_i}-F)+(1+\Sigma\alpha_i\beta_i)(F'_{\alpha_1}-F'_{\beta_1})}{2(\Sigma\alpha_i\beta_i+1)},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
X_{n-1} &= \frac{\Sigma\alpha_i F'_{\alpha_i}+\Sigma\beta_i F'_{\beta_i}-F}{\Sigma\alpha_i\beta_i+1}.
\end{aligned}\right\} \quad (22)$$

We shall call this surface the *surface of reference* of the complex. It is the envelope of the ∞^{n-2} flats

$$(\alpha_1+\beta_1)X_1+i(\alpha_1-\beta_1)X_2+\ldots.+(1-\Sigma\alpha_i\beta_i)X_{n-1}+F=0, \quad (23)$$

whose direction cosines are

$$\xi_1=\frac{\alpha_1+\beta_1}{1+\Sigma\alpha_i\beta_i},\quad \xi_1=\frac{i(\alpha_1-\beta_1)}{1+\Sigma\alpha_i\beta_i},\quad \ldots,\quad \xi_{n-1}=\frac{1-\Sigma\alpha_i\beta_i}{1+\Sigma\alpha_i\beta_i}; \quad (24)$$

hence, the lines α_i, β_i on the surface of reference correspond to the rectilinear generators of the sphere

$$\sum_1^{n-1}\xi_i^2=1. \quad (25)$$

These generators are minimal straight lines; in fact the linear element of the sphere is found to be

$$d\sigma^2=\frac{2}{(1+\Sigma\alpha_i\beta_i)^2}\,[d\alpha_1\,d\beta_1+d\alpha_2\,d\beta_2+\ldots.+d\alpha_{\frac{n-2}{2}}\,d\beta_{\frac{n-2}{2}}].$$

We shall say that the sphere (25) is the *spherical representation* of the surface of reference. Let M be a point on the surface (22) and m its spherical representation, and consider a rectilinear generator D of the sphere; it meets the sphere at infinity in a point P. To it will correspond on the surface a curve which is the locus of points of contact of tangent flats parallel to the generator D; that is to say, the curve of contact of a cone whose vertex is P and which is tangent to the surface; thus, to the rectilinear generator $\alpha_1=\alpha_2=\ldots.$ $=-\alpha_{\frac{n-2}{2}}=$ const., $\beta_1=\beta_2=\ldots.=\beta_{\frac{n-4}{2}}=$ const. corresponds on the surface the curve determined by the identical equations. We may also note here that the line-element of the surface of reference takes the form

$$ds^2=\sum_1^{\frac{n-2}{2}}{}_i(X_{n-1}\,d\beta_i-dF'_{\alpha_i})(X_{n-1}\,d\alpha_i-dF'_{\beta_i}).$$

We may here mention another property of the expression on the right side of (22). If the expression

$$a\Sigma\alpha_i\beta_i + \Sigma b_i\alpha_i + \Sigma c_i\beta_i + d$$

be added to F the effect amounts to a translation and a stretching or dilation; in fact we have, on putting for F the function

$$F + a\Sigma\alpha_i\beta_i + \Sigma b_i\alpha_i + \Sigma c_i\beta_i + d$$

in (22),

$$\left.\begin{aligned}
\bar{X}_1 &= X_1 - \frac{b_1+c_1}{2} - \frac{a+d}{2}\xi_1,\\
\bar{X}_2 &= X_2 - \frac{i(b_1-c_1)}{2} - \frac{a+d}{2}\xi_2,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
\bar{X}_{n-1} &= X_{n-1} - \frac{d-a}{2} - \frac{a+d}{2}\xi_{n-1},
\end{aligned}\right\} \quad (26)$$

from which it follows that

$$\left[\bar{X}_1 - \left(X_1 - \frac{b_1+c_1}{2}\right)\right]^2 + \left[\bar{X}_2 - \left(X_2 - \frac{i(b_1-c_1)}{2}\right)\right]^2 + \cdots\cdot$$
$$+\left[\bar{X}_{n-1} - \left(X_{n-1} - \frac{d-a}{2}\right)\right]^2 = \left(\frac{a+d}{2}\right)^2 \Sigma\xi_i^2 = \left(\frac{a+d}{2}\right)^2,$$

and this identity proves the proposition.

We shall now prove the following important

THEOREM: *To the edges of regression of the focal surface* (19) *of the isotropic hypercomplex* C_{n-2} *in* S_n *correspond the lines of curvature on the surface of reference in the space* S_{n-1}.

Proof: The differential equations of the lines of curvature are obtained as follows:* a point $\bar{X}_i$ on the normal at a distance λ from the point of tangency is determined by the following system of equations,

$$\bar{X}_1 = X_1 + \lambda\xi_1, \quad \bar{X}_2 = X_2 + \lambda\xi_2, \quad \ldots, \quad \bar{X}_{n-1} = X_{n-1} + \lambda\xi_{n-1}, \qquad (27)$$

the ξ's being the direction cosines of the normal. Substituting the values of

*This method is a generalization of one used by Darboux in his *Leçons*, Vol. I, pp. 238–240, for $n=3$.

X_i obtained from these equations in (23) and differentiating partially with respect to α_i and β_i, we have the system

$$\left.\begin{aligned}
(\alpha_1+\beta_1)\bar{X}_1 + i(\alpha_1-\beta_1)\bar{X}_2 + \ldots + (1-\Sigma\alpha_i\beta_i)\bar{X}_{n-1} + F &= (1+\Sigma\alpha_i\beta_i)\lambda,\\
\bar{X}_1 + i\bar{X}_2 \qquad\qquad -\beta_1\bar{X}_{n-1} + \frac{\partial F}{\partial\alpha_1} &= \beta_1\lambda,\\
\bar{X}_1 - i\bar{X}_2 \qquad\qquad -\alpha_1\bar{X}_{n-1} + \frac{\partial F}{\partial\beta_1} &= \alpha_1\lambda,\\
\bar{X}_3 + i\bar{X}_4 \qquad\qquad -\beta_2\bar{X}_{n-1} + \frac{\partial F}{\partial\alpha_2} &= \beta_2\lambda,\\
\bar{X}_3 - i\bar{X}_4 \qquad\qquad -\alpha_2\bar{X}_{n-1} + \frac{\partial F}{\partial\beta_2} &= \alpha_2\lambda,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,&\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,&\\
\bar{X}_{n-3} + \bar{X}_{n-2} - \beta_{\frac{n-2}{2}}\bar{X}_{n-1} + \frac{\partial F}{\partial\alpha_{\frac{n-2}{2}}} &= \beta_{\frac{n-2}{2}}\lambda,\\
\bar{X}_{n-3} - i\bar{X}_{n-2} - \alpha_{\frac{n-2}{2}}\bar{X}_{n-1} + \frac{\partial F}{\partial\beta_{\frac{n-2}{2}}} &= \alpha_{\frac{n-2}{2}}\lambda.
\end{aligned}\right\} \quad (28)$$

If now we wish to obtain the lines of curvature, we must express the condition that there exists a deplacement on the surface such that a point on the normal, for some specific value of λ conveniently chosen, describes a curve tangent to the normal. We must therefore have

$$\frac{d\bar{X}_1}{\alpha_1+\beta_1} = \frac{d\bar{X}_2}{i(\alpha_1-\beta_1} = \ldots = \frac{d\bar{X}_{n-1}}{1-\Sigma\alpha_i\beta_i} = d\theta. \quad (29)$$

Differentiating then (28), we get from the first equation

$$\begin{aligned}
(\alpha_1+\beta_1)d\bar{X}_1 + i(\alpha_1-\beta_1)d\bar{X}_2 + \ldots + (1+\Sigma\alpha_i\beta_i)d\bar{X}_{n-1} + dF\\
+ \bar{X}_1(d\alpha_1+d\beta_1) + i\bar{X}_2(d\alpha_1-d\beta_1) + \ldots .\\
- \bar{X}_{n-1}(\alpha_1 d\beta_1 + \beta_1 d\alpha_1 + \ldots + \beta_i d\alpha_i)\\
= \lambda d(1+\Sigma\alpha_i\beta_i) + (1+\Sigma\alpha_i\beta_i)d\lambda.
\end{aligned}$$

This equation reduces, when account is taken of the remaining equations (28) and the equation (29), to the following simple one:

$$d\theta(1+\Sigma\alpha_i\beta_i) = d\lambda. \quad (30)$$

Differentiating the last $n-3$ equations (28), we have

$$\begin{aligned}
d\bar{X}_1 + id\bar{X}_2 - \beta_1 d\bar{X}_{n-1} - \bar{X}_{n-1}d\beta_1 + d\frac{\partial F}{\partial\alpha} &= \beta_1 d\lambda + \lambda d\beta_1,\\
d\bar{X}_1 - id\bar{X}_2 - \alpha_1 d\bar{X}_{n-1} - \bar{X}_{n-1}d\alpha_1 + d\frac{\partial F}{\partial\beta} &= \alpha_1 d\lambda + \lambda d\alpha_1,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,&\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,&
\end{aligned}$$

$$d\bar{X}_{n-2} + id\bar{X}_{n-2} - \beta_{\frac{n-2}{2}} d\bar{X}_{n-1} - \bar{X}_{n-1} d\beta_{\frac{n-2}{2}} + d\frac{\partial F}{\partial \alpha_{\frac{n-2}{2}}} = \beta_{\frac{n-2}{2}} d\lambda + \lambda d\beta_{\frac{n-2}{2}},$$

$$d\bar{X}_{n-2} - id\bar{X}_{n-2} - \alpha_{\frac{n-2}{2}} d\bar{X}_{n-1} - \bar{X}_{n-1} d\alpha_{\frac{n-2}{2}} + d\frac{\partial F}{\partial \beta_{\frac{n-2}{2}}} = \alpha_{\frac{n-2}{2}} d\lambda + \lambda d\alpha_{\frac{n-2}{2}},$$

or, by virtue of (29) and (30),

$$\begin{aligned}
&-\bar{X}_{n-1} d\beta_1 + dF'_{\alpha_1} - \lambda d\beta_1 = 0,\\
&-X_{n-1} d\alpha_1 + dF'_{\beta_1} - \lambda d\alpha_1 = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&-\bar{X}_{n-1} d\alpha_{\frac{n-2}{2}} + dF'_{\beta_{\frac{n-2}{2}}} - \lambda d\alpha_{\frac{n-2}{2}} = 0.
\end{aligned}$$

Eliminating λ and X_{n-1} from these equations, we have the following system of differential equations:

$$\frac{d\beta_i}{d\alpha_{\frac{n-2}{2}}} = \frac{dF'_{\alpha_i}}{dF'_{\beta_{\frac{n-2}{2}}}}, \quad \frac{d\alpha_i}{d\alpha_{\frac{n-2}{2}}} = \frac{dF'_{\beta_i}}{dF'_{\beta_{n-1}}}, \qquad \begin{matrix} i = 1, 2, \ldots, \frac{n-4}{2}, \\ k = 1, 2, \ldots, \frac{n-2}{2}. \end{matrix}$$

But this system is precisely the same as the one obtained for the edges of regression on the focal surface in S_n. ((11)b on p. 204.) Hence, the above theorem follows.

Consider now a point M on the focal surface (19) and let E be an edge of regression, L the minimal line tangent to E at M. The line L determines a minimal space Q_{n-1} which is normal to it and at the same time contains L. Cutting this space by the space $X_n = 0$, we obtain a space Q_{n-2} in S_{n-1} which passes through the point P, the intersection of $X_n = 0$ with L. Hence, to the minimal element in S_n corresponds a surface-element in S_{n-1}. To the edge E corresponds a curve E' which is the intersection of the minimal developable determined by E with the space $X_n = 0$. As the line L moves along E the surface-element determined by the point P and the space Q_{n-2} moves along the curve E', which therefore may be considered as an element-band.

We now project M orthogonally on $X_n = 0$ by means of the transformation:*

$$x_1 = X_1, \quad x_2 = X_2, \quad \ldots, \quad x_{n-1} = X_{n-1}, \quad R = iX_n; \tag{31}$$

the point M is projected into a point m which will describe a curve e. But since MP is perpendicular to Q_{n-1}, mP is also perpendicular to Q_{n-2}. The

* For details concerning this transformation see Lie-Scheffers *Berührungstr.*, and also the author's paper in Vol. XII of *Trans. Amer. Math. Soc.*, p. 415, where the case $n = 4$ is treated.

curve e, the projection of E, is therefore tangent to Pm; that is, e is an evolute of the curve E'. To the focal sheet in S_n will correspond in S_{n-1} a focal sheet of the normal hypercomplex determined by the ∞^{n-2} normals along the lines E' which are lines of curvature on the surface of reference. Since through a point on this surface there pass $n-2$ lines of curvature, the normal complex will have $n-2$ sheets. The principal radii of curvature may be obtained from the system (19) by using the orthogonal transformation (31), viz.:

$$\left.\begin{array}{ll} \text{a)} & \left\{\begin{array}{l} x_1 = \frac{1}{2}[(\alpha_1+\beta_1)\rho_k - F'_{\alpha_1} - F'_{\beta_1}], \\ x_2 = \frac{i}{2}[(\alpha_1-\beta_1)\rho_k + F'_{\alpha_1} - F'_{\beta_1}], \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ x_{n-1} = \frac{1}{2}[(1-\Sigma\alpha_i\beta_i)\rho_k - F + \Sigma\alpha_i F'_{\alpha_i} + \Sigma\beta_i F'_{\beta_i}], \end{array}\right. \\ \text{b)} & R_k = \frac{1}{2}[(1+\Sigma\alpha_i\beta_i)\rho_k + F - \Sigma\alpha_i F'_{\alpha_i} - \Sigma\beta_i F'_{\beta_i}], \\ & \qquad\qquad k = 1, 2, \ldots, n-2, \end{array}\right\} \quad (32)$$

which is a generalization of the formulæ given by Darboux in his *Leçons*, Vol. I, p. 245. *To the edges of regression will correspond geodetic lines on the surface* (32)a, *which, in general, has real sheets provided* α_i *and* β_i *are chosen conjugate imaginaries.*

§ 5. *A Flat Spread-Sphere Geometry.*

We proved above (p. 203) that to all the ∞^n hyperboloids in space $\bar{S}_{n-1}$

$$F = a\Sigma\alpha_i\beta_i + \Sigma b_i\alpha_i + \Sigma c_i\beta_i + d \quad (6)$$

there correspond in S_n the ∞^n minimal cones (7). We shall now employ the generalized Euler's transformation:

$$x_i = F'_{\alpha_i}, \quad y_i = \beta_i, \quad p_i = -\alpha_i, \quad q_i = F'_{\beta_i}, \quad z = F - \Sigma\alpha_i F'_{\alpha_i}, \quad \left(i = 1, 2, \ldots, \frac{n-2}{2}\right), \quad (33)$$

by means of which the hyperboloids are transformed into ∞^n flat spreads of $\frac{n-2}{2}$ dimensions

$$x_i = ay_i + b_i, \quad z = \Sigma c\, y_i + d, \quad q_i = -ap_i + c_i; \quad (34)$$

the last set of equations give the relations between the coordinates of the

surface-elements of the flat. We have here a hypercomplex of flats that satisfy the following system of total differential equations:

$$\left.\begin{aligned} &dz - p_1\,dx_1 - p_2\,dx_2 - \ldots - q_1\,dy_1 - q_2\,dy_2 - \ldots - q_{\frac{n-2}{2}}\,dy_{\frac{n-2}{2}} = 0,\\ &dx_i\,dp_{\frac{n-2}{2}} + dy_i\,dq_{\frac{n-2}{2}} = 0, \qquad i = 1, 2, \ldots, \frac{n-2}{2},\\ &dq_k\,dp_{\frac{n-2}{2}} - dp_k\,dq_{\frac{n-2}{2}} = 0, \qquad k = 1, 2, \ldots, \frac{n-4}{2}. \end{aligned}\right\} \quad (35)$$

We may also note that the condition that any two flats shall intersect in a point is unique, viz.:

$$\Sigma(c_i - c_i')(b_i - b_i') + (d' - d)(a - a') \equiv 0, \quad (36)$$

and that to a flat (a, c_i, b_i, d) corresponds in S_n a minimal cone whose vertex is

$$\xi_1 = -\frac{b_1 + c_1}{2}, \quad \xi_2 = \frac{i(b_1 - c_1)}{2}, \quad \ldots, \quad \xi_{n-1} = \frac{a - d}{2}, \quad \xi_n = -\frac{i(a + d)}{2}. \quad (37)$$

Further, to flats that intersect correspond two minimal cones that are tangent to each other; that is, they have a minimal line-element in common.

Suppose now we transform the hypercomplex of minimal lines (5) by means of the transformation (33); it takes the following form:

$$\left.\begin{aligned} X_1 &= \frac{-i(y_1 - p_1)}{1 - \Sigma p_i y_i} X_n + \frac{(y_1 - p_1)(\Sigma q_i y_i - z) - (1 - \Sigma p_i y_i)(x_1 + q_1)}{2(1 - \Sigma p_i y_i)},\\ X_2 &= \frac{-(y_1 + p_1)}{1 - \Sigma p_i y_i} X_n + i\,\frac{(y_1 + p_1)(2 - \Sigma q_i y_i) + (1 - \Sigma p_i y_i)(x_1 - q_1)}{2(1 - \Sigma p_i y_i)},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ X_{n-1} &= \frac{-i(1 + \Sigma p_i y_i)}{1 - \Sigma p_i y_i} X_n + \frac{\Sigma q_i y_i - z}{1 - \Sigma p_i y_i}, \end{aligned}\right\} \quad (38)$$

and the system (4) becomes

$$\left.\begin{aligned} (1 - \Sigma p_i y_i) X_{n-1} + i(1 + \Sigma p_i y_i) X_n + z - \Sigma q_i y_i &= 0,\\ X_1 + iX_2 - y_1 X_{n-1} + iy_1 X_n + x_1 &= 0,\\ X_1 - iX_2 + p_1 X_{n-1} - ip_1 X_n + q_1 &= 0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ X_{n-3} - iX_{n-2} + p_{\frac{n-2}{2}} X_{n-1} - ip_{\frac{n-2}{2}} X_n + q_{\frac{n-2}{2}} &= 0. \end{aligned}\right\} \quad (39)$$

The condition that any two consecutive generators of the complex (38) or (39) shall intersect is

$$\left.\begin{aligned} &\text{a)}\quad dz - p_1\,dx_1 - p_2\,dx_2 - \ldots - q_1\,dy_1 - q_2\,dy_2 - \ldots - q_{\frac{n-2}{2}}\,dy_{\frac{n-2}{2}} = 0, \\ &\text{b)}\quad \begin{cases} dx_i\,dp_{\frac{n-2}{2}} + dy_i\,dq_{\frac{n-2}{2}} = 0, & i = 1, 2, \ldots, \dfrac{n-2}{2}, \\ dq_k\,dp_{\frac{n-2}{2}} - dp_k\,dq_{\frac{n-2}{2}} = 0, & k = 1, 2, \ldots, \dfrac{n-4}{2}. \end{cases} \end{aligned}\right\} \quad (40)$$

Consider now the transformed surface of reference obtained by putting $X_n = 0$ in (38). We have seen that to the edges of regression in S_n correspond lines of curvature on the surface of reference (22) in S_{n-1}. By means of Euler's transformation (33) it is transformed into the following:

$$\left.\begin{aligned} X_1 &= \frac{(y_1 - p_1)(\Sigma y_i q_i - z) - (1 - \Sigma p_i y_i)(x_1 + q_1)}{2(1 - \Sigma p_i y_i)}, \\ X_2 &= i\,\frac{(y_1 + p_1)(z - \Sigma q_i y_i) + (1 - \Sigma p_i y_i)(x_1 - q_1)}{2(1 - \Sigma p_i y_i)}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ X_{n-1} &= \frac{\Sigma y_i q_i - z}{1 - \Sigma p_i y_i}. \end{aligned}\right\} \quad (41)$$

The lines of curvature on (22) are of course also transformed into a set of curves on (41), which we shall now investigate. By Euler's transformation the system of differential equations ((11)b, p. 204) which defines the system of lines of curvature is transformed into the system (40)b, the integration of which will give us a system of curves on (41) which correspond to the lines of curvature on (22). We shall call these lines *asymptotic curves.** Through any point on the surface pass $n-2$ such curves corresponding to the $n-2$ directions, orthogonal to each other on the original surface, which determine the lines of curvature. Their geometric properties are evident from the following considerations:

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*The asymptotic curves defined above are a *special set* belonging to the system

$$\Sigma x_i\,p_i = 0,$$

which has been studied by several others. Cf. Bertini, *Geom. proj. degli Iperspazi*, p. 173. Also, E. O. Lovett, "Contact Transformations between the Fundamental Elements of Space," *Annali di Matematica*, 3d ser., Vol. VII, No. 1, pp. 39–98, and C. L. E. Moore, in a recent paper in the *Annals*, Vol. XIII, p. 94. Of the $\lfloor n-1$ directions on an $(n-1)$-spread in S_n we have chosen $n-1$ special ones, namely, those that correspond on the transform to the $n-1$ directions of curvature. From the standpoint of a generalized flat-sphere geometry, such as has been worked out in this paper, these lines must be designated as *asymptotic lines.*

The condition that the tangent space to a surface $z = f(x_1, \ldots, x_{\frac{n-2}{2}}, y_1, \ldots, y_{\frac{n-2}{2}})$ shall osculate a curve on the surface is

$$\Sigma(dx_i dp_i + dy_i dq_i) = 0, \tag{42}$$

but from (40)b we have

$$dx_i dp_i + dy_i dq_i = 0, \qquad i = 1, 2, \ldots, \frac{n-2}{2}; \tag{43}$$

hence (42) vanishes identically. Therefore, *the hyperplanes of the surface-elements of the surface osculate the asymptotic curves defined by the system of differential equations* (40)b. Again, two consecutive hyperplanes of the surface-elements along the curve contain flat spreads of the form

$$x_i = ay_i + b_i, \quad z = \Sigma c_i y_i + d; \tag{44}$$

in fact, we have on substituting in the equation

$$dz = \Sigma p_i dx_i + q_i dy_i,$$

$$\Sigma c_i dy_i \equiv a\Sigma p_i dy_i + \Sigma q_i dy_i,$$

from which it follows that

$$c_i = ap_i + q_i,$$

which system satisfies the differential equations

$$dq_k dp_{\frac{n-2}{2}} - dp_k dq_{\frac{n-2}{2}} = 0. \tag{44'}$$

But the systems (45′) and (43) are precisely the same as (40)b. The flats (44) are therefore tangents to the surface along the curve, and any one flat passes through the two consecutive points of tangency determined by the two consecutive tangent hyperplanes which osculate the curve.

Remark. For $n = 4$, the flats (44) become the straight lines $x_1 = ay + b$, $z = cy + d$; the two consecutive tangent planes intersect in a line which passes through the two consecutive points of tangency; the curve becomes therefore an asymptotic line on a surface in ordinary space. The only equation remaining in the system (40)b is $dxdp + dydq = 0$, which is the differential equation of asymptotic lines on a surface.

If we consider the space $S_{n-1}(F_1 \alpha_i \beta_i)$ and a surface $F = \phi(\alpha_i \beta_i)$, the system of curves (11)b, which we shall call Euler's lines,* have by Euler's transformation been transformed into asymptotic lines on the transformed

*These curves have been considered for 3-space in an article published by the author in *Trans. Amer. Math. Soc.*, Vol. VI, No. 4, pp. 450–471, entitled: "On a Certain System of Conjugate Lines on a Surface Connected with Euler's Transformation."

surface $z = f(x_i, z_i)$. The hyperplanes of the surface-elements of the surface are common with those of the hyperboloids

$$F = a\Sigma\alpha_i\beta_i + \Sigma b_i\alpha_i + \Sigma c_i\beta_i + d,$$

which are therefore tangents to the surface along the curves. The tangent spaces osculate the curve, since Euler's transformation is a contact-transformation. We shall denote the different spaces used previously as follows:

$$S_n(X_1, X_2, \ldots, X_n), \quad S_{n-1}(X_1, X_2, \ldots, X_{n-1}, P_1, \ldots, P_{n-2}),$$
$$\bar{S}_{n-1}(\alpha_i, \beta, F, F'_{\alpha_i}, F'_{\beta_i}), \quad \bar{\bar{S}}_{n-1}(z, p_i, q_i, x_i, y_i),$$

and we may put in evidence the correspondence between these spaces as follows:

S_n	S_{n-1}	$\bar{S}_{n-1}$	$\bar{\bar{S}}_{n-1}$
Edges of regression on focal surface	Lines of curvature	Euler's lines	Asymptotic lines
∞^n minimal cones $X_n = 0$	∞^n spheres Surface of reference	∞^n hyperboloids Surface of reference	∞^n flats Surface of reference

We may consider the system (41) as a transformation establishing a one-to-one correspondence between the surface-elements of S_{n-1} and $\bar{\bar{S}}_{n-1}$. It is only necessary to add to it the following:

$$P_1 = -\frac{y_1 - p_1}{1 + \Sigma p_i y_i}, \quad P_2 = \frac{i(y_1 + p_1)}{1 + \Sigma p_i y_i}, \quad \ldots, \quad P_{n-2} = \frac{i(y_{\frac{n-2}{2}} + p_{\frac{n-2}{2}})}{1 + \Sigma p_i y_i}, \quad (41')$$

obtained from (3) by putting $X_n = 0$, by writing it in the form

$$X_{n-1} = -\frac{\alpha_1 + \beta_1}{1 - \Sigma a_i\beta_i} X_1 - \frac{i(\alpha_1 - \beta_1)}{1 - \Sigma a_i\beta_i} X_2 - \ldots - \frac{i(\alpha_{\frac{n-2}{2}} - \beta_{\frac{n-2}{2}})}{1 - \Sigma a_i\beta_i}$$

and using Euler's transformation. From the correspondence between the spaces S_{n-1} and $\bar{\bar{S}}_{n-1}$ it is clear that *the transformation* (41) *and* (41') *transforms spheres into flats and lines of curvature on a surface in* S_{n-1} *into asymptotic lines on the transformed surface in* $\bar{\bar{S}}_{n-1}$*; it may therefore be regarded as the extension proper of Lie's classical transformation;* in fact, for $n = 4$ we obtain this latter transformation. It may be noted that the rôle which the line-element plays in Lie's line-sphere geometry is here represented by the surface-elements of ∞^n flats; further, although any space S_n has a sphere-geometry, only an odd space can have a sphere-flat geometry of the kind developed in these pages.

We shall study in detail the system (39) when cut by the space $X_n = 0$. We have then, writing the system in a slightly modified form:

$$\left.\begin{array}{ll}
\text{a)} & \left\{\begin{array}{rcl}
x_1 &=& y_1X_{n-1}-(X_1+iX_2),\\
x_2 &=& y_2X_{n-1}-(X_3+iX_4),\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
x_{\frac{n-2}{2}} &=& y_{\frac{n-2}{2}}X_{n-1}-(X_{n-3}+iX_{n-2}),\\
z &=& -y_1(X_1-iX_2)-y_2(X_3-iX_4)-\cdots.\\
& & \qquad -y_{\frac{n-2}{2}}(X_{n-3}-iX_{n-2})-X_{n-1};
\end{array}\right.\\
\text{b)} & \left\{\begin{array}{rcl}
q_1 &=& -p_1X_{n-1}-(X_1-iX_2),\\
q_2 &=& -p_2X_{n-1}-(X_3-iX_4),\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
q_{\frac{n-2}{2}} &=& -p_{\frac{n-2}{2}}X_{n-1}-(X_{n-3}-iX_{n-2}).
\end{array}\right.
\end{array}\right\} \quad (45)$$

If now we consider (45)a, we see that to the ∞^{n-1} points in S_{n-1} there correspond ∞^{n-1} flats of the complex $a+d=0$; in fact, if we compare (45)a with the system

$$x_i = ay_i + b_i, \quad z = \Sigma c_i y_i + d,$$

the statement is at once proven. The point-geometry in S_{n-1} is therefore equivalent to a flat-geometry in $\bar{\bar{S}}_{n-1}$, namely, the geometry of the flat-complex $a+d=0$. To a point (x_i, y_i, z) in $\bar{\bar{S}}_{n-1}$ corresponds an isotropic flat $I_{\frac{n-2}{2}}$ in in S_{n-1}; in fact, we find from (45)a:

$$\Sigma dX_i^2 = dX_{n-1}[y_1(dX_1 - idX_2) + y_2(dX_3 - idX_4) + \cdots + dX_{n-1}] = 0.$$

To a flat of the system

$$x_i = ay_i + b_i, \quad z = \Sigma c_i y_i + d$$

corresponds a sphere in S_{n-1} which may be written

$$\left[X_1 + \frac{b_1+c_1}{2}\right]^2 + \left[X_2 - i\frac{b_1-c_1}{2}\right]^2 + \cdots + \left[X_{n-1} - \frac{a-d}{2}\right]^2 = \left(\frac{a+d}{2}\right)^2.$$

Denoting the center of this sphere and the radius by $X_1', X_2', \ldots, X_n', R$ we have

$$\left.\begin{array}{l}
X_1' = -\dfrac{b_1+c_1}{2}, \quad X_2' = i\dfrac{b_1-c_1}{2}, \quad \ldots, \quad X_{n-2} = \dfrac{i(b_{\frac{n-2}{2}} - c_{\frac{n-2}{2}})}{2},\\
X_{n-1}' = \dfrac{a-d}{2}, \quad R = \pm\dfrac{a+d}{2},
\end{array}\right\} \quad (46)$$

or, solving,

$$\left.\begin{array}{l}
b_1 = -(X_1' + iX_2'), \quad b_2 = -(X_3' + iX_3'), \quad \ldots, \quad a = X_{n-1}' \pm R;\\
c_1 = -(X_1' - iX_2'), \quad c_2 = -(X_3' - iX_3'), \quad \ldots, \quad d = -(X_{n-1}' \mp R),
\end{array}\right\} \quad (46')$$

from which it follows that conversely to a given sphere correspond two flats which are conjugates with respect to the flat-complex $a+d=0$, that is to say, the complex (45)a. The correspondence is therefore a one-to-two.*

*For a proof of this and similar theorems in the case $n=4$ (3-space) see Lie, "Über Complexe," *Math. Annalen*, Vol. V, pp. 167–175.

A general complex of flat spreads in $\bar{\bar{S}}_{n-1}$ may be written

$$Aa + \Sigma B_i b_i + \Sigma C_i c_i + Dd + Ee + F = 0, \tag{47}$$

using as coordinates of the complex the following:

$$a,\ b_i,\ c_i,\ d,\ e = ad - \Sigma c_i b_i.$$

We shall call such a complex a *flat-complex.* The ∞^{n-1} flats of (47) are transformed into ∞^{n-1} spheres. We shall prove that *these spheres all intersect a given sphere at a constant angle.**

Proof: Let the sphere be written

$$\Sigma (X_i - X_i')^2 = R^2. \tag{48}$$

Substituting the values of b_i, c_i, a and d from (46′) in (47) we find that the coordinates of the center and the radius must satisfy the following relation:

$$\Sigma \left(X'_{2i-1} + \frac{B_i + C_i}{2E}\right)^2 + \Sigma \left(X_{2i} + i\frac{B_i - C_i}{2E}\right)^2 + \ldots + \left(X'_{n-1} - \frac{A - D}{2E}\right)^2$$
$$= R^2 \pm R\frac{D + A}{E} + \frac{(A - D)^2 + 4\Sigma B_i C_i + 4EF}{4E^2}. \tag{49}$$

Let the coordinates of the fixed sphere be denoted by $X'_{10}, X'_{20}, \ldots, X'_{n-10}, R_0$. We have then

$$\sum_1^{n-1} (X_i' - X_{i0}')^2 + (iR - iR_0)^2 = 2R_0 R\,(1 + \cos\phi), \tag{50}$$

where ϕ is the angle between the two spheres. This relation is also true for the null-spheres $R = 0$ contained in the complex, and since in this case the null-sphere is a point on the fixed sphere, the equation of this sphere may be obtained by putting $R = 0$ in (50) and considering X_i' as running coordinates. The fixed sphere may therefore be written

$$\Sigma (X_i' - X_{i0}')^2 = R_0^2. \tag{51}$$

But X_i' must satisfy (49) when in this equation R is put equal to zero; hence (51) becomes

$$\sum_1^{\frac{n-2}{2}} \left(X_{2i-1} + \frac{B_i + C_i}{2E}\right)^2 + \sum_1^{n-2} \left[X_{2i} + \frac{i(B_i - C_i)}{2E}\right]^2 + \left(X_{n-1} - \frac{A - D}{2E}\right)^2$$
$$= \frac{(A - D)^2 + 4\Sigma B_i C_i + 4EF}{4E^2}.$$

The coordinates of the fixed sphere are therefore

$$X_{2i-10} = -\frac{B_i + C_i}{2E},\quad X_{2i0} = -\frac{i(B_i - C_i)}{2E},$$
$$R_0 = \frac{1}{2E}\sqrt{(A - D)^2 + 4\Sigma B_i C_i + 4EF},$$

*This theorem is a generalization of Lie's theorem on sphere-complexes consisting of all the spheres in 3-space that intersect a given sphere at a constant angle. See Lie-Scheffers *Berührungstr.*, pp. 651–653. For $n = 4$, 3-space, the flat-complex becomes a linear complex, or null-system.

so that the equation (49) of the sphere-complex may be written

$$\Sigma(X'_{2i-1} - X_{2i-1\,0})^2 + \Sigma(X'_{2i} - X_{2i\,0})^2 + (iR - iR_0)^2 - 2RR_0(1 + \cos\phi) = 0,$$

where

$$\cos\phi = \frac{\pm(D + A)}{\sqrt{(A - D)^2 + 4\Sigma B_i C_i + 4EF}};$$

ϕ has therefore the same value for all spheres of the complex. Q. E. D.

If the ∞^{n-1} spheres of the complex all *touch* the given sphere, we have $\phi = 0$; that is, we must have

$$(D + A)^2 = (A - D) + 4\Sigma C_i B_i + 4EF,$$

or,

$$\Sigma B_i C_i + EF - AD = 0. \tag{52}$$

In the space $\overline{\overline{S}}_{n-1}$ we have a flat-complex consisting of all the ∞^{n-1} flats that intersect a given flat. We shall call this complex a *special flat-complex.* We may also extend the idea of *involution* to flat-complexes just as in ordinary space to line-complexes. Let there be given two complexes C and C', and consider the family of complexes $C + kC'_1 = 0$, which we may write

$$(A + kA')a + \Sigma(B_i + kB'_i)b_i + \Sigma(C_i + kC'_i)c_i + (D + kD')d + (E + kE')e + F + kF) = 0.$$

In this family there are two special complexes determined by the quadratic equation

$$k^2 L + kM + N = 0, \tag{53}$$

where L, M and N have the following values:

$$L = \Sigma C'_i B'_i + E'F' - A'D',$$
$$M = \Sigma(C_i B'_i + C'_i B_i) + E'F + EF' - A'D - AD',$$
$$N = \Sigma C_i B_i + EF - AD.$$

If now k_1 and k_2 be the roots of the quadratic equation (53), we shall say that the complexes are in involution whenever the double ratio $(0, k_1, \infty, k_2)$ equals -1, that is, whenever $k_1 + k_2 = 0$; *hence the condition for involution is*

$$\Sigma(C_i B'_i + C'_i B_i) + E'F + EF' - A'D - AD' = 0.$$

If $\phi = 90°$, we have $A + D = 0$, and the complexes

$$Aa + \Sigma B_i b + \Sigma C_i c_i - Ad + Ee + F = 0$$

and $a + d = 0$ are in involution.

Conversely, the ∞^{n-1} spheres which cut a given sphere at a constant angle are transformed into the flats of two complexes which are conjugate to each other with respect to the complex $a + d = 0$. The proof is not difficult and need not be given here.

The flats of the linear complex $a + d = 0$ which intersect any given flat f_1 (and also its reciprocal, or polar f_2 with respect to the complex) are trans-

formed into the points of a sphere; the points of the flats f_1 and f_2 are transformed into the flat spread generators of the sphere. These are minimal flats of $\frac{n-2}{2}$ dimensions, and they are seen to be divided into two distinct families just like the rectilinear generators of the sphere. Any two generators of different families intersect in a point on the sphere which is the image of the flat joining two points on f_1 and f_2.

A surface-element in S_{n-1} contains ∞^1 flats passing through its point; to these flats are associated another set of ∞^1 flats which are reciprocal with respect to the complex $a + d = 0$, and which also pass through the point of a second surface-element. In S_{n-1} we get two intersecting minimal flats and ∞^1 spheres touching each other at this point. Hence, to a surface F and all the ∞^1 flats which touch it at a given point there correspond in S_{n-1} a surface and all the ∞^1 spheres which touch it at a point.

If a flat has a singly infinite number of points in common with the surface, we get in S_{n-1} a surface which is touched by a sphere along a curve. To a surface generated by moving a flat so that it is tangent to a given curve (corresponding to developables in ordinary space) corresponds the envelope of ∞^1 spheres which move in such a way that two consecutive spheres touch. To "ruled" surfaces correspond canal surfaces in S_{n-1}, etc. It is evident from the above that the geometry of flat-complexes in $\bar{S}_{n-1}$ is equivalent to the geometry of sphere-complexes in S_{n-1} ($n-1$ odd).

We found above that there are two special flat-complexes contained in a family $C + kC' = 0$. Each of these has a flat as directrix. If we consider all the flats which intersect these two directrices, we get a complex of ∞^{n-2} flats which we may denote by the symbol C_{n-2}. Hence *to the ∞^{n-2} flats which intersect two fixed flats there correspond the ∞^{n-2} spheres which intersect two fixed spheres at constant angles ϕ_1 and ϕ_2.*

We may continue in this manner building up a theory of sphere- and flat-complexes, just as has been done by Lie in his paper on Complexes in *Mathematische Annalen*, Vol. V, for ordinary space.

PART II.

§ 1. *The Flat-Sphere Transformations.*

It remains now to find all the contact-transformations that have the same property as (41) and (41′). We know that the transformations of the conform group in S_n leave invariant the ∞^n minimal cones and that to this group there corresponds in S_{n-1} a group of contact-transformations which leaves invariant the ∞^n spheres of S_{n-1}. We may obtain the characteristic functions of this group in the following manner:

The conform group in S_n is*

$$\frac{\partial f}{\partial X_\nu},\quad X_\mu\frac{\partial f}{\partial X_\nu}-X_\nu\frac{\partial f}{\partial X_\mu},\quad \sum_1^n X_j\frac{\partial f}{\partial X_j},\quad 2X_\nu\sum_1^n X_j\frac{\partial f}{\partial X_\nu}-\frac{\partial f}{\partial X}\sum_1^n X_j^2, \tag{1}$$

$$\nu,\mu=1,2,\ldots,n.$$

The sphere corresponding to a minimal cone whose vertex is $(X_1, X_2, \ldots, X_n)$ we found to be (p. 203)

$$(x_1-X_1)^2+(x_2-X_2)^2+\ldots+(x_{n-1}-X_{n-1})^2+X_n^2=0, \tag{2}$$

which it will be convenient to put in the following form, introducing auxiliary variables $\phi_1, \phi_2, \ldots, \phi_{n-1}$,

$$x_1=X_1+iX_n\cos\phi_1,\quad x_2=X_2+iX_n\cos\phi_2,\quad \ldots,\quad x_{n-1}=X_{n-1}+iX_n\cos\phi_{n-1}, \tag{3}$$

where $\sum_1^{n-1}\cos^2\phi_i=1$.

Differentiating (2) partially with respect to $x_1, x_2, \ldots, x_{n-2}$ in succession, we have

$$\left.\begin{aligned} x_1-X_1+(x_{n-1}-X_{n-1})p_1=0,\quad x_2-X_2+(x_{n-1}-X_{n-1})p_2=0,\quad \ldots,\\ x_{n-2}-X_{n-2}+(x_{n-1}-X_{n-1})p_{n-2}=0. \end{aligned}\right\} \tag{4}$$

From (3) and (4) we find

$$p_1=-\frac{\cos\phi_1}{\cos\phi_{n-1}},\quad p_2=-\frac{\cos\phi_2}{\cos\phi_{n-1}},\quad \ldots,\quad p_{n-2}=-\frac{\cos\phi_{n-2}}{\cos\phi_{n-1}}, \tag{5}$$

from which we obtain the following equations:

$$\left.\begin{aligned} \cos\phi_1=\frac{-p_1}{\sqrt{1+\Sigma p_i^2}},\quad \cos\phi_2=\frac{-p_2}{\sqrt{1+\Sigma p_i^2}},\quad \ldots,\quad \cos\phi_{n-2}=\frac{-p_{n-2}}{\sqrt{1+\Sigma p_i^2}},\\ 1+\Sigma p_i^2=\frac{1}{\cos^2\phi_{n-1}}. \end{aligned}\right\} \tag{5'}$$

From (3) and (5) we have, making x_i, X_i and ϕ_i vary,

$$\left.\begin{aligned} \delta x_1 &= \delta X_1+i\cos\phi_1\,\delta X_n-iX_n\sin\phi_1\,\delta\phi_1,\\ \delta x_2 &= \delta X_2+i\cos\phi_2\,\delta X_n-iX_n\sin\phi_2\,\delta\phi_2,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ \delta x_{n-1} &= \delta X_{n-1}+i\cos\phi_{n-1}\,\delta X_n-iX_n\sin\phi_{n-1}\,\delta\phi_{n-1}, \end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned} \delta p_1 &= \sqrt{1+\Sigma p_i^2}\sin\phi_1\,\delta\phi_1-\frac{\cos\phi_1\sum_1^{n-2}p_i\,\delta p_i}{\sqrt{1+\Sigma p_i^2}},\\ \delta p_2 &= \sqrt{1+\Sigma p_i^2}\sin\phi_2\,\delta\phi_2-\frac{\cos\phi_2\sum_1^{n-2}p_i\,\delta p_i}{\sqrt{1+\Sigma p_i^2}},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ \delta p_{n-2} &= \sqrt{1+\Sigma p_i^2}\sin\phi_{n-2}\,\delta\phi_{n-2}-\frac{\cos\phi_{n-2}\sum_1^{n-2}p_i\,\delta p_i}{\sqrt{1+\Sigma p_i^2}}. \end{aligned}\right\} \tag{7}$$

* See Lie, *Theorie der Transformationsgruppen*, Vol. III, p. 334.

If now by means of the equations (7) we eliminate $\delta\phi_i$ from (6) we have the following system of equations:

$$\left.\begin{aligned}
\delta x_1 &= \delta X_1 + i\cos\phi_1\,\delta X_n - iX_n\frac{\sqrt{1+\Sigma p_i^2}\,\delta p_1 + \cos\phi_1\,\Sigma p_i\,\delta p_i}{1+\Sigma p_i^2},\\
\delta x_2 &= \delta X_2 + i\cos\phi_2\,\delta X_n - iX_n\frac{\sqrt{1+\Sigma p_i^2}\,\delta p_2 + \cos\phi_2\,\Sigma p_i\,\delta p_i}{1+\Sigma p_i^2},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
\delta x_{n-1} &= dX_{n-1} + i\cos\phi_{n-1}\,\delta X_n - i\,X_n\frac{\overset{n-2}{\Sigma}\,p_i\,\delta p_i}{[1+\Sigma p_i^2]^{\frac{1}{2}}}\,.
\end{aligned}\right\}\qquad(8)$$

Introducing in (8) the values of $\delta X_1, \delta X_2, \ldots, \delta X_n$ obtained from (1) and eliminating $X_1, X_2, \ldots, X_{n-1}$ from the resulting equations, we obtain a set of equations which must be identically satisfied for all values of X_n. We thus obtain the values of the increments $\delta x_1, \ldots, \delta x_{n-1}, \delta p_1, \ldots, \delta p_{n-2}$. Thus, suppose we take the transformation

$$Uf = X_2\frac{\partial f}{\partial X_1} - X_1\frac{\partial f}{\partial X_2}\,;$$

we have here

$$\delta X_1 = X_2\,\delta t,\quad \delta X_2 = -X_1\,\delta t,\quad \delta X_i = 0,\quad (i = 3, 4, \ldots, n).$$

The system (8) now becomes

$$\begin{aligned}
\delta x_1 &= X_2\,\delta t - iX_n\frac{\sqrt{1+\Sigma p_i^2}\,\delta p_1 + \cos\phi_1\,\Sigma p_i\,\delta p_i}{1+\Sigma p_i^2},\\
\delta x_2 &= -X_1\,\delta t - iX_n\frac{\sqrt{1+\Sigma p_i^2}\,\delta p_2 + \cos\phi_2\,\Sigma p_i\,\delta p_i}{1+\Sigma p_i^2},\\
\delta x_3 &= -iX_n\frac{\sqrt{1+\Sigma p_i^2}\,\delta p_3 + \cos\phi_3\,\Sigma p_i\,\delta p_i}{1+\Sigma p_i^2},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
\delta x_{n-1} &= -iX_n\frac{\Sigma p_i\,\delta p_i}{[1+\Sigma p_i^2]^{\frac{1}{2}}}.
\end{aligned}$$

Introducing in this system the values of X_1 and X_2 from (3), we have

$$\begin{aligned}
\delta x_1 &= (x_2 - iX_n\cos\phi_2)\,\delta t - iX_n\frac{\sqrt{1+\Sigma p_i^2}\,\delta p_1 + \cos\phi_1\,\Sigma p_i\,\delta p_i}{1+\Sigma p_i^2},\\
\delta x_2 &= -(x_1 - iX_n\cos\phi_1)\,\delta t - iX_n\frac{\sqrt{1+\Sigma p_i^2}\,\delta p_2 + \cos\phi_2\,\Sigma p_i\,\delta p_i}{1+\Sigma p_i^2},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
\delta x_{n-1} &= -iX_n\frac{\Sigma p_i\,\delta p_i}{(1+\Sigma p_i^2)^{\frac{1}{2}}}.
\end{aligned}$$

From the last equation we get $\Sigma p_i \delta p_i = 0$, and from the first $n-2$

$$\delta x_1 = x_2 \delta t, \quad \delta x_2 = -x_1 \delta t, \quad \delta x_i = 0, \quad i = 3, \ldots, n-1,$$
$$\delta p_1 = p_2 \delta t, \quad \delta p_2 = -p_1 \delta t, \quad \delta p_i = 0, \quad i = 3, \ldots, n-1,$$

and the characteristic function

$$W = p_1 \xi_1 + p_2 \xi_2 + \ldots + p_{n-2} \xi_{n-2} - \xi_{n-1} = p_1 x_2 - p_2 x_1.$$

In this manner we obtain the following characteristic functions of the required group of contact-transformations, using capital letters in order to conform with the notation of the formulæ (41),

$$\left.\begin{array}{l} 1, \quad P_1, P_2, \ldots, P_{n-2}, \quad X_\mu P_\nu - X_\nu P_\mu, \quad X_\nu + P_\nu X_{n-1}, \\ X_\nu \sqrt{1+\Sigma P_j^2}, \quad X_{n-1}\sqrt{1+\Sigma P_j^2}, \quad \sqrt{1+\Sigma P_j^2}, \quad X_{n-1} - \sum_1^{n-2} P_j X_j, \\ 2X_n X_{n-1} + P_\nu \sum_1^{n-1} X_j^2 - 2X_\nu \sum_1^{n-2} P_j X_j, \\ -2X_{n-1}\Sigma P_j X_j - \sum_1^{n-2} X_j^2 + X_{n-1}^2, \quad \sum_1^{n-1} X_j^2 \sqrt{1+\Sigma P_j^2}. \end{array}\right\} \quad (9)$$

There exist therefore, in the space S_{n-1}, $\frac{n+1 \cdot n+2}{2}$ *infinitesimal contact-transformations which leave invariant the* ∞^n *spheres of that space.* This theorem has been proved by Lie.*

If now we superpose the generalized Lie's transformation (41) and (41′) on each one of these transformations, we shall obtain a group of transformations in $\overline{\overline{S}}_{n-1}$ which leaves invariant the ∞^n flats (44) and transforms the asymptotic lines of an $(n-2)$-spread into asymptotic lines on the transformed spread. We shall show how to obtain the characteristic functions of this group, which is similar to the group whose characteristic functions are given by (9).

If $U(X_i, P_i)$ represents any characteristic function of the set (9) and $V(x_i, y_i, p_i, q_i, z)$ the required function, the following relation exists between them :†

$$\rho V(x_i, y_i, p_i, q_i, z) = U(X_i, P_i),$$

where ρ is determined by the relation

$$dX_{n-1} - \sum_1^{n-2} P_j dX_j = \rho (dz - \Sigma p_j dx_j - \Sigma q_j dy_j).$$

* Lie, *Götting. Nachr.*, May, 1871, *Math. Ann.*, Bd. V (1872). The above method of deriving the characteristic functions of the general group of contact-transformations which change spheres into spheres is a generalization of the one used in Lie-Scheffers *Berührungstr.*, pp. 441–443, for $n=3$, or, rather, a generalization of the method when reversed, that is, going from 3-space back to 2-space.

† Lie, *Theorie der Transformationsgruppen*, Vol. II, pp. 527–528.

From the equations (41) and (41′) we find, after an easy calculation, that $\rho = \dfrac{1}{1+\Sigma p_j y_j}$, so that we have

$$V(x_i, y_i, p_i, q_i, z) = U(X_i, P_i)\ (1 + \Sigma p_j y_j),$$

and V may be obtained by expressing U in terms of the coordinates x_i, y_i, p_i, q_i and z. We shall not repeat the rather long and tedious calculations here, but write the functions in order corresponding to (9):

$$\left.\begin{array}{c}
1, \quad y_i - p_i, \quad y_i + p_i, \quad x_k p_k + y_k q_k, \\
x_k y_i - x_i y_k + q_i p_k - p_i q_k + p_i x_k - p_k x_i + q_i y_k - q_k y_i, \\
x_i y_k - y_i x_k + p_i q_k - q_i p_k + p_i x_k - p_k x_i + q_i y_k - y_i q_k, \\
y_i x_k - x_i y_k + p_i q_k - q_i p_k - p_i x_k - q_k y_i - q_i y_k - x_i p_k, \\
x_i y_k - x_k y_i + q_i p_k - q_k p_i - p_i x_k - q_i y_k - q_k y_i - x_i p_k, \\
z y_k - y_k \Sigma q_j y_j - x_k \Sigma p_j y_j - x_k - z p_k + p_k \Sigma q_j y_j - p_k \Sigma p_j y_j - p_k, \\
-z y_k + y_k \Sigma q_j y_j + x_k \Sigma p_j y_j + x_k - z p_k + p_k \Sigma q_j y_j - p_k \Sigma p_j y_j - p_k, \\
-z y_k + y_k \Sigma y_j q_j + x_k \Sigma p_j y_j - x_k + z p_k - p_k \Sigma y_j q_j + q_k \Sigma p_j y_j - q_k, \\
-z y_k + y_k \Sigma y_j q_j + x_k \Sigma p_j y_j - x_k - z p_k + p_k \Sigma y_j q_j - q_k \Sigma p_j y_j + q_k, \\
\Sigma q_j y_j - z, \quad 1 - \Sigma p_j y_j, \quad \Sigma x_j p_j - z, \\
z x_k - y_k \Sigma x_j q_j - x_k \Sigma x_j p_j + z q_k + p_k \Sigma x_j p_j - q_k \Sigma x_j p_j, \\
z x_k - y_k \Sigma x_j q_j - x_k \Sigma x_j p_j - z q_k - p_k \Sigma x_j q_j + q_k \Sigma x_j p_j, \\
z^2 - z \Sigma q_j y_j - z \Sigma x_j p_j + \Sigma x_j p_j \Sigma q_j y_j - \Sigma x_j q_j \Sigma y_j p_j - \Sigma x_j q_j, \\
z^2 - z \Sigma q_j y_j - z \Sigma x_j p_j + \Sigma x_j p_j \Sigma q_j y_j - \Sigma x_j q_j \Sigma y_j p_j + \Sigma x_j q_j, \\
\left(i, k = 1, 2, \ldots, \dfrac{n-2}{2}, i \neq k\right).
\end{array}\right\} \quad (10)$$

By properly combining, that is, adding and subtracting, we may reduce this set to the following simpler one:

$$\left.\begin{array}{l}
1, \quad p_i, \quad y_i, \quad x_k p_k + y_k q_k, \quad x_k y_i - x_i y_k, \quad q_i p_k - p_i q_k, \\
p_i x_k + q_i y_k, \quad x_i p_k + y_i q_k, \quad x_k, \quad q_k, \quad z y_k - y_k \Sigma q_j y_j - x_k \Sigma p_j y_j, \\
z p_k - p_k \Sigma q_j y_j + q_k \Sigma p_j y_j, \quad \Sigma q_j y_j, \quad \Sigma p_j y_j, \quad 2z - \Sigma (x_j p_j + q_j y_j), \\
z x_k - y_k \Sigma x_j q_j - x_k \Sigma x_j p_j, \quad z q_k + p_k \Sigma x_j q_j - q_k \Sigma x_j p_j, \\
z^2 - z \Sigma q_j y_j - z \Sigma x_j p_j + \Sigma x_j p_j \Sigma q_j y_j - \Sigma x_j q_j \Sigma y_j p_j \Sigma x_j q_j.
\end{array}\right\} \quad (11)$$

As a check on the accuracy of these calculations we may observe that for $n = 4$ we get the following functions:

$$\begin{array}{c}
1, \quad p, \quad y, \quad xp + qy, \quad x, \quad q, \quad zy - y^2 q - xyp, \quad zp, \quad qy, \quad py, \\
2z - xp - qy, \quad zx - xyq - x^2 p, \quad zq, \quad z^2 - zyq - zxp, \quad xq,
\end{array}$$

which are the characteristic functions of the projective group in 3-space considered as a contact-transformation.

The corresponding group of contact-transformations is now

$$\left.\begin{array}{l}
\dfrac{\partial f}{\partial z},\ \dfrac{\partial f}{\partial x_i},\ \dfrac{\partial f}{\partial q_i}+y_i\dfrac{\partial f}{\partial z},\ x_k\dfrac{\partial f}{\partial x_k}+y_k\dfrac{\partial f}{\partial y_k}-p_k\dfrac{\partial f}{\partial p_k}-q_k\dfrac{\partial f}{\partial q_k},\\
x_k\dfrac{\partial f}{\partial q_i}-x_i\dfrac{\partial f}{\partial q_k}+y_i\dfrac{\partial f}{\partial p_k}-y_k\dfrac{\partial f}{\partial p_i}+(x_ky_i-x_iy_k)\dfrac{\partial f}{\partial z},\\
q_i\dfrac{\partial f}{\partial x_k}-q_k\dfrac{\partial f}{\partial x_i}+p_k\dfrac{\partial f}{\partial y_i}-p_i\dfrac{\partial f}{\partial y_k}+(q_ip_k-q_kp_i)\dfrac{\partial f}{\partial z},\\
x_k\dfrac{\partial f}{\partial x_i}+y_k\dfrac{\partial f}{\partial y_i}-p_i\dfrac{\partial f}{\partial p_k}-q_i\dfrac{\partial f}{\partial q_k},\ x_i\dfrac{\partial f}{\partial x_k}+y_i\dfrac{\partial f}{\partial y_k}-p_k\dfrac{\partial f}{\partial p_i}\\
-q_k\dfrac{\partial f}{\partial q_i},\ \dfrac{\partial f}{\partial p_i}+x_i\dfrac{\partial f}{\partial z},\ \dfrac{\partial f}{\partial y_i},\ x_k\Sigma y_j\dfrac{\partial f}{\partial x_j}+y_k\Sigma y_j\dfrac{\partial f}{\partial y_j}\\
+y_kz\dfrac{\partial f}{\partial z}-y_k\Sigma p_j\dfrac{\partial f}{\partial p_j}+x_k\Sigma p_j\dfrac{\partial f}{\partial q_j}+(\Sigma q_jy_j-q_ky_k-z)\dfrac{\partial f}{\partial q_k}\\
+\Sigma p_iy_i\dfrac{\partial f}{\partial p_k},\ q_k\Sigma y_j\dfrac{\partial f}{\partial x_j}-p_k\Sigma y_j\dfrac{\partial f}{\partial y_j}+(z-\Sigma q_jy_j)\dfrac{\partial f}{\partial x_k}\\
+\Sigma p_iy_i\dfrac{\partial f}{\partial y_k}+(q_k\Sigma y_jp_j-p_k\Sigma y_jq_j)\dfrac{\partial f}{\partial z}-p_k\Sigma p_j\dfrac{\partial f}{\partial p_j}-q_k\Sigma p_j\dfrac{\partial f}{\partial q_j},\\
\Sigma y_j\dfrac{\partial f}{\partial y_j}-\Sigma q_j\dfrac{\partial f}{\partial q_j},\ \Sigma y_j\dfrac{\partial f}{\partial x_j}-\Sigma p_j\dfrac{\partial f}{\partial q_j},\ \Sigma\Big(x_j\dfrac{\partial f}{\partial x_j}+y_j\dfrac{\partial f}{\partial y_j}\\
+p_j\dfrac{\partial f}{\partial p_j}+q_j\dfrac{\partial f}{\partial q_j}\Big)+2z\dfrac{\partial f}{\partial z},\ p_k\Sigma x_j\dfrac{\partial f}{\partial y_j}-q_k\Sigma x_j\dfrac{\partial f}{\partial x_j}\\
+(z-\Sigma x_jp_j)\dfrac{\partial f}{\partial y_k}+\Sigma q_jx_j\dfrac{\partial f}{\partial x_k}+(p_k\Sigma x_jq_j-q_k\Sigma x_jp_j)\dfrac{\partial f}{\partial z}\\
-q_k\Sigma q_j\dfrac{\partial f}{\partial q_j}-p_k\Sigma q_j\dfrac{\partial f}{\partial p_j},\ y_k\Sigma x_j\dfrac{\partial f}{\partial y_j}+x_k\Sigma x_j\dfrac{\partial f}{\partial x_j}+x_kz\dfrac{\partial f}{\partial z}\\
-x_k\Sigma q_j\dfrac{\partial f}{\partial q_j}+y_k\Sigma q_j\dfrac{\partial f}{\partial q_j}+(\Sigma p_jx_j-p_kz_k-z)\dfrac{\partial f}{\partial p_k}+\Sigma q_jx_j\dfrac{\partial f}{\partial q_k},\\
(\Sigma q_jy_j-z)\Sigma\Big(x_j\dfrac{\partial f}{\partial x_j}+q_j\dfrac{\partial f}{\partial q_j}\Big)+(\Sigma x_jp_j-z)\Sigma\Big(y_j\dfrac{\partial f}{\partial y_j}+p_j\dfrac{\partial f}{\partial p_j}\Big)\\
-\Sigma x_jq_j\Sigma y_j\dfrac{\partial f}{\partial x_j}-\Sigma y_jp_j\Sigma x_j\dfrac{\partial f}{\partial y_j}+\Sigma y_jp_j\Sigma q_j\dfrac{\partial f}{\partial p_j}+\Sigma x_jp_j\Sigma p_j\dfrac{\partial f}{\partial q_j},\\
\Sigma x_j\dfrac{\partial f}{\partial y_j}-\Sigma q_j\dfrac{\partial f}{\partial p_j}.\qquad \Big(i,k=1,2,\ldots,\dfrac{n-2}{2},\ i\neq k.\Big)
\end{array}\right\}\quad(12)$$

We shall denote the $\infty^{\frac{n+1.n+2}{2}}$ finite transformations of this group by the symbol $G_{\frac{n+1.n+2}{2}}$. We have then proved the following important

THEOREM: *There exist in* $\bar{\bar{S}}_{n-1}$ $\infty^{\frac{n+1.n+2}{2}}$ *contact-transformations which transform the* ∞^n *flat spreads*

$$\left.\begin{aligned} x_i &= ay_i + b_i, \\ z &= \Sigma c_i y_i + d, \\ \{q_i &= -ap_i + c_i\}, \end{aligned}\right\} \quad (13)$$

into the ∞^n *spheres in* S_{n-1}. *These transformations are obtained by superposing the inverse of the generalized Lie transformation* L *on the* $\infty^{\frac{n+1.n+2}{2}}$ *transformations defined by the group of infinitesimal contact-transformations* (12). *The inverse of* L *is*

$$\left.\begin{aligned} x_i &= -(X_{2i-1} + iX_{2i}) - \frac{P_{2i-1} + iP_{2i}}{1 \pm \sqrt{1 + \Sigma P_j^2}} X_{n-1}, \\ y_i &= -\frac{P_{2i-1} + iP_{2i}}{1 \pm \sqrt{1 + \Sigma P_j^2}}, \quad p_i = \frac{P_{2i-1} - iP_{2i}}{1 \pm \sqrt{1 + \Sigma P_j^2}}, \\ q_i &= -(X_{2i-1} - iX_{2i}) - \frac{P_{2i-1} - iP_{2i}}{1 \pm \sqrt{1 + \Sigma P_j^2}} X_{n-1}, \\ z &= \frac{\Sigma (P_{2j-1} + iP_{2j})(X_{2j-1} - iX_{2j})}{1 \pm \sqrt{1 + \Sigma P_j^2}} - X_{n-1}.^* \end{aligned}\right\} \quad (14)$$

For $n = 4$ *the group* $G_{\frac{n+1.n+2}{2}}$ *becomes the general projective group in 3-space and* L *the classical transformation of Lie.*

The same transformations $L^{-1}\, G_{\frac{n+1.n+2}{2}}$ *transform the asymptotic curves on an* $(n-2)$*-spread* M_{n-2} *in* $\bar{\bar{S}}_{n-1}$ *into lines of curvature on the transform of* M_{n-2} *in the space* S_{n-1}.

The differential equations of the asymptotic curves on M_{n-2} *are*

$$\left.\begin{aligned} &dx_i\, dp_{\frac{n-2}{2}} + dy_i\, dq_{\frac{n-2}{2}} = 0, \quad i = 1, 2, \ldots, \frac{n-2}{2}, \\ &dq_k\, dp_{\frac{n-2}{2}} - dp_k\, dq_{\frac{n-2}{2}} = 0, \quad k = 1, 2, \ldots, \frac{n-4}{2}, \end{aligned}\right\} \quad (15)$$

while those of the lines of curvature on the transform of M_{n-2} *are*

$$(dX_i + P_i\, dX_{n-1})\, dP_{n-2} - (dX_{n-2} + P_{n-2}\, dX_{n-1})\, dP_i = 0, \quad i = 1, 2, \ldots, n-3. \quad (16)$$

The last statement may be proved as follows:

* These formulæ are generalizations of those obtained by Darboux in his *Leçons*, Vol. IV, p. 175, for $n=4$. We note that the correspondence is *one-to-two* on account of the double sign of the radical; *i. e.*, to a flat corresponds *one* sphere, while to a sphere correspond *two* flats.

On the $(n-2)$-spread in S_{n-1} the surface-elements along a line of curvature must be functions of a single parameter. If two consecutive surface-elements $(X_i + dX_i,\ P_i + dP_i)$ and $(X_i,\ P_i)$ shall belong to the same curvature-sphere, it is necessary and sufficient that the corresponding consecutive normals at the two points intersect. We have then

$$\bar{X}_i = X_i + tP_i, \quad \bar{X}_{n-1} = X_{n-1} - t, \quad (i = 1, 2, \ldots, n-2),$$

where $\bar{X}_i$, $\bar{X}_{n-1}$ are the running coordinates of the normal. But two consecutive normals will intersect if

$$dX_i + P_i dt + t dP_i = 0,$$
$$dX_{n-1} = dt;$$

eliminating t and dt from these equations, we obtain the system (16) written above. Q. E. D.

If we introduce Euler's transformation E,

$$x_i = F'_{\alpha' i}, \quad y_i = \beta_i, \quad p_i = -\alpha_i, \quad q_i = F'_{\beta_i}, \quad z = F - \Sigma \alpha_i F'_{\alpha_i},$$

and superpose it on the group $G_{\frac{n+1 \cdot n+2}{2}}$, we obtain $\infty^{\frac{n+1 \cdot n+2}{2}}$ contact-transformations $E\,G_{\frac{n+1 \cdot n+2}{2}}$ which transform the asymptotic lines on M_{n-2} into the Euler lines on the transform in $\bar{S}_{n-1}$. Further, if we introduce the variables α_i, β_i, F'_{α_i}, F'_{β_i} in the group (12), we obtain a group which has the property of leaving invariant the ∞^n quadrics

$$F = a \Sigma \alpha_j \beta_j + \Sigma b_j \alpha_j + \Sigma c_j \beta_j + d;$$

it is a subgroup of the group (56) in Vol. II, p. 512, of Lie's *Theorie der Transformationsgruppen.*

If we inspect more closely the group (12), we find that it contains a subgroup of linear transformations, which we write as follows, omitting the terms due to the extension of the transformation:

$$\frac{\partial f}{\partial z},\ \frac{\partial f}{\partial x_i},\ \frac{\partial f}{\partial y_i},\ x_i \frac{\partial f}{\partial z},\ y_i \frac{\partial f}{\partial z},\ x_k \frac{\partial f}{\partial x_k} + y_k \frac{\partial f}{\partial y_k},\ x_k \frac{\partial f}{\partial x_i} + y_k \frac{\partial f}{\partial y_i},$$
$$\Sigma y_i \frac{\partial f}{\partial y_i},\ \Sigma y_i \frac{\partial f}{\partial x_i},\ \Sigma x_i \frac{\partial f}{\partial y_i},\ 2z \frac{\partial f}{\partial z} + \Sigma \left(x_i \frac{\partial f}{\partial x_i} + y_i \frac{\partial f}{\partial y_i} \right).$$

The remaining point-transformations are not projective except in the case $n=4$. Hence, *if* $n>4$ *there exists a group of* $\infty^{\left(\frac{n+2}{2}\right)^2+1}$ *linear transformations which will leave invariant the* ∞^n *flat spreads* (13), *and which transform the*

asymptotic lines on an M_{n-2} into asymptotic lines on its transform. In these transformations are contained the following $\frac{(n-2)\,(n-4)}{8}$ rotations:

$$x_k\frac{\partial f}{\partial x_i}-x_i\frac{\partial f}{\partial x_k}+y_k\frac{\partial f}{\partial y_i}-y_i\frac{\partial f}{\partial y_k},$$

$$k=1,2,\ldots,\frac{n-2}{2},\quad i=1,2,\ldots,\frac{u-2}{2},\ k\neq i,$$

and the $n-1$ translations $\frac{\partial f}{\partial z}$, $\frac{\partial f}{\partial x_i}$, $\frac{\partial f}{\partial y_i}$.

Application of the above theory to differential equations will be made in another paper.

WEST VIRGINIA UNIVERSITY, *February* 22, 1912.

The Primitive Groups of Class Twelve.

By W. A. Manning.

The list of the primitive groups of the first thirteen classes prepared by Jordan has long been known to be defective in the case of class 12; in fact, Jordan states explicitly that the calculations for that case, having been gone through but once, may very well contain errors.* In spite of the considerable advances in substitution theory in recent years, the complete *à priori* determination of all the groups of this class still involves a good deal of labor. In order therefore not to intrude too far upon the space of this journal, the writer avoids a redetermination of the primitive groups of class 12 on less than 21 letters.† In a series of memoirs and in his lectures Professor Miller has gone over this ground without use of the list of the groups of class 12. He has thus checked the results of Miss Martin, of Jordan, and of Miss Bennett on the degrees 18, 19, and 20, respectively, according to which there is no primitive group of class 12 on either of these three degrees.

The method here followed will be much the same as that used by the author in treating the classes 6, 8, and 10, with the advantage of having at hand the processes employed in the paper entitled "On the Limit of the Degree of Primitive Groups." This last paper, as well as that "On Multiply Transitive Groups," and that "On the Order of Primitive Groups," will be used freely without specific reference.‡

In addition to the 25 primitive groups of class 12 of degree less than 18, there are four others, one of degree 27, two of degree 28, and one of degree 36.

* C. Jordan, *Comptes Rendus*, Vol. LXXV (1872), p. 1754.

† Miller, Degrees 13 and 14, *Quarterly Journal*, Vol. XXIX (1897), p. 224; Degree 15, *Proceedings of the London Mathematical Society*, Vol. XXVIII (1897), p. 533; Degree 16, American Journal of Mathematics, Vol. XX (1898), p. 229; Degree 17, *Quarterly Journal*, Vol. XXXI (1899), p. 49.

E. N. Martin, Degree 18, American Journal of Mathematics, Vol. XXIII (1901), p. 259.

E. R. Bennett, Degree 20, American Journal of Mathematics, Vol. XXXIV (1912), p. 1.

‡ Classes 6 and 8, American Journal of Mathematics, Vol. XXXII (1910), p. 235; Class 10, *op. cit.*, Vol. XXVIII (1906), p. 226; "On the Limit of the Degree of Primitive Groups," *Transactions of the American Mathematical Society*, Vol. XII (1911, p. 375); "On Multiply Transitive Groups," *op. cit.*, Vol. VII (1906), p. 499; "On the Order of Primitive Groups," *op. cit.*, Vol. X (1909), p. 247.

Only one of these is doubly transitive. It, and its maximal subgroup, which is primitive of degree 27, seem not to have been before noticed.

A regular group of degree and order 12 can not be multiply imprimitive in a sufficiently great number of ways to be a subgroup of a primitive group of higher degree than 20.

Let G contain the substitution

$$s_1 = a_1 a_2 a_3 \cdot b_1 b_2 b_3 \cdot c_1 c_2 c_3 \cdot d_1 d_2 d_3.$$

If it is impossible to say that among the totality of substitutions in G similar to s_1 there are two substitutions such that one connects cycles of the other and has at most one new letter in a cycle, we at any rate know that we can find two substitutions s_1 and s_2 of such a nature that $\{s_1, s_2\}$ has one imprimitive constituent on at least nine letters all of which are displaced by both s_1 and s_2.

If there are just two letters new to s_1 in one cycle of s_2, the group $H_2 \equiv \{s_1, s_2\}$ has an alternating constituent of order 60, which, however, can not be in isomorphism to a transitive group of degree 9.

Then let s_2 have a cycle of new letters $(\alpha\beta\gamma)$. At once, on forming the commutator,

$$s_1 s_2 = s_2 s_1.$$

Hence

$$s_2 = a_1 b_1 c_1 \cdot a_2 b_2 c_2 \cdot a_3 b_3 c_3 \cdot \alpha\beta\gamma.$$

Since the constituent in the letters $a_1, \ldots$. is of degree greater than 4, there is a substitution s_3 similar to s_1, which replaces a_1 by a letter of another set, by the letter d_1, as we may assume without loss of generality, and which has at most one new letter to a cycle. The group H_2 can not be a subgroup of a transitive group of degree less than 20. To confirm this statement only the degrees 16 and 18 require examination. In the case of the degree 16, let δ be the letter fixed by H_2. Now the transitive group that is to include H_2 must be five-fold imprimitive in systems of four letters. But the only possible systems that can have δ in common are $d_1 d_2 d_3 \delta$ and $\alpha\beta\gamma\delta$. If the degree is 18, let δ_1, δ_2 and δ_3 be the three letters fixed by H_2. The only systems of two letters that can have δ_1 in common are $\delta_1\delta_2$ and $\delta_1\delta_3$, and the only system of three letters that can contain δ_1 is $\delta_1\delta_2\delta_3$. Then H_3 is intransitive. The constituent on the letters $\alpha, \ldots$. is not of degree greater than 4, because the letters α, β, γ can not occur in more than one cycle of s_3. Since the other constituent can not be of degree 13, 14, or 15, the degrees of the two constituents are fixed as 12 and 4. Now H_4 must be an imprimitive group of degree 20, so that

$$s_4 = (a_1 \omega \varepsilon_1)(--\varepsilon_2)(--\varepsilon_3)(--\varepsilon_4),$$

where ω is one of the letters $\alpha, \beta, \gamma, \delta$ of H_3. If ω is δ, transform s_1 and s_2 by s_4; since both these transforms fix ε_1, and since s_4 can not replace all four letters a_2, a_3, b_1, c_1 by letters not included among these four, we see that ω can not be δ. Let ω be α; returning to s_3, that substitution has a cycle new to s_1 and we may assume $s_3 = (a_1 - d_1) \ldots . (- \delta)$. Now if s_1 connects two cycles of s_3, s_1 has in some cycle two letters new to s_3. Then $s_1^{-1} s_3^{-1} s_1 s_3$, or else $s_1 s_3^{-1} s_1^{-1} s_3$, is of degree less than 12. If s_1 does not connect two cycles of s_3, the first cycle $(a_1 - d_1)$ of s_3 is the only cycle of s_3 which contains any one of the six letters a_1, a_2, a_3, d_1, d_2, or d_3, so that s_3 fixes two a's or two d's, and again the degree of $s_1^2 s_3^2 s_1 s_3$ (or $s_1 s_3^2 s_1^2 s_3$) is less than 12. Then s_3 and s_1 are commutative:

$$s_3 = (a_1 b d_1)(a_2 b d_1)(a_3 b d_3)(- - \delta).$$

If s_3 were to connect two cycles of s_2, there would be two letters d in one cycle of s_3, hence

$$s_3 = (a_1 b_1 d_1)(a_2 b_2 d_2)(a_3 b_3 d_3)(- - \delta).$$

Since $s_4^2 s_1 s_4$ fixes ε_1, s_4 can not replace a_2 or a_3 by any letter of the set $a_1 \ldots .$ of H_3. If $s_4 = (a_1 \alpha \varepsilon_1)(- c_1 \varepsilon_2)(- - \varepsilon_3)(- - \varepsilon_4)$, $s_4^2 s_3 s_4$ fixes ε_2, and therefore s_4 can not replace b_1 or d_1 by any letter of the first set $a_1 \ldots .$ of H_3. This is in contradiction with the preceding statement regarding a_2 and a_3. Likewise $s_4 = (- b_1 \varepsilon_2) \ldots .$ is impossible because of the transform of $s_2 s_3 s_2^2$ by s_4, which fixes ε_2 and requires that s_4 replace c_1 and d_1 by letters not in the set $a_1 \ldots .$ of H_3. But we can show that s_4 must replace b_1 or c_1 by a letter ε. One of the substitutions $s_4^2 s_1 s_4$ or $s_4^2 s_2 s_4$ has in its first cycle the letter α and one of the letters a_2, a_3, b_1 or c_1. This transform (σ) must displace the four letters $\varepsilon_1, \ldots, \varepsilon_4$. If two of the letters $\varepsilon_1, \ldots .$ are in the same cycle, the group $\{H_3, \sigma^{-1} H_3 \sigma\}$ has a generator joining the two sets of H_3, with less than four new letters ε. Then in σ the four letters $\varepsilon_1, \ldots .$ are in different cycles and σ is the transform of s_2 by s_4. Hence s_4 replaces b_1 or c_1 by a letter ε_2.

Then there are in G some two substitutions s_1 and s_2 of degree 12 and order 3, such that s_2 connects cycles of s_1 and has at most one new letter in a cycle. Since $H_2 \equiv \{s_1, s_2\}$ has a set of at least six letters, we form the successive groups H_3, H_4, enlarging the set a of H_i (say) by union with some other set of H_i by means of a substitution s_{i+1} which has at most one new letter in any cycle $(i = 2, 3)$.

If H_4 is the first transitive group of the series, H_2 has three sets of degrees ranging from 6,3,3 to 8,4,4. The constituent of highest degree is in all cases in simple isomorphism (to keep up the class) with one of the others, which are of order 3 or 12. But this is impossible. If H_i $(i = 2$ or $3)$ is transitive, the

only degrees to be considered are 16, 18 and 20. Let H_i be of degree 16. Then it is at least five-fold imprimitive in systems of four letters. But if α is one of the new letters $\alpha\beta\gamma\delta$, s_1 transforms the systems to which α belongs into themselves and therefore they are only

$$a_1 a_2 a_3 \alpha, \quad b_1 b_2 b_3 \alpha, \quad c_1 c_2 c_3 \alpha, \quad d_1 d_2 d_3 \alpha, \quad \beta\gamma\delta\alpha.$$

But consider s_2 (or s_3), which displaces all the letters $\alpha\beta\gamma\delta$, and suppose α chosen so that in the cycle of s_2 (or s_3) with α the two remaining letters are from different cycles of s_1. This is legitimate since some generator must permute systems. Now s_2 (or s_3) can transform into itself a system to which a letter left fixed (say a_1) belongs only if the two other letters making up that system (a_2, a_3) are in the same cycle with α. Then the degree of H_i is greater than 16 and i is equal to 3 only. Let the degree of H_3 be 18. The systems consist of two or three letters. Letters in H_{i+1} and not in H_i ($i = 1, 2$) can not be in a system with letters of H_i, because of the smallness of the systems. For some value of i, H_{i+1} introduces at most three new letters. Then one of them is not in more than one system of three letters, nor in more than two systems of two letters. Let H_3 be of degree 20. It is imprimitive. The systems are of two or four letters. Systems of two letters can not be permuted according to a primitive group, because of the substitutions of order 3 and degree 12 which permute systems, and cause the group in the systems to be of class 6 or less and hence alternating. This brings in a substitution of order 7. Hence there are in all cases systems of four letters permuted according to G_{60}^5. Now H_2 has just two sets of letters and is of degree 16. Suppose first that one constituent is of degree 12 and the other of degree 4. Since the latter is generated by two circular substitutions of order 3, it is of order 12. The isomorphism between the two constituents can not be simple. The head of degree and class 12 can only be of order 3, as is seen by recalling that the average number of letters in the intransitive group H_2 is 14. Then H_2 is of order 36. Assume the substitution of the head to be

$$t = a_1 b_1 c_1 \cdot a_2 b_2 c_2 \cdot a_3 b_3 c_3 \cdot \alpha\beta\gamma$$

and

$$s_1 = a_1 a_2 a_3 \cdot b_1 b_2 b_3 \cdot c_1 c_2 c_3 \cdot d_1 d_2 d_3.$$

Now t is in each of the four subgroups of order 9 of H_2, and H_2 is doubly transitive in the systems. Hence

$$s_2 = a_1 \alpha b_2 \cdot b_1 \beta c_2 \cdot c_1 \gamma a_2 \cdot d\, d\, \delta,$$

or

$$s_2' = (a_1 \alpha c_2) \ldots\ldots$$

But since $\{s_1, s_2'\}$ is the transform of $\{s_1, s_2\}$ by $b_1c_1 \cdot b_2c_2 \cdot b_3c_3 \cdot \beta\gamma$, s_2' may be dropped. Now

$$s_1 s_2 = (a_1 c_1 b_1)(a_2 a_3 \alpha b_2 b_3 \beta c_2 c_3 \gamma)(\ldots.),$$

and $(s_1 s_2)^3$ is of degree 9, thus reducing the class of H_2. Let H_2 have two unconnected sets of eight letters each. The group in the systems is again G_{12}^4. The isomorphism between the two groups is simple. The two constituents permute corresponding systems of two letters each. Consider one constituent: it is generated by

$$\bar{s}_1 = a_1 a_2 a_3 \cdot b_1 b_2 b_3$$

and

$$\bar{s}_2 = a_1 \alpha b_2 \cdot b_1 \beta a_2$$

uniquely;

$$(\bar{s}_1 \bar{s}_2)^3 = a_1 b_1 \cdot a_2 b_2 \cdot a_3 b_3 \cdot \alpha\beta$$

is an invariant operator. The group $\overline{H}_2 \equiv \{\bar{s}_1, \bar{s}_2\}$ is of order 24. Then H_2^{16} has an invariant substitution t of order 2 and degree 16. Let $\alpha', \beta', \gamma', \delta'$ be the four letters of H_3 new to H_2. Since H_3 is triply transitive in its systems, we may say that s_3 permutes the systems $a_1 b_1 c_1 d_1$, $\alpha\beta\gamma\delta$, $\alpha'\beta'\gamma'\delta'$. Then t and $s_3^2 t s_3$ have in common four cycles in the letters of the two other systems $a_2 b_2 c_2 d_2$, $a_3 b_3 c_3 d_3$, so that $t s_3^2 t s_3$ is of degree 12. Again, $t s_3^2 t s_3$ and $s_1^2 t s_3^2 t s_3 s_1$ have in common four cycles of the Greek letters, so that $t s_3^2 t s_3 s_1^2 t s_3^2 t s_3 s_1$ is of degree 8.

In conclusion, no primitive group of degree exceeding 20 contains a substitution of order 3 and degree 12.

There is in G a substitution

$$s_1 = a_1 a_2 \cdot b_1 b_2 \cdot c_1 c_2 \cdot d_1 d_2 \cdot e_1 e_2 \cdot f_1 f_2.$$

We shall first set up a list of the diedral rotation groups which two substitutions similar to s_1 may generate and which G may possibly include. Now G can contain no substitution of order 3 and degree 12, no substitution of order 5 and degree less than 20, no substitution of order 7, 11, 13 or 17 and of degree less than 21. Nor has it a circular substitution of degree 18 or less. The group generated by all the substitutions of G which are similar to s_1 is positive. Then the only types of substitutions of degree less than 19 in $\{s_1, \ldots.\}$ are

$$(a_1 a_2 a_3 \ldots. a_{16})(b_1 b_2), \qquad (a_1 a_2 \ldots. a_9)(b_1 b_2 \ldots. b_9),$$
$$(a_1 a_2 \ldots. a_{12})(b_1 b_2 \ldots. b_6), \quad (a_1 a_2 \ldots. a_{12})(b_1 b_2 b_3)(c_1 c_2),$$

and their powers, as can easily be verified directly. And the only types that

can occur in the diedral rotation groups generated by two substitutions similar to s_1 are powers of

$$a_1 a_2 a_3 a_4 \cdot b_1 b_2 b_3 b_4 \cdot c_1 c_2 c_3 c_4 \cdot d_1 d_2,$$
$$a_1 a_2 a_3 a_4 a_5 a_6 \cdot b_1 b_2 b_3 b_4 b_5 b_6 \cdot c_1 c_2 c_3,$$
$$a_1 a_2 a_3 a_4 \cdot b_1 b_2 b_3 b_4 \cdot c_1 c_2 c_3 c_4 \cdot d_1 d_2 d_3 d_4,$$
$$a_1 a_2 a_3 a_4 \cdot b_1 b_2 b_3 b_4 \cdot c_1 c_2 c_3 c_4 \cdot d_1 d_2 \cdot e_1 e_2 \cdot f_1 f_2,$$
$$a_1 a_2 a_3 \cdot b_1 b_2 b_3 \cdot c_1 c_2 c_3 \cdot d_1 d_2 d_3 \cdot e_1 e_2 e_3 \cdot f_1 f_2 f_3.$$

In this statement only two points seem to require special mention. First, if the diedral group in question is of degree greater than 18, it is Abelian. Suppose $\{s_1, s_2\}$ of degree 19 and non-Abelian. Since s_2 now has at least one cycle of letters new to s_1, and vice-versa, $(s_1 s_2)^2$ is of degree 15 at most. If the degree is 15, $s_1 s_2$ has five cycles of three letters each, so that $(s_1 s_2)^3$ is of degree 4. Now $(s_1 s_2)^2$ must displace an odd number of letters and is not of degree 13. Suppose $\{s_1, s_2\}$ of degree 20. If the product $s_1 s_2$ has four transpositions (cycles of order 2), $(s_1 s_2)^2$ is of order 3 and degree 12. The class is lower than 12 if there are more than four cycles of order 2 in the product, and so many of these cycles certainly occur. If $\{s_1, s_2\}$ is of degree 21 or more, the commutator $(s_1 s_2)^2$ is of degree 9 at most. In the second place, it may be asked why the type $(a_1 a_2 \ldots . a_8)(b_1 b_2 \ldots . b_8)$ is excluded from the second list. Since the two constituent diedral groups are generated by an odd and an even substitution of order 2, substitutions of degree 6 do not correspond to substitutions of degree 6, but to substitutions of degree 8, so that this product belongs to a diedral group of class 14.

From these substitutions we now construct the following list of diedral groups. Each is generated by

$$a_1 a_2 \cdot b_1 b_2 \cdot c_1 c_2 \cdot d_1 d_2 \cdot e_1 e_2 \cdot f_1 f_2$$

and a substitution s_2 which is written out along with a numbered symbol "D" to distinguish the group.

$$D_1 : \quad a_1 b_1 \cdot a_2 b_2 \cdot c_1 d_1 \cdot c_2 d_2 \cdot e_1 f_1 \cdot e_2 f_2 ;$$
$$D_2 : \quad a_1 b_1 \cdot a_2 b_2 \cdot c_1 d_1 \cdot c_2 d_2 \cdot e_1 e_2 \cdot \alpha_1 \alpha_2 ;$$
$$D_3 : \quad a_1 b_1 \cdot a_2 b_2 \cdot c_1 d_1 \cdot c_2 d_2 \cdot \alpha_1 \alpha_2 \cdot \beta_1 \beta_2 ;$$
$$D_4 : \quad a_1 b_1 \cdot a_2 b_2 \cdot c_1 c_2 \cdot d_1 d_2 \cdot \alpha_1 \alpha_2 \cdot \beta_1 \beta_2 ;$$
$$D_5 : \quad a_1 b_1 \cdot a_2 b_2 \cdot c_1 c_2 \cdot \alpha_1 \alpha_2 \cdot \beta_1 \beta_2 \cdot \gamma_1 \gamma_2 ;$$
$$D_6 : \quad a_1 b_1 \cdot a_2 b_2 \cdot \alpha_1 \alpha_2 \cdot \beta_1 \beta_2 \cdot \gamma_1 \gamma_2 \cdot \delta_1 \delta_2 ;$$
$$D_7 : \quad a_1 a_2 \cdot b_1 b_2 \cdot c_1 c_2 \cdot \alpha_1 \alpha_2 \cdot \beta_1 \beta_2 \cdot \gamma_1 \gamma_2 ;$$
$$D_8 : \quad a_1 a_2 \cdot b_1 b_2 \cdot \alpha_1 \alpha_2 \cdot \beta_1 \beta_2 \cdot \gamma_1 \gamma_2 \cdot \delta_1 \delta_2 ;$$
$$D_9 : \quad a_1 a_2 \cdot \alpha_1 \alpha_2 \cdot \beta_1 \beta_2 \cdot \gamma_1 \gamma_2 \cdot \delta_1 \delta_2 \cdot \varepsilon_1 \varepsilon_2 ;$$

$$
\begin{aligned}
D_{10}: &\quad \alpha_1\alpha_2 \cdot \beta_1\beta_2 \cdot \gamma_1\gamma_2 \cdot \delta_1\delta_2 \cdot \varepsilon_1\varepsilon_2 \cdot \zeta_1\zeta_2 ;\\
D_{11}: &\quad a_1 b_2 \cdot b_1 c_2 \cdot c_1 d_2 \cdot d_1 a_2 \cdot e_1\alpha_1 \cdot e_2\alpha_2 ;\\
D_{12}: &\quad a_1 b_2 \cdot b_1 c_2 \cdot c_1 d_2 \cdot d_1 a_2 \cdot e_1 f_1 \cdot \alpha_1\alpha_2 ;\\
D_{13}: &\quad a_1 b_1 \cdot a_2 c_1 \cdot b_2 c_2 \cdot d_1\delta \cdot e_1\varepsilon \cdot f_1\zeta ;\\
D_{14}: &\quad a_1 b_1 \cdot a_2 c_1 \cdot d_1 e_1 \cdot d_2\alpha_1 \cdot e_2\alpha_2 \cdot f_1\zeta ;\\
D_{15}: &\quad a_1 b_1 \cdot c_1 d_1 \cdot e_1\varepsilon_1 \cdot e_2\varepsilon_2 \cdot f_1\zeta_1 \cdot f_2\zeta_2 ;\\
D_{16}: &\quad a_1 b_1 \cdot c_1 c_2 \,.\, d_1\delta_1 \cdot d_2\delta_2 \cdot e_1\varepsilon_1 \cdot e_2\varepsilon_2 ;\\
D_{17}: &\quad a_1 b_1 \cdot c_1 d_1 \cdot e_1 e_2 \cdot f_1\zeta_1 \cdot f_2\zeta_2 \cdot \eta_1\eta_2 ;\\
D_{18}: &\quad a_1\alpha_1 \cdot a_2\alpha_2 \cdot b_1\beta_1 \cdot b_2\beta_2 \cdot c_1\gamma_1 \cdot c_2\gamma_2 ;\\
D_{19}: &\quad a_1 b_1 \cdot c_1 d_1 \cdot e_1 f_1 \cdot \alpha_1\alpha_2 \cdot \beta_1\beta_2 \cdot \gamma_1\gamma_2 ;\\
D_{20}: &\quad a_1 b_1 \cdot c_1\gamma_1 \cdot c_2\gamma_2 \cdot d_1\delta_1 \cdot d_2\delta_2 \cdot \varepsilon_1\varepsilon_2 ;\\
D_{21}: &\quad a_1 b_1 \cdot c_1 d_1 \cdot e_1\alpha_1 \cdot e_2\alpha_2 \cdot \beta_1\beta_2 \cdot \gamma_1\gamma_2 ;\\
D_{22}: &\quad a_1\alpha \cdot b_1\beta \cdot c_1\gamma \cdot d_1\delta \cdot e_1\varepsilon \cdot f_1 f_2 ;\\
D_{23}: &\quad a_1\alpha \cdot b_1\beta \cdot c_1\gamma \cdot d_1\delta \cdot e_1\varepsilon \cdot f_1\zeta .
\end{aligned}
$$

Of these groups the first ten only are Abelian. The two groups D_{11} and D_{12} are the same, the order in which the two generators are taken being reversed. The same is true of the three pairs: D_{16} and D_{17}; D_{18} and D_{19}; D_{20} and D_{21}. We shall show that of all these groups only D_6, D_{10} and D_{23} are actually subgroups of a primitive group G of class 12 and of degree greater than 20.

The first of these groups which we take up for discussion is D_{14}. Those substitutions of D_{14} of which we shall make use are:

$$
\begin{aligned}
s_1 &= a_1 a_2 \cdot b_1 b_2 \cdot c_1 c_2 \cdot d_1 d_2 \cdot e_1 e_2 \cdot f_1 f_2 ,\\
s_2 &= a_1 b_1 \cdot a_2 c_1 \cdot d_1 e_1 \cdot d_2\alpha_1 \cdot e_2\alpha_2 \cdot f_1\zeta ,\\
t_1 &= a_2 b_2 \cdot a_1 c_2 \cdot d_2 e_2 \cdot d_1\alpha_1 \cdot e_1\alpha_2 \cdot f_2\zeta ,\\
t_2 &= b_1 c_2 \cdot b_2 c_1 \cdot d_1 e_2 \cdot d_2 e_1 \cdot \alpha_1\alpha_2 \cdot f_1 f_2 ,\\
t_3 &= a_1 b_2 \cdot a_2 c_2 \cdot b_1 c_1 \cdot d_1\alpha_2 \cdot e_1\alpha_1 \cdot f_2\zeta ,\\
t_4 &= a_1 c_1 \cdot a_2 b_1 \cdot b_2 c_2 \cdot d_2\alpha_2 \cdot e_2\alpha_1 \cdot f_1\zeta ,\\
t_5 &= a_1 a_2 \cdot b_1 c_1 \cdot b_2 c_2 \cdot d_1 e_1 \cdot d_2 e_2 \cdot \alpha_1\alpha_2 ,\\
(s_1 s_2)^2 &= a_1 c_2 b_1 \cdot a_2 b_2 c_1 \cdot d_1 d_2\alpha_2 \cdot e_1 e_2\alpha_1 \cdot f_2 f_1\zeta .
\end{aligned}
$$

To the set $a_1 a_2 b_1 b_2 c_1 c_2$ is joined another set of D_{14} by a substitution s_3, similar to s_1, with at most one new letter in any cycle. The degree of the resulting transitive constituent is 9, 12, 16, 18 or 20. If the degree is 18 or 20, since D_{14} can contribute at most 12 letters to a transitive constituent of an intransitive $H_3 \equiv \{D_{14}, s_3\}$, the group H_3 is transitive. However, if s_3 introduces five new letters it can connect only two sets of D_{14}. Hence the degree 20 is not to be considered. In this connection it is well to remark that D_{14} can not be contained in a transitive subgroup H_i of $\{s_1, \ldots .\}$ of degree less than 20. First,

such a subgroup being generated by $s_1, s_2, \ldots$ can have no system of imprimitivity of 8 or 9 letters. Second, it can have no system of two letters because of $(s_1 s_2)^2$. Third, because of $(s_1 s_2)^2$, s_1, s_2, jointly, a system of three, four or six letters involving a common letter can not be chosen in more than one way,

Hence H_3 is intransitive, and we assume for the moment that the larger constituent is of degree 16. Let

$$s_3 = (a_1 d_1)(\beta_1 -)(\beta_2 -)(\beta_3 -)(\beta_4 -)(--).$$

Since s_3 generates with each of the substitutions s_1, s_2, t_1, t_3, t_5 a group of degree 16 or 18, s_3 fixes $a_2, d_2, b_1, e_1, c_2, \alpha_1, b_2, \alpha_2$, and only c_1 and e_2 remain to fill the four places with $\beta_1, \ldots, \beta_4$. Let

$$s_3' = (a_1 \alpha_1)(\beta_1 -)(\beta_2 -)(\beta_3 -)(\beta_4 -)(---).$$

For the same reason as before s_3' fixes $b_1, d_2, c_2, d_1, b_2, e_1, c_1, e_2, a_2, \alpha_2$. The substitutions $d_1 e_1 \cdot d_2 e_2 \cdot \alpha_1 \alpha_2$ and $b_1 c_2 \cdot b_2 c_1 \cdot d_1 d_2 \cdot e_1 e_2 \cdot f_1 f_2$ transform D_{14} into itself and consequently $s_3 = (a_1 d_1) \ldots$. and $s_3' = (a_1 \alpha_1) \ldots$. carry with them $s_3'' = (a_1 d_2) \ldots$, etc. Likewise $(a_1 f_1) \ldots$. and $(a_1 f_2) \ldots$. are conjugate. However, $(a_1 f_1) \ldots$. can not in this case give a constituent of degree 16.

Assume the larger constituent to be of degree 12. Let

$$s_3 = (a_1 f_1)(\beta_1 -)(\beta_2 -)(\beta_3 -)(---)(---),$$

where no letter $d_1, d_2, e_1, e_2, \alpha_1, \alpha_2$ is replaced by β_1, β_2 or β_3. The second constituent is of degree 6 or 8.

Suppose the second constituent of degree 8:

$$s_3 = (a_1 f_1)(\beta_1 -)(\beta_2 -)(\beta_3 -)(\gamma_1 -)(\gamma_2 -).$$

As before, s_3 fixes $a_2, f_2, b_1, \zeta, c_1$, leaving only b_2 and c_2 to fill the three places with β_1, β_2 and β_3. If

$$s_3 = (a_1 \zeta_1)(\beta_1 -)(\beta_2 -)(\beta_3 -)(\gamma_1 -)(\gamma_2 -),$$

s_3 fixes $b_1, f_1, c_2, f_2, b_2, c_1$.

Suppose the second constituent of degree 6, and let

$$s_3 = (a_1 f_1) \ldots\ldots$$

Neither $\{s_1, s_3\}$ nor $\{t_4, s_3\}$ can be of degree 15. For were they of degree 15, the letters $d_1 d_2 e_1 e_2 \alpha_1 \alpha_2$ would form one of the constituents of degree 6 in $\{s_1, s_3\}$ and $\{t_4, s_3\}$. Not being united to new letters they can not form a set of three letters. But since s_1 and t_4 have only two cycles in these letters (as has s_3), this is not possible. If $\{s_2, s_3\}$ is of degree 15, s_3 must displace either c_1 or ζ, but from $\{t_4, s_3\}$, s_3 fixes both c_1 and ζ. Hence a_2, f_2, b_1, ζ and c_1 are fixed by s_3, leaving only b_2 and c_2 for the cycles with β_1, β_2 and β_3. Let

$$s_3 = (a_1 \zeta) \ldots\ldots$$

Since, as before, $\{t_3, s_3\}$ and $\{t_4, s_3\}$ can not be of degree 15, s_3 fixes f_1 and f_2, and therefore neither $\{s_2, s_3\}$ nor $\{t_1, s_3\}$ is of degree 15. For since s_3 has only four letters of one constituent of degree 6, it must have six letters of the other. Then $b_1, f_1, c_2, f_2, b_2, c_1$ are fixed, leaving only a_2 free to enter s_3.

If $s_3 = (a_1 d_1) \ldots$, the second set is of degree 3, 4 or 6. Let

$$s_3 = (a_1 d_1)(--)(--)(f_1 \beta_1)(f_2 \beta_2)(\zeta \beta_3).$$

From $\{s_1, s_3\}$, s_3 fixes a_2 and d_2. Hence $\{t_5, s_3\}$ is such a group as D_{19}, and s_3 displaces one but not both the letters α_1 and α_2. But this would make $\{s_1, s_3\}$ of degree 17. Let

$$s_3 = (a_1 \alpha_1)(--)(--)(f_1 \beta_1)(f_2 \beta_2)(\zeta \beta_3).$$

From $\{s_1, s_3\}$, s_3 must have the cycle $(a_2 \alpha_2)$. Now $\{s_2, s_3\}$ is of degree 16 or 18, so that s_3 fixes b_1, d_2, c_1, and e_2. From $\{t_1, s_3\}$, s_3 fixes c_2, d_1, b_2, and e_1, so that no letter is left for the third cycle of s_3. Now let the second set have just one new letter β. This constituent of degree 4 is of order 24. Since H_3 contains no substitution of degree 12 connecting the set $a_1 \ldots$ and the set $d_1 \ldots$ of D_{14} without a new letter β, the constituent of degree 12 is of order 48, and t_5 is invariant in H_3. Then

$$s_3 = (a_1 d_1)(a_2 e_1) \ldots (\beta_1 -).$$

Now ζ is new to β_1, f_2 to s_2, f_1 to t_1, and with s_1, s_2, and t_1, severally, s_3 generates a group that has a constituent on six or more letters. In one of the groups $\{s_1, s_3\}$, $\{s_2, s_3\}$, $\{t_1, s_3\}$, s_3 has a cycle of "new letters," either $(f_1 \beta)$, $(f_2 \beta)$ or $(\zeta \beta)$. That group can not be the D_{12} because s_3 can not join the cycle $(f_1 f_2)$ to another cycle of s_1; similar statements hold for $(f_1 \zeta)$, $(f_2 \zeta)$ and s_2, t_1, respectively. Hence that group can only be D_{17}. If

$$s_3 = a_1 d_1 \cdot a_2 e_1 \cdot \zeta \beta \ldots,$$

$\{s_1, s_3\}$ is not D_{17}. Nor if

$$s_3 = a_1 d_1 \cdot a_2 e_1 \cdot f_2 \beta \ldots,$$

can $\{s_2, s_3\}$ be D_{17}. Then

$$s_3 = a_1 d_1 \cdot a_2 e_1 \cdot f_1 \beta \ldots,$$

and fixes $c_2, \alpha_1, b_2, \alpha_2$. Now $\{s_1, s_3\}$ is of degree 14, that is, it is D_{11}, and therefore

$$s_3 = (a_1 d_1)(a_2 c_1)(f_1 \beta)(f_2 \zeta)(e_2 -)(d_2 -);$$

the two undetermined letters must be from the same cycle of s_1, but because $\{t_1, s_3\}$ is of degree 16, those two letters are b_1 and c_1. Let

$$s_3 = (a_1 \alpha_1)(a_2 \alpha_2) \ldots (\beta -).$$

Since $(a_1 a_2)$ is a cycle of s_1, $\{s_1, s_3\}$ is not of degree 15, and therefore s_3 displaces ζ. Let us try

$$s_3 = a_1\alpha_1 \cdot a_2\alpha_2 \cdot f_1\zeta \cdot f_2\beta \ldots\ldots$$

To this is conjugate $a_1\alpha_1 \cdot a_2\alpha_2 \cdot f_2\zeta \cdot f_1\beta \ldots\ldots$ under $b_1c_2 \cdot b_2c_1 \cdot b_1d_2 \cdot e_1e_2 \cdot f_1f_2$. Now $\{s_2, s_3\}$, as above, is of degree 16, and hence s_3 fixes b_1, d_2, c_1, e_2. It has d_1 and e_1 in different cycles, accompanied by the only letters we have new to s_2: b_2 and c_2. Then

$$s_3' = a_1\alpha_1 \cdot a_2\alpha_2 \cdot f_1\zeta \cdot f_2\beta \cdot d_1b_2 \cdot e_1c_2,$$

or

$$s_3'' = a_1\alpha_1 \cdot a_2\alpha_2 \cdot f_1\zeta \cdot f_2\beta \cdot d_1c_2 \cdot e_1b_2.$$

Before further study of s_3' and s_3'', let us consider

$$s_3 = (a_1\alpha_1)\,(a_2\alpha_2)\,(\zeta\beta)\ldots\ldots$$

Since $\{s_1, s_3\}$ is D_{17}, s_1 and s_3 have a cycle in common. If this common cycle is not (f_1f_2), s_3 replaces an f by a letter of another cycle of s_1. In the two remaining cycles we have one letter each from the cycles (b_1b_2), (c_1c_2), (d_1d_2), $e_1e_2)$. Now $\{s_2, s_3\}$ can not be of degree 15. If $\{s_2, s_3\}$ is of degree 14, s_3 displaces b_1, c_1, d_2, e_2. If $\{s_2, s_3\}$ is of degree 16, s_3 displaces b_2, c_2, d_1, e_1. We may have

$$s_3''' = a_1\alpha_1 \cdot a_2\alpha_2 \cdot f_1f_2 \cdot \zeta\beta \cdot b_1c_1 \,.\, d_2e_2,$$
$$s_3^{IV} = a_1\alpha_1 \cdot a_2\alpha_2 \cdot f_1f_2 \cdot \zeta\beta \cdot b_1d_2 \cdot c_1e_2,$$
$$s_3^{V} = a_1\alpha_1 \cdot a_2\alpha_2 \cdot f_1f_2 \cdot \zeta\beta \,.\, b_1e_2 \,.\, c_1d_2,$$

and their conjugates under $b_1c_2 \cdot b_2c_1 \cdot d_1d_2 \cdot e_1e_2 \cdot f_1f_2$.

Now pass to H_4; we have only to consider the degrees 20 and 21. Let

$$s_4 = (a_1f_1)\,(\gamma_1 -)\,(\gamma_2 -)\,(\gamma_3 -)\,(\gamma_4 -)\,(\gamma_5 -).$$

From $\{s_1, s_4\}$, $\{s_2, s_4\}$, $\{t_4, s_4\}$, s_4 fixes $a_2, f_2, b_1, \zeta, c_1$. Now $\{t_5, s_4\}$ is of degree 18; it can not be D_{18} because a_2 is fixed by s_4, nor can it be D_{23} because both b_1 and c_1 are fixed. These are the only possibilities. Let

$$s_4 = (a_1\zeta)\,(\gamma_1 -)\,(\gamma_2 -)\,(\gamma_3 -)\,(\gamma_4 -)\,(\gamma_5 -).$$

This s_4 fixes $b_1, f_1, c_2, f_2, b_2, c_1$. Because of $\{s_1, s_4\}$,

$$s_4 = a_1\zeta \cdot a_2\gamma_1 \cdot d_1\gamma_2 \cdot d_2\gamma_3 \cdot e_1\gamma_4 \cdot e_2\gamma_5;$$

but thus s_4 has eight letters new to t_3 and is not commutative with it. Let

$$s_4 = (a_1\beta)\,(\gamma_1 -)\,(\gamma_2 -)\,(\gamma_3 -)\,(\gamma_4 -)\,(\gamma_5 -).$$

Here $\{t_2, s_4\}$ is of higher degree than 18, and non-Abelian.

Next let H_4 be of degree 20, and let

$$s_4 = (a_1f_1)\,(--)\,(\gamma_1 -)\,(\gamma_2 -)\,(\gamma_3 -)\,(\gamma_4 -).$$

The groups $\{s_1, s_4\}$, $\{s_2, s_4\}$, $\{t_4, s_4\}$, $\{s_3, s_4\}$ (s_3 may be any one of the five substitutions s_3', ... given above) are of degree 16 or 18. Hence s_4 leaves fixed $a_2, f_2, b_1, \zeta, c_1, \alpha_1$. Now $\{t_2, s_4\}$ is of degree 17 and hence is of order 6; $\{t_5, s_4\}$ is also of degree 17. But s_4 can not have a cycle in common with both t_2 and t_5. Let

$$s_4 = (a_1\zeta)\,(--)\,(\gamma_1 -)\,(\gamma_2 -)\,(\gamma_3 -)\,(\gamma_4 -).$$

This substitution fixes $b_1, f_1, c_2, f_2, b_2, c_1, \alpha$. The diedral group $\{s_1, s_4\}$ is D_{18}, so that s_4 displaces a_2. This fact makes $\{t_2, s_4\}$ impossible. Let

$$s_4 = (a_1\beta)\,(--)\,(\gamma_1 -)\,(\gamma_2 -)\,(\gamma_3 -)\,(\gamma_4 -).$$

From $\{t_2, s_4\}$ it is clear that s_4 fixes ζ and a_2. Since the products $s_1 s_4$ and $t_5 s_4$ have the cycle $(a_1 a_2 \beta)$, they are of order 3. Hence s_4 displaces at least one f and at least one α, so that $\{s_1, s_4\}$ and $\{t_5, s_4\}$ are each of degree 18. Further, s_4 fixes one letter from each cycle of s_1 and t_5. Then $\{t_2, s_4\}$ is D_{20}, and since from $\{s_3, s_4\}$ α_1 is seen to be fixed by s_4, we have

$$s_4 = (a_1\beta)\,(f_2\alpha_2)\,(\gamma_1 -)\,(\gamma_2 -)\,(\gamma_3 -)\,(\gamma_4 -)\,(\alpha_1)\,(f_1)\,(a_2)\,(\zeta),$$

or else $(a_1\beta)\,(f_1\alpha_2)$ ...., which we transform into $s_4 = (f_2\alpha_2)$ .... by

$$b_1 c_2 \cdot b_2 c_1 \cdot d_1 d_2 \cdot e_1 e_2 \cdot f_1 f_2.$$

From $\{t_1, s_4\}$, s_4 fixes e_1; and from $\{t_3, s_4\}$, s_4 fixes d_1, so that

$$s_4 = (a_1\beta)\,(f_2\alpha_2)\,(d_2\gamma_1)\,(e_2\gamma_2) \ldots,$$

inconsistent with the remark that $t_5 s_4$ is of order 3.

While the larger constituent of H_3 is of degree 12, the other may be of degree 3. Pass at once to H_4. Let

$$s_4 = (a_1 f_1)\,(\gamma_1 -)\,(\gamma_2 -)\,(\gamma_3 -)\,(\gamma_4 -)\,(\gamma_5 -).$$

Now s_4 fixes $a_2, f_2, b_1, \zeta, c_1$; the product $t_5 s_4 = (a_1 a_2 f_1)$ .... is of order 3 and of degree 18, which is not possible since both b_1 and c_1 are fixed. Let

$$s_4 = (a_1\zeta)\,(\gamma_1 -)\,(\gamma_2 -)\,(\gamma_3 -)\,(\gamma_4 -)\,(\gamma_5 -).$$

Here s_4 fixes $b_1, f_1, c_2, f_2, b_2, c_1$. From $\{s_1, s_4\}$ we have

$$s_4 = a_1\zeta \cdot a_2\gamma_1 \cdot d_1\gamma_2 \cdot d_2\gamma_3 \cdot e_1\gamma_4 \cdot e_2\gamma_5,$$

but now $\{t_2, s_4\}$ is impossible.

We return to D_{14} and seek s_3 under the assumption that H_3 has a constituent of degree 9. The other set has 6, 8, or 9 letters. First suppose that there are two constituents of degree 9. Let

$$s_3 = (a_1 f_1)\,(--)\,(--)\,(\beta_1 -)\,(\beta_2 -)\,(\beta_3 -).$$

With β_1, ... are united only letters $d_1, d_2, e_1, e_2, \alpha_1, \alpha_2$. Since the letters with $\beta_1, \beta_2, \beta_3$ can not come from three different cycles of s_1 or t_4, the groups

$\{s_1, s_3\}$ and $\{t_4, s_3\}$ are not of degree 15. Then s_3 fixes a_2, f_2, c_1, ζ. Because ζ and f_2 are fixed, both $\{s_1, s_3\}$ and $\{t_4, s_3\}$ are of degree 16. Then s_3 fixes α_1 or α_2, and d_1 or e_1. It follows that s_3 displaces b_1, b_2, d_2, e_2 in addition to an α and one of the letters d_1, e_1. But s_3 must displace an odd number of the letters $d_1, d_2, e_1, e_2, \alpha_1, \alpha_2$. Let

$$s_3 = (a_1\zeta)(--)(--)(\beta_1 -)(\beta_2 --)(\beta_3 -).$$

Again $\{t_3, s_3\}$ and $\{t_4, s_3\}$ are not of degree 15; hence s_3 fixes b_2, c_1, f_1, f_2, in consequence of which each group is of degree 16. Then s_3 fixes d_1 or e_1, and fixes also one of the letters d_2 and e_2. Hence the remaining five letters displaced by s_3 are $a_2, b_1, c_2, \alpha_1, \alpha_2$. But as before, this gives an even number of letters $d_1, d_2, e_1, e_2, \alpha_1, \alpha_2$ in s_3.

Suppose that there are not more than two new letters in the second constituent of H_3. The isomorphism between the two constituents is simple. If a transitive group of degree less than 9 has a subgroup of order 9, it contains a cycle of three letters. Then to this circular substitution of order 3 there corresponds in the other constituent a substitution of degree 9. Since any substitution of degree 12 in G is regular, here is a substitution of order 3 and degree 12.

In conclusion D_{14} is not a subgroup of a primitive group of class 12 of which the degree exceeds 20.

It is now an easy matter to show that G may not include D_{22}. The three substitutions of order 2 in D_{22} are s_1,

$$s_2 = a_1a_3 \cdot b_1b_3 \cdot c_1c_3 \cdot d_1d_3 \cdot e_1e_3 \cdot f_1f_2,$$
$$t \;= a_2a_3 \cdot b_2b_3 \cdot c_2c_3 \cdot d_2d_3 \cdot e_2e_3 \cdot f_1f_2.$$

Among the substitutions of G similar to s_1 there is a substitution s_3 which replaces f_1 by a letter of another set of D_{22}. There is not imposed upon this s_3 any other condition. Let

$$s_3 = (a_1f_1)\ldots.$$

First suppose $s_1s_3 = s_3s_1$. The constituent on the letters $a_1, \ldots.$ of $\{s_2, s_3 = (a_1f_1)(a_2f_2)\ldots.\}$ is non-regular of degree 6 or more. But now that D_{14} has been rejected, this is not possible. Hence $s_1s_3 \neq s_3s_1$, and likewise $s_2s_3 \neq s_3s_2$. Suppose that $s_3 = (a_1f_1)\ldots.$ fixes f_2. Then the letters a_2 and a_3, which form a cycle of t, are also fixed. The product $ts_3 = (a_1f_1f_2)\ldots.$ requires that $\{t, s_3\}$ be of order 6. But a characteristic of the groups of order 6 in our list is that each generator displaces at least one letter from each cycle of the other. If s_3 displaces f_2, it displaces also a_1 and a_3. The set $a_1, f_1, f_2, \ldots.$ of $\{s_1, s_3\}$ can not include the letter a_3. For that would put a non-regular

constituent of degree greater than four in $\{s_1, s_3\}$, and the presence of f_1 and f_2 in different cycles requires that the constituent on the letters $a_1, \ldots$ be regular of degree 6 or 8. Then

$$s_3 = (a_1 f_1)(f_2 b_1)(b_2 —) \ldots .$$

To justify the cycle $(f_2 b_1)$: obviously the four sets b, c, d, e can be interchanged at will; and if b_2 were to follow f_2, $\{s_2, s_3\}$ would be non-regular in the set $a_1, \ldots$. First suppose the set $a_1, \ldots$ to consist of 6 letters; then

$$s_3 = (a_1 f_1)(f_2 b_1)(b_2 a_2) \ldots ,$$

and from $\{s_2, s_3\}$ the fourth cycle is $(b_3 a_3)$ or else

$$s_3 = (a_1 f_1)(f_2 b_1)(b_3 c_1)(c_3 a_3)(b_2 a_2) \ldots .$$

or

$$(a_1 f_1)(f_2 b_1)(b_3 c_3)(c_1 a_3)(b_2 a_2) \ldots ,$$

and in all three cases s_3 has a cycle entirely new to s_1, which D_{13} does not admit. Then suppose $\{s_1, s_3\}$ and $\{s_2, s_3\}$ both of order 8:

$$s_3 = (a_1 f_1)(f_2 b_1)(b_2 —) \ldots .$$

Now s_3 will clearly leave fixed the letters of one set of D_{22}, the letters e_1, e_2, e_3, say. Hence both $\{s_1, s_3\}$ and $\{s_2, s_3\}$ require that s_3 have the form of the "s_2" in D_{11}, that is, s_3 has no cycle new to s_1 or to s_2. Let us see if this is possible. A consequence is that c_1 and not c_3 follows b_2 in s_3. Then

$$s_3 = (a_1 f_1)(f_2 b_1)(b_2 c_1)(c_2 a_2) \ldots ,$$

which after all has a cycle new to s_2.

Thus D_{22} is no longer to be considered.

The group D_{12} is of order 8 and has but three transitive sets. The substitutions of order 2 and degree 12 in it are s_1,

$$\begin{aligned} s_2 &= a_1 b_2 \cdot b_1 c_2 \cdot c_1 d_2 \cdot d_1 a_2 \cdot e_1 f_1 \cdot \alpha_1 \alpha_2 , \\ t_1 &= a_1 d_2 \cdot b_1 a_2 \cdot c_1 b_2 \cdot d_1 c_2 \cdot e_2 f_2 \cdot \alpha_1 \alpha_2 , \\ t_2 &= a_1 c_2 \cdot b_1 d_2 \cdot c_1 a_2 \cdot d_1 b_2 \cdot e_1 f_2 \cdot e_2 f_1 , \\ t_3 &= a_1 c_1 \cdot b_1 d_1 \cdot a_2 c_2 \cdot b_2 d_2 \cdot e_1 f_1 \cdot e_2 f_2 . \end{aligned}$$

There is a substitution s_3, similar to s_1, which unites the set $a_1 a_2 b_1 b_2 c_1 c_2 d_1 d_2$ to one of the other sets of D_{12}, and which has not more than one new letter in any cycle. This s_3 is subject to the condition that it displaces as few letters as any substitution of the series $s_1, \ldots$ which replaces a_1 by a letter of one of the other two sets. The degree of the extended constituent is 12, 16 or 18. If the degree is 18, H_3 is transitive. Let

$$s_3 = (a_1 e_1)(— —)(\beta_1 —)(\beta_2 —)(\beta_3 —)(\beta_4 —).$$

Now s_3 is not commutative with any of the five substitutions of order 2 above, and since H_3 is transitive, displaces both α_1 and α_2. Therefore

$$s_3 = (a_1 e_1)(\alpha_1 —)(\alpha_2 \beta_1)(\beta_2 —)(\beta_3 —)(\beta_4 —)(a_2)(e_2)(b_2)(f_1)(c_2)(f_2)(c_1),$$

and but b_1, d_1 and d_2 are left for the remaining four places. Since D_{12} is invariant under all the substitutions of the group

$$\{b_1 d_1 \cdot b_2 d_2 \cdot e_1 e_2 \cdot f_1 f_2, \quad e_1 f_1 \cdot e_2 f_2 \cdot \alpha_1 \alpha_2\},$$

there remains in this connection only

$$s_3 = (a_1 \alpha_1)(— —)(\beta_1 —)(\beta_2 —)(\beta_3 —)(\beta_4 —).$$

Now s_3 fixes α_2, so that $\{s_1, s_3\}$ is of degree 17.

Suppose H_3 transitive of degree 16, and first let

$$s_3 = (a_1 e_1)(\beta_1 —)(\beta_2 —)(\alpha_1 —)(— —)(— —).$$

If $\{s_1, s_3\}$, $\{t_2, s_3\}$ and $\{t_3, s_3\}$ are of degree 15 (D_{13}), s_3 displaces a_2, e_2, c_2, f_2, c_1 and f_1 in its last two cycles, an absurdity. Hence s_3 displaces α_2, and in consequence fixes a_2, e_2, c_2, f_2, c_1, f_1. Now s_3 can not be formed for lack of letters. Let

$$s_3 = (a_1 \alpha_1)(\alpha_2 —)(\beta_1 —)(\beta_2 —) \ldots\ldots$$

One of the letters e_1, e_2, f_1, f_2 is in the cycle with α_2. If necessary, transform s_3 by a substitution of

$$\{b_1 d_1 \cdot b_2 d_2 \cdot e_1 e_2 \cdot f_1 f_2, \quad e_1 f_1 \cdot e_2 f_2\},$$

so that s_3 may be written

$$(a_1 \alpha_1)(e_1 \alpha_2)(\beta_1 --)(\beta_2 —) \ldots\ldots$$

Since $\{s_1, s_3\}$ and $\{t_4, s_3\}$ are of degree 16, they require the contradictory forms for s_3:

$$(\beta_1 a_2)(\beta_2 e_2) \ldots\ldots \quad \text{and} \quad (\beta_1 c_1)(\beta_2 f_1) \ldots\ldots$$

Then H_3 is transitive.

Suppose the larger constituent of degree 16. Then

$$s_3 = (a_1 e_1)(\beta_1 —)(\beta_2 —)(\beta_3 —)(\beta_4 —)(— —),$$

where α_1 or α_2, if displaced, is in the last cycle only. Now s_3 fixes a_2, e_2, b_2, f_1, c_2, f_2, c_1, leaving only the five letters b_1, d_1, d_2, α_1, α_2 with the aid of a new letter γ to fill the six places vacant in s_3. But if γ is displaced, α_1 or α_2 is certainly fixed.

Suppose the first constituent to be of degree 12. Let s_3 unite the sets $a_1, \ldots\ldots$ and $e_1, \ldots\ldots$ of D_{12}. There is no new letter in this extended constituent, and s_3 can not displace more than two new letters. Assume then

$$s_3 = (a_1 e_1)(\alpha_1 \beta_1)(\alpha_2 \beta_2) \ldots\ldots$$

The three groups $\{s_1, s_3\}$, $\{t_2, s_3\}$ and $\{t_3, s_3\}$ are Abelian. Hence

$$s_3 = (a_1 e_1)(a_2 e_2)(c_1 f_1)(c_2 f_2)(\alpha_1 \beta_1)(\alpha_2 \beta_2),$$

which is inconsistent with s_2. Let

$$s_3 = (a_1 e_1)(\alpha_1 \beta) \ldots .,$$

leaving α_2 fixed. One constituent of H_3 is of order 6, and to identity corresponds an invariant subgroup of degree 12 and class 12: the group $\{s_1, t_2\}$. Hence the larger constituent is of order 24. Its class is 10. Its positive subgroup has four subgroups of order 3 and hence is the regular tetrahedral group. Hence s_3 is not commutative with t_3, but is commutative with s_1 or else transforms s_1 into t_3. First assume $s_3 s_1 = s_1 s_3$, and let

$$s_3 = a_1 e_1 \cdot a_2 e_2 \cdot \alpha_1 \beta \ldots . (\alpha_2) \ldots . .$$

Since $(a_2 \alpha_1)$ is not a cycle of s_2, $\{s_2, s_3\}$ is not of degree 14, and s_3 displaces f_2:

$$s_3 = (a_1 e_1)(b_2 —)(f_1 —)(a_2 e_2)(\alpha_1 \beta)(f_2 —)(\alpha_2)(d_1).$$

Now that cycle which s_3 has in common with s_1 is evidently $(b_1 b_2)$, and since $\{t_1, s_3\}$ is of degree 15, d_2 is fixed. Then $(c_1 f_2)$ is a cycle of s_3. Therefore when $s_1 s_3 = s_3 s_1$, s_3 is determined as

$$a_1 e_1 \cdot b_1 b_2 \cdot f_1 c_2 \cdot a_2 e_2 \cdot \alpha_1 \beta \cdot c_1 f_2.$$

The substitution $a_2 c_2 \cdot b_2 d_2 \cdot e_1 f_2 f_1 e_2$ transforms s_1 into t_2, s_2 into t_1 and $s_1 s_3$ into a substitution $s_3' = a_1 e_1 \cdot c_2 f_2 \cdot \alpha_1 \beta \ldots .$ commutative with t_2 and not with s_1. Hence s_3 as determined above is unique.

We take with H_3 a substitution s_4, which with it generates a transitive group on 16, 18 or 20 letters. If the degree is 16 or 18, systems of imprimitivity of 8 or 9 letters are impossible. Nor, because of

$$s_2 s_3 = a_1 f_1 b_1 \cdot a_2 e_2 d_1 \cdot b_2 c_2 e_1 \cdot c_1 f_2 d_2 \cdot \alpha_1 \beta \alpha_2,$$

are systems of two letters possible. There may not be more than one system containing a given letter (γ) of three, four or six letters. Then we have only to study the transitive group of degree 20. Let

$$s_4 = (a_1 \alpha_1)(\gamma_1 —)(\gamma_2 —)(\gamma_3 —)(\gamma_4 —)(\gamma_5 —).$$

At once we note that s_4 fixes b_2, α_2, d_2, e_1, β, and from $s_1 s_3$, e_2. Then $\{s_1, s_4\}$ is D_{18}, so that

$$s_4 = a_1 \alpha_1 \cdot a_2 \gamma_1 \cdot c_1 \gamma_2 \cdot c_2 \gamma_3 \cdot d_1 \gamma_4 \cdot d_2 \gamma_5,$$

but $\{t_3, s_4\}$ is also D_{18}, now impossible.

Now H_3 can not be a group with one constituent of degree 12 and one of degree 2. For the subgroup leaving fixed the two letters of the second set is of class 12 and can only be of order 4.

If s_3 joins the sets $a_1, \ldots$ and $\alpha_1, \ldots$ of D_{12} (to form a constituent of degree 12 in H_3), there are just two new letters in this set, while the degree of the other constituent is 4 or 6. Let

$$s_3 = (a_1 \alpha_1)(\beta_1 -)(\beta_2 -)(--)(\gamma_1 -)(\gamma_2 -).$$

Clearly s_3 fixes α_2 and $\{s_1, s_3\}$ is of degree 17. Let

$$s_3 = (a_1 \alpha_1)(\beta_1 -)(\beta_2 -) \ldots,$$

with no new letters other than β_1 and β_2. In case this substitution s_3 can be set up without contradiction, there is a substitution $s_4 = (a_1 e_1) \ldots$ with at most one new letter in any cycle; and if

$$s_4 = (a_1 e_1)(\beta_1 \delta_1) \ldots,$$

for example, consider the transform of D_{12} by s_4, bearing in mind that no substitution of the series $s_4, \ldots$ that connects the two sets of H_3 displaces fewer new letters than s_4. The transform of D_{12} by s_4 fixes β_1 and δ_1. Then with the substitutions s_1, s_2, t_1, t_2, t_3 before us it is seen that s_4 must replace the letters a_2, b_2, d_2, c_2, c_1 by letters not a part of the set $a_1, a_2, \ldots, d_2$; that is, that these five letters must be in different cycles of s_4, which is impossible. Hence s_4 does not replace β_1 or β_2 by a new letter, but is simply a substitution uniting $a_1, \ldots$ and $e_1, \ldots$ in a set of degree 12 or more, and with at most one letter new to D_{12} in any cycle. It has already been seen that this is not possible.

With D_{12}, D_{11} disappears from our list.

Consider D_{13}. We shall make use of the substitutions s_1,

$$s_2 = a_1 b_1 \cdot a_2 c_1 \cdot b_2 c_2 \cdot d_1 \delta \cdot e_1 \varepsilon \cdot f_1 \zeta, \quad t = a_1 c_2 \cdot a_2 b_2 \,.\, b_1 c_1 \cdot d_2 \delta \cdot e_2 \varepsilon \cdot f_2 \zeta,$$
$$s_1 s_2 = a_1 c_1 b_2 \cdot a_2 b_1 c_2 \cdot d_1 d_2 \delta \cdot e_1 e_2 \varepsilon \,.\, f_1 f_2 \zeta.$$

Since there is in D_{13} a set of six letters, there is a substitution s_3 similar to s_1 which replaces a_1 by a letter of some of the three other sets and which has at most one new letter in any cycle. The substitutions of the group

$$\{d_1 e_1 \cdot d_2 e_2 \cdot \delta \varepsilon, \quad e_1 f_1 \cdot e_2 f_2 \cdot \varepsilon \zeta, \quad b_1 c_2 \cdot b_2 c_1 \cdot d_1 d_2 \cdot e_1 e_2 \cdot f_1 f_2, \quad a_2 b_1 \cdot b_2 c_1 \cdot d_2 \delta \cdot e_2 \varepsilon \cdot f_2 \zeta\}$$

transform D_{13} into itself and hence we need only put

$$s_3 = (a_1 d_1) \ldots.$$

If $\{s_1, s_2, \ldots\}$ is a transitive group on 16 or 18 letters, systems of imprimitivity of two, eight or nine letters are not possible. A system of three letters

can be chosen in but one way, and a system of four or six letters in three ways at most. Then D_{13} can not lead to a transitive group of lower degree than 20. Let

$$s_3 = (a_1 d_1)(\gamma_1 -)(\gamma_2 -)(\gamma_3 -)(\gamma_4 -)(\gamma_5 -),$$

without condition upon the number of constituents in H_3. There is no cycle of new letters in s_3. Since $\{s_1, s_3\}$ and $\{s_2, s_3\}$ are non-Abelian, s_3 fixes a_2, d_2, b_1 and ζ, and again from s_1, b_2. Both a_2 and b_2 being fixed, $\{t, s_3\}$ is D_{18}, so that

$$s_3 = a_1 d_1 \cdot c_2\gamma_1 \cdot d_2\gamma_2 \cdot \delta\gamma_3 \cdot e_2\gamma_4 \cdot \varepsilon\gamma_5,$$

which, however, generates with s_1 a group of degree 17.

Let us remove the condition that there is no cycle of new letters in s_3. Suppose γ_5 replaced by a letter β. The group $\{s_1, s_3\}$ is of degree 18 and non-Abelian. Then s_3 fixes a_2, d_2, b_1, ζ, b_2, ε and δ, so that $\{t, s_3\}$ is of degree greater than 18, an impossibility. Nor could there be more cycles new to D_{13} in s_3. In the same way let

$$s_3 = (a_1 d_1)(--)(\gamma_1 -)(\gamma_2 -)(\gamma_3 -)(\gamma_4 -),$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are the only letters new to D_{13} in s_3. Since $\{s_1, s_3\}$ and $\{s_2, s_3\}$ are non-Abelian, s_3 fixes a_2, d_2, b_1 and δ. The group $\{t, s_3\}$ is of degree 18 (d_2 and δ being fixed by s_3), so that s_3 displaces e_1 or f_1, not both, and replaces c_2 by a letter new to t. If s_3 replaces c_2 by e_1 or f_1, s_4 fixes c_1, b_2, e_2 or f_2, and ε or ζ. Not enough letters are left to replace the four $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. Hence

$$s_3 = (a_1 d_1)(c_2\gamma_2) \ldots\ldots$$

From s_1, c_1 is replaced by a letter new to s_1, and from t, b_1 being fixed, c_1 is replaced by ε or ζ, not by $\gamma_1, \ldots\ldots$ Then s_3 fixes e_2 or f_2, and since $\{s_2, s_3\}$ can not be of degree 17, s_4 fixes both e_2 and f_2. But from $\{s_1, s_3\}$, if f_1 is replaced by a γ, so also is f_2. Since now e_1, e_2, f_1, f_2 and b_1 are fixed, $\{s_1, s_3\}$ is impossible. Suppose s_3 has one other new letter (β) in the cycle $(\beta\gamma_4)$. If $s_3 s_1 = s_1 s_3$,

$$s_3 = a_1 d_1 \cdot a_1 d_2 \cdot \delta\gamma_1 \cdot \varepsilon\gamma_2 \cdot \zeta\gamma_3 \cdot \beta\gamma_4,$$

which is not consistent with s_2. Then $\{s_1, s_3\}$ is of degree 18, and s_3 fixes a_2 and d_2. From s_2, s_3 fixes b_1 and δ; from t, it fixes e_1 and f_1. Now from s_1, s_3 displaces one, not both, the letters ε and ζ, while from s_2, either $s_3 = (\varepsilon\zeta)\ldots\ldots$ or fixes both ε and ζ. If s_3 has two or more letters $\beta_1, \beta_2, \ldots\ldots$ in the last four cycles, $\{t, s_3\}$ is impossible.

We now are in position to say that if s is a substitution of G similar to s_1, which replaces a_1 by a letter of one of the three remaining sets of D_{13}, s has new letters in not more than three of its cycles.

Let s_3 displace just three new letters in three cycles:

$$s_3 = (a_1 d_1)(---)(---)(\gamma_1 -)(\gamma_2 -)(\gamma_3 -).$$

If s_1 and s_3 are commutative,

$$s_3 = (a_1 d_1)(a_2 d_2)(---)(\delta\gamma_1)(\varepsilon\gamma_2)(\zeta\gamma_3),$$

but since $\{s_2, s_3\}$ is of degree 16 or more, s_3 fixes δ. Then s_3 is not commutative with s_1, nor, in like manner, with s_2. If $\{s_1, s_3\}$ is of degree 15,

$$s_3 = (a_1 d_1)(a_2 -)(d_2 -)(\gamma_1 -)(\gamma_2 -)(\gamma_3 -),$$

and fixes, since $\{s_2, s_3\}$ is of higher degree, b_1 in addition to δ, ε, and ζ. If c_2 is fixed, the second and third cycles of s_3, because of t, contain e_1 and f_1, which s_1 does not admit. From t, c_2 follows γ_1; from s_2 and s_1, b_2 is replaced by a letter new to s_2 and to s_1, by γ_2; a_2 is replaced by a letter of s_2 (c_1 being fixed) that is new to t, by e_1:

$$s_3 = (a_1 d_1)(a_2 e_1)(d_2 e_2)(\gamma_1 c_2)(\gamma_2 b_2)(\gamma_3 -).$$

From s_2 the remaining letter should be f_2, and from t, f_1. After transforming $\{D_{13}, (a_1 d_1)\ldots.\}$ by $a_2 b_1 \cdot b_2 c_1 \cdot d_2\delta \cdot e_2\varepsilon \cdot f_2\zeta$, we conclude that both $\{s_1, s_3\}$ and $\{s_2, s_3\}$ are of degree 16 or more. The letters a_2, d_2, b_1, δ, ε or ζ, e_2 or f_2 are fixed by s_3. As there remain seven places to fill in s_3, no other letter of D_{13} is fixed by s_3. Now c_2 must be replaced by a letter new to t, by γ_1 or by e_1. If $s_3 = (c_2 e_1)\ldots.$, c_1 is fixed. Now if $s_3 = (a_1 d_1)(c_2\gamma_1)\ldots.$, c_1 is followed by a letter of t, not in s_1: by ε or ζ, whence e_1 or f_1 is fixed.

It is proved that s_3 has not just three new letters, one to a cycle.

Let s_3 have just two new letters in distinct cycles. If s_3 and s_1 are commutative we can not have

$$s_3 = (a_1 d_1)(a_2 d_2)(\gamma_1\delta)(\gamma_2 -)\ldots.,$$

because then the set $a_1, \ldots.$ of $\{s_2, s_3\}$ is non-regular of degree 5 or more. Then put

$$s_3 = (a_1 d_1)(a_2 d_2)(\gamma_1\varepsilon)(\gamma_2\zeta)\ldots.(\delta).$$

Both $\{s_2, s_3\}$ and $\{t, s_3\}$ are of degree 16, so that e_1, f_1, e_2, f_2 are replaced by letters new to s_2 and t. But this makes $\{s_2, s_3\}$ of degree 17. Then s_3 is not commutative with s_1 or s_2. Suppose both $\{s_1, s_3\}$ and $\{s_2, s_3\}$ of degree 16. Now s_3 fixes a_2, d_2, b_1, δ, and e_1, say, and can fix no other letter of D_{13}. From s_1, b_2 must be replaced by e_2, which, since $\{t, s_3\}$ is also of degree 16, is not possible. Suppose $\{s_1, s_3\}$ of degree 15 and $\{s_2, s_3\}$ of degree 16: s_3 fixes

b_1, δ, e_2, say, ε or ζ, e_1 or f_1. No other letter of D_{13} is fixed by s_3. The group $\{t, s_3\}$ is of degree 16.

$$s_3 = (a_1 d_1)(a_2 —)(d_2 —)(— —)(\gamma_1 —)(\gamma_2 —).$$

From t, c_1 is followed by d_2 or e_1. But

$$s_3 = (a_1 d_1)(a_2 c_2)(d_2 c_1) \ldots .$$

requires that s_3 fixes c_1 and b_2, and

$$s_3 = (a_1 d_1)(a_2 —)(d_2 —)(c_1 e_2)(\gamma_1 —)(\gamma_2 —)$$

has but two letters new to s_1. Then both $\{s_1, s_3\}$ and $\{s_2, s_3\}$ are of degree 15; s_3 fixes two of the letters δ, ε, ζ, both e_2 and f_2, and only one of the letters e_1 and f_1, e_1, say. Hence $\{s_1, s_3\}$ is not of degree 15 (e_1 and e_2 being fixed).

Let H_3 be intransitive and of degree 16. The constituent $a_1, \ldots, d_1, \ldots$ is of degree 9 or 12. Then the letter γ is in the set f_1, f_2, ζ on four letters. Three constituents of the degrees 9, 3, and 4 entail the presence of a substitution of degree 12 and order 3 in H_3. Hence H_3 has just two constituents. Now s_4 must have letters new to H_3 in at least four cycles, and can therefore only be

$$s_4 = (a_1 \gamma_1) \ldots .$$

Unless s_3 replaces a_1, a_2, b_1, b_2, c_1, c_2 by letters not belonging to this set of six letters, it is possible to choose a substitution σ from D_{13} such that $s_3 \sigma^{-1} s_4 \sigma s_3$ replaces a letter of the set $a_1, \ldots$ of D_{13} by a letter of another set $(f_1, f_2, \zeta)'$ and has letters new to D_{13} in more than three cycles. The condition that s_3 fixes a letter $a_1, \ldots, c_2$ or transposes two of them is satisfied, because s_3 uses one cycle to transpose an $f_1, \ldots$ and γ.

If H_3 is of degree 15, s_4 certainly displaces letters new to D_{13}, without a cycle of new letters. But we have shown that s_4 does not exist.

Any non-Abelian diedral rotation group generated by two substitutions similar to s_1, and contained in a primitive group of class 12 and of degree greater than 20, is therefore of degree 16 or 18.

The group which now seems to offer least difficulty is D_1. Other than the identity its substitutions are s_1,

$$s_2 = a_1 b_1 \cdot a_2 b_2 \cdot c_1 d_1 \cdot c_2 d_2 \cdot e_1 f_1 \cdot e_2 f_2,$$
$$t = a_1 b_2 \cdot a_2 b_1 \cdot c_1 d_2 \cdot c_2 d_1 \cdot e_1 f_2 \cdot e_2 f_1.$$

All substitutions similar to s_1, which unite sets of D_1, may be transformed into $s_3 = (a_1 c_1) \ldots .$ by

$\{a_1 c_1 \cdot a_2 c_2 \cdot b_1 d_1 \cdot b_2 d_2,\ a_1 e_1 \cdot a_2 e_2 \cdot b_1 f_1 \cdot b_2 f_2,\ a_2 b_1 \cdot c_2 d_1 \cdot e_2 f_1,\ b_1 b_2 \cdot d_1 d_2 \cdot f_1 f_2\}$,

under which D_1 is invariant.

First suppose that no s_3 which joins sets of D_1 is free from a cycle of new letters. Then s_3 displaces all the letters of the sets united. Since H_3 can not be regular of order 12, all three sets are not united. Now s_3 displaces the eight letters $a_1, \ldots, c_1, \ldots$, and is therefore commutative with s_1 and s_2. Hence it fixes each of the four letters e_1, e_2, f_1, f_2, and is uniquely

$$s_3 = a_1 c_1 \cdot a_2 c_2 \cdot b_1 d_1 \cdot b_2 d_2 \cdot \alpha_1 \alpha_2 \cdot \beta_1 \beta_2.$$

This s_3 is obtained under the assumption that it displaces the minimum number of new letters. For later developments consider any substitution

$$s = (a_1 c_1)(\alpha_1 \alpha_2) \ldots .$$

of degree 12. If s is commutative with both s_1 and s_2, it is identical with s_3 above. Let $s_1 s_2 \neq s_2 s_1$. Then s has at least four letters new to D_1, and fixes a_2 and c_2. The substitution $s\, s_1\, s$ has at most, and hence exactly, four letters new to s_1, and since it joins sets of D_1, has a cycle of new letters; but any substitution, as $s\, s_1\, s$, that joins sets of D_1, displaces the minimum number of new letters, and has a cycle of new letters, is of the same type as s_3. Then

$$s\, s_1\, s = a_2 c_1 \cdot a_1 c_2 \cdot b_1 d_2 \cdot b_2 d_1 \cdot \gamma_1 \gamma_2 \cdot \gamma_3 \gamma_4,$$

so that

$$s = a_1 c_1 \cdot \alpha_1 \alpha_2 \cdot e_1 \gamma_1 \cdot e_2 \gamma_2 \cdot f_1 \gamma_3 \cdot f_2 \gamma_4.$$

Then

$$s\, s_2\, s = c_1 b_1 \cdot a_2 b_2 \cdot a_1 d_1 \cdot c_2 d_2 \cdot \gamma_1 \gamma_3 \cdot \gamma_2 \gamma_4,$$

not, as required, of the form of s_3. Hence any substitution $(a_1 c_1)(\alpha_1 \alpha_2) \ldots .$ of degree 12 coincides with s_3.

Let us next study a substitution

$$s = (a_1 \alpha) \ldots .$$

of degree 12. This substitution $(a_1 \alpha) \ldots .$ can join no sets of D_1. The number of new letters displaced by it is 6, one to a cycle. One and only one of the groups $\{s_1, s\}$, $\{s_2, s\}$, $\{t, s\}$ is of order 8 (D_{18}).

We now seek the substitution s_4, $(a_1 e_1) \ldots .$ or $(a_1 \alpha_1) \ldots .$, with at most one new letter in any cycle. Every substitution similar to s_1, which joins sets of such a group as D_1, has, like s_3, four new letters. Now $\{D_1, s_4\}$ is such a group, and hence the first four cycles of s_4 are fixed as $a_1 e_1 \cdot a_2 e_2 \cdot b_1 f_1 \cdot b_2 f_2$. The product $s_2 s_4$ is of order 3, so that

$$s_4 = a_1 e_1 \cdot a_2 e_2 \cdot b_1 f_1 \cdot b_2 f_2 \cdot \alpha_1 \alpha_3 \cdot \beta_1 \beta_3.$$

Before considering s_5, let $s_4 = (a_1 \alpha_1) \ldots .$ Suppose $\{s_1, s_4\}$ of order 8. If $s_3 s_4 = s_4 s_3$,

$$s_4 = a_1 \alpha_1 \cdot c_1 \alpha_2 \cdot a_2 \beta_1 \cdot c_2 \beta_2 \cdot e_1 \gamma_1 \cdot e_2 \gamma_2,$$

after having transformed H_4 by $e_1 f_1 \cdot e_2 f_2$, if necessary. If $s_3 s_4 \neq s_4 s_3$, s_4 fixes c_1, a_2, b_1, b_2, c_2, both e's or else both f's. Hence

$$s_4 = (a_1 \alpha_1)(a_2 -)(d_1 -)(d_2 -)(e_1 -)(e_2 -).$$

Now $\{s_3, s_4\}$ must be of degree 18, and hence a_2, d_1 or d_2 is replaced by a letter new to s_3, requiring s_4 to displace c_2, b_1 or b_2. The substitution $b_1 b_2 a_2 \cdot d_1 d_2 c_2 \cdot f_1 f_2 e_2$ permutes s_1, s_2, t cyclically, transforms s_3 into itself, and fixes the cycle $(a_1 \alpha_1)$ of s_4. Hence it is not necessary to investigate $(a_1 \alpha_1) \ldots .$ under the assumption that $\{s_2, s_4\}$ is of order 8.

There are two forms s_4 may have:

$$s_4' = a_1 \alpha_1 \cdot c_1 \alpha_2 \cdot a_2 \beta_1 \cdot c_2 \beta_2 \cdot e_1 \gamma_1 \cdot e_2 \gamma_2,$$
$$s_4'' = a_1 e_1 \cdot a_2 e_2 \cdot b_1 f_1 \,.\, b_2 f_2 \cdot \alpha_1 \alpha_3 \cdot \beta_1 \beta_3.$$

The group $\{H_3, s_4'\} \equiv H_4'$ has but two constituents. If $s_5 = (a_1 e_1) \ldots .$, from $\{s_4', s_1 s_4' s_1\}$ and $\{s_1, s_2\}$, at once

$$s_5 = a_1 e_1 \cdot a_2 e_2 \cdot b_1 f_1 \cdot b_2 f_2 \cdot \alpha_1 \gamma_1 \cdot \beta_1 \gamma_2.$$

We now have before us a transitive group of degree 18, in which no system of imprimitivity of q letters involving γ_1 can be chosen in $q+1$ ways. Hence we are not concerned with this group. If $s_5 = (a_1 f_1) \ldots .$, from D_1,

$$s_5 = a_1 f_1 \cdot a_2 f_2 \cdot b_1 e_1 \cdot b_2 e_2 \ldots .,$$

and if $s_5 = (a_1 \gamma_1) \ldots .$, from $\{s_4', s_1 s_4' s_1\}$,

$$s_5 = a_1 \gamma_1 \cdot e_1 \alpha_1 \cdot a_2 \gamma_2 \cdot e_2 \beta_2 \ldots .;$$

then $\{H_4', s_5\}$ is transitive of degree 20 at most. Hence G is more than simply transitive. We may transform s_1 into $(a_1 e_1) \ldots .$ and this substitution must coincide with that form of s_5 above which gave the transitive group of degree 18. Having thrown out s_4', we may remark that H_3 can not be a subgroup of a doubly transitive group. For were that the case s_1 could be transformed into $(a_1 \alpha_1) \ldots .$. In particular $\{s_1, s_2, s_3, \ldots .\}$ must not lead us to a transitive group of degree less than 21. Consider

$$s_4'' = a_1 e_1 \cdot a_2 e_2 \cdot b_1 \cdot f_1 \cdot b_2 f_2 \cdot \alpha_1 \alpha_3 \cdot \beta_1 \beta_3,$$

and $s_5 = (a_1 \alpha_3) \ldots .$. Now s_5 bears the same relation to $\{s_1, s_2, s_4''\}$ as s_4' does to $\{s_1, s_2, s_3\}$, so that $s_5 s_4'' = s_4'' s_5$, and s_5 replaces the letters of the last two cycles of s_4'' by the letters of two of the preceding cycles. Then $\{s_1, s_2, s_3, s_4'', s_5\}$ is transitive of degree less than 21.

Finally assume that at least one substitution s_3 is to be found in the series

$s_1, s_2, \ldots$. which connects two or more sets of D_1 and has no cycle of new letters. Let

$$s_3 = (a_1 c_1) \ldots . :$$

s_3 is not commutative with s_1, s_2 or t. Hence s_3 fixes the six remaining letters of the sets $a_1, \ldots .$ and $c_1, \ldots .$ of D_1, requiring six new letters, while five is the most it can displace.

Since D_1 is a subgroup of D_{18} and D_{19}, these two groups are, with D_1, to be struck from our list.

Now consider D_2. The substitutions of order 2 are s_1,

$$s_2 = a_1 b_1 \cdot a_2 b_2 \cdot c_1 d_1 \cdot c_2 d_2 \cdot e_1 e_2 \cdot \alpha_1 \alpha_2, \quad t = a_1 b_2 \cdot a_2 b_1 \cdot c_1 d_2 \cdot c_2 d_1 \cdot f_1 f_2 \cdot \alpha_1 \alpha_2.$$

Since D_2 is invariant under all the substitutions of the group

$$\{c_1 c_2 \cdot d_1 d_2,\ c_1 d_1 \cdot c_2 d_2,\ e_1 e_2,\ f_1 f_2,\ \alpha_1 \alpha_2,\ b_1 b_2 \cdot d_1 d_2 \cdot e_1 f_1 \cdot e_2 f_2,\ a_2 b_1 \cdot c_2 d_1 \cdot f_1 \alpha_1 \cdot f_2 \alpha_2\},$$

we reduce all substitutions which replace a_1 by a letter of one of the other sets to $(a_1 c_1) \ldots .$ or to $(a_1 e_1) \ldots .$. Without condition upon the number of new letters involved, let

$$s_3 = (a_1 e_1) \ldots .$$

Obviously it is not commutative with either s_1 or s_2. Then it fixes a_2, b_1 or e_2, and replaces b_2 by a letter new to t, but in s_1 and s_2, which is absurd. Hence:

No substitution of G, similar to s_1, transposes one of the letters a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 with e_1, e_2, f_1, f_2, α_1 or α_2.

Now assume that s_3 displaces the minimum number of letter new to D_2, and that every such substitution has a cycle of new letters. Then s_3 displaces all the letters of the set $a_1 \ldots .$ of H_3, and in consequence is commutative with both s_1 and s_2. Hence

$$s_3' = a_1 c_1 \cdot a_2 c_2 \cdot b_1 d_1 \cdot b_2 d_2 \cdot e_1 e_2 \cdot \gamma_1 \gamma_2$$

or

$$s_3 = a_1 c_1 \cdot a_2 c_2 \cdot b_1 d_1 \cdot b_2 d_2 \cdot \gamma_1 \gamma_2 \cdot \delta_1 \delta_2.$$

The next substitution $s_4 = (a_1 \gamma_1) \ldots .$ has no cycle of letters new to H_3.

Let us first attend to s_3'. Since $\{s_1, s_3'\}$ is a group D_2, the substitution $s_4 = (a_1 \gamma_1) \ldots .$ is not possible.

Consider s_3, with the assumption that $s_3 s_4 = s_4 s_3$. Then

$$s_4 = (a_1 \gamma_1)(c_1 \gamma_2) \ldots .$$

If s_4 displaces a_2, from s_1, a_2 and c_2 are replaced by letters new to s_1, so that s_4 interchanges the cycles $(a_2 c_2)$ and $(\delta_1 \delta_2)$ of s_3. Now $s_2 s_4$ and $t s_4$ can only be of order 3; hence

$$s_4 = a_1 \gamma_1 \cdot c_1 \gamma_2 \cdot a_2 \delta_1 \cdot c_2 \delta_2 \cdot e_1 f_1 \cdot \alpha_1 \alpha_3.$$

Suppose a_2 fixed. If b_1 (or b_2) is displaced we can transform the resulting substitution into the s_4 just given. Let s_4 fix a_2, b_1 and b_2. Then $\{s_1, s_4\}$ is impossible. Next assume that s_3 and s_4 are not commutative. The letters c_1 and γ_2 are fixed. Suppose a_2 displaced by s_4. It is followed by a letter new to s_1: by α_1, δ_1 or ε. The first

$$(a_1\gamma_1)(a_2\alpha_1)\ldots.$$

is impossible because from s_2, b_1 is displaced by s_4, while from t, it is seen to be fixed. Let

$$s_4 = (a_1\gamma_1)(a_2\delta_1)\ldots..$$

The letters c_1, c_2, γ_2, δ_2 are fixed. If s_4 and s_3 have a cycle in common it is $(b_1 d_1)$ or $(b_2 d_2)$. In either event $\{s_2, s_4\}$ is rendered impossible. Now since $\{s_3, s_4\}$ is D_{15} or D_{21}, and since $b_1 d_1 \cdot b_2 d_2 \cdot e_1 f_1 \cdot e_2 f_2$ transforms H_3 into itself,

$$s_4 = (a_1\gamma_1)(a_2\delta_1)(b_1—)(d_1—)\ldots.,$$

where b_1 and d_1 are followed by letters not merely new to s_3, but also new to D_2. On consulting s_1, this is seen to be impossible. If

$$s_4 = (a_1\gamma_1)(a_2\varepsilon)\ldots.(c_1)\ldots.,$$

c_2 is replaced by a letter of s_1 new to s_3, an impossibility. Assume that s_4 fixes a_2. We may add that it fixes b_1 and b_2. Now $\{s_1, s_4\}$ is impossible.

The unique substitution

$$s_4 = a_1\gamma_1 \cdot c_1\gamma_2 \cdot a_2\delta_1 \cdot c_2\delta_2 \cdot e_1 f_1 \cdot \alpha_1\alpha_3$$

generates with H_3 a group with three transitive constituents. We need consider only

$$s_5 = (a_1\alpha_3)\ldots.;$$

and since

$$s_1 s_5 s_1 = (b_1\alpha_3)\ldots.,$$

and

$$s_4 s_1 s_5 s_1 s_4 = (b_1\alpha_1)\ldots.,$$

this case is complete.

There remains the hypothesis that a substitution s_3 exists without a cycle of new letters. Let s_3 be commutative with s_1. It is not commutative with s_2. Then b_1, b_2', d_1, d_2 are fixed. If s_2 and s_3 have the cycle $(\alpha_1\alpha_2)$ in common, s_4 has a cycle of letters new to D_2. If s_2 and s_4 have $(e_1 e_2)$ in common,

$$s_3' = a_1 c_1 \cdot a_2 c_2 \cdot e_1 e_2 \cdot f_1 f_2 \cdot \alpha_1\beta_1 \cdot \alpha_2\beta_2.$$

If s_3 has α_1 and α_2 in different cycles, and no cycle in common with s_2,

$$s_3'' = a_1 c_1 \cdot a_2 c_2 \cdot \alpha_1\beta_1 \,.\, \alpha_2\beta_2 \cdot e_1 f_1 \cdot e_2 f_2.$$

Let s_3 be non-commutative with s_1. Since

$$b_1 b_2 a_2 \cdot d_1 d_2 c_2 \cdot e_1 \alpha_1 f_1 \cdot e_2 \alpha_2 f_2$$

permutes s_1, s_2 and t cyclically, we assume s_3 non-commutative with s_2 and t. Then s_3 fixes a_2, c_2, b_1, b_2, d_1, d_2, and must displace e_1, e_2, f_1, f_2, α_1 and α_2, an impossibility.

The next step is to set up a substitution s_4 with s_3' or s_3''. If we put $s_4 = (a_1 \beta_1) \ldots$, $s_3 s_4 s_3 = (c_1 \alpha_1) \ldots$.

Then D_2 does not occur in G, and with it go D_{16}, D_{17}, D_{20} and D_{21}, in which D_2 is a subgroup.

Any two substitutions of G, similar to s_1, which have a common cycle, or such that one has a cycle new to the other, are commutative.

The next group we shall study is D_{15}. The substitutions of order 2 are

$$s_1 = a_1 a_2 \cdot b_1 b_2 \cdot c_1 c_2 \cdot d_1 d_2 \cdot e_1 e_2 \cdot f_1 f_2, \quad s_2 = a_1 b_1 \cdot c_1 d_1 \cdot e_1 \varepsilon_1 \cdot e_2 \varepsilon_2 \cdot f_1 \zeta_1 \cdot f_2 \zeta_2,$$
$$t_1 = a_2 b_2 \cdot c_2 d_2 \cdot e_2 \varepsilon_1 \cdot e_1 \varepsilon_2 \cdot f_2 \zeta_1 \cdot f_1 \zeta_2, \quad t_2 = a_1 b_2 \cdot a_2 b_1 \cdot c_1 d_2 \cdot c_2 d_1 \cdot \varepsilon_1 \varepsilon_2 \cdot \zeta_1 \zeta_2.$$

D_{15} is invariant under the substitutions of the group

$$\{c_1 d_1 \cdot c_2 d_2, \quad e_1 e_2 \cdot \varepsilon_1 \varepsilon_2 \cdot f_1 f_2 \cdot \zeta_1 \zeta_2, \quad a_2 b_2 \cdot c_2 d_2 \cdot e_1 \varepsilon_1 \cdot e_2 \varepsilon_2 \cdot f_1 \zeta_1 \cdot f_2 \zeta_2,$$
$$a_1 c_1 \cdot a_2 c_2 \cdot b_1 d_1 \cdot b_2 d_2, \quad e_1 f_1 \cdot e_2 f_2 \cdot \varepsilon_1 \zeta_1 \cdot \varepsilon_2 \zeta_2,$$
$$a_1 e_1 \cdot a_2 \varepsilon_1 \cdot b_1 e_2 \cdot b_2 \varepsilon_2 \cdot c_1 f_1 \cdot c_2 \zeta_1 \cdot d_1 f_2 \cdot d_2 \zeta_2\}.$$

The substitutions s_3 to be considered are

$$s_3' = (a_1 c_1) \ldots, \quad s_3'' = (a_1 e_1) \ldots, \quad s_3''' = (a_1 c_2) \ldots.$$

Assume that any substitution similar to s_1 which connects two or more sets of D_{15} has a cycle of new letters. Now $s_3'' = (a_1 e_1) \ldots$ and $s_3''' = (a_1 c_2) \ldots$ can not be commutative with both s_1 and s_2. Then we have uniquely

$$s_3 = a_1 c_1 \cdot a_2 c_2 \cdot b_1 d_1 \cdot b_2 d_2 \cdot \alpha_1 \alpha_2 \cdot \beta_1 \beta_2.$$

Now s_4 must be $(a_1 \alpha_1) \ldots$. This s_4 can have no cycle new to D_{15} and hence does not unite two sets of D_{15}. Suppose s_4 not commutative with s_3. Since now s_4 fixes c_1, it must displace d_1 and d_2, whence $\{s_1, s_4\}$ is of degree 16. Now s_4 should replace a_2 by a letter new to D_{15}, but this may not be, because s_4 and t_1 are commutative. Let s_4 and s_3 be commutative. If s_4 displaces a_2, it replaces a_2 by b_2, which $\{s_1, s_4\}$ does not permit. Then s_4 fixes a_2 and b_2. Both b_1 and b_2 can not be fixed by s_4, hence

$$s_4 = a_1 \alpha_1 \cdot c_1 \alpha_2 \cdot b_1 \beta_1 \cdot d_1 \beta_2 \cdot e_1 \varepsilon_2 \cdot f_1 \zeta_2.$$

We determined e_1 and f_1 by means of the transformations $e_1 e_2 \cdot \varepsilon_1 \varepsilon_2$ and $f_1 f_2 \cdot \zeta_1 \zeta_2$.

The substitution s_5 is now obviously out of the question.

There is in the series $s_1, \ldots$ at least one substitution which connects sets of D_{15} and has no cycle of new letters. Let

$$s_3 = (a_1 c_1) \ldots .$$

Since $s_3 t_1 = t_1 s_3$, if s_3 is commutative with s_1 or t_2, it is commutative with all the substitutions of D_{15}. Then it is not commutative with s_1 or t_2. Suppose it is commutative with s_2. It fixes a_2, b_2, c_2, d_2, and displaces the same letters as s_2. Then s_3 fixes a_2 and b_1, leading to the absurd conclusion that it is commutative with t_2. Let

$$s_3 = (a_1 e_1) \ldots .$$

If s_3 is commutative with s_1, it is not commutative with s_2, and vice-versa. Suppose

$$s_3 = (a_1 e_1)(a_2 e_2) \ldots ,$$

fixing b_1, b_2, ε_1. Since $t_2 s_3$ is of order 3, one cycle of s_3 is $(\varepsilon_2 \gamma_1)$, and another (we may transform by $f_1 f_2 \cdot \zeta_1 \zeta_2$) is $(\zeta_2 \gamma_2)$. Then we also have $(f_2 c_2)$ or $(f_2 d_2)$, determined arbitrarily by means of $c_1 d_1 \cdot c_2 d_2$ as $(f_2 c_2)$. Then this substitution

$$s_3 = a_1 e_1 \cdot a_2 e_2 \cdot c_1 f_1 \cdot c_2 f_2 \cdot \varepsilon_2 \gamma_1 \cdot \zeta_2 \gamma_2$$

is unique. If s_3 is commutative with s_2, we transform s_3 into the above by means of $a_1 e_1 \cdot a_2 \varepsilon_1 \cdot b_1 e_2 \cdot b_2 \varepsilon_2 \cdot c_1 f_1 \cdot c_2 \zeta_1 \cdot d_1 f_2 \cdot d_2 \zeta_2$. Now assume that s_3 is non-commutative with both s_1 and s_2. The letters a_2, e_2, b_1, ε_1 are fixed, so that s_3 is commutative with both t_1 and t_2, which generate D_{15}. Let

$$s_3 = (a_1 c_2) \ldots .$$

We first note that s_3 is not commutative with both s_1 and t_2. If s_3 is commutative with s_1,

$$s_3 = a_1 c_2 \cdot a_2 c_1 \cdot \varepsilon_1 \alpha_1 \cdot \varepsilon_2 \alpha_2 \cdot \zeta_1 \beta_1 \cdot \zeta_2 \beta_2 .$$

Since $a_2 b_2 \cdot c_1 d_1 \cdot e_1 \varepsilon_1 \cdot e_2 \varepsilon_2 \cdot f_1 \zeta_1 \cdot f_2 \zeta_2$ transposes s_1 and t_2, we next assume that s_3 is commutative with neither s_1 nor t_2. Now s_3 fixes c_1 and d_1, which would make it commutative with s_2, an absurdity.

There are now two groups H_3 that require attention. Let us first investigate that one generated by D_{15} and

$$s_3 = a_1 e_1 \cdot a_2 e_2 \cdot c_1 f_1 \cdot c_2 f_2 \cdot \varepsilon_2 \gamma_1 \cdot \zeta_2 \gamma_2 .$$

If G is doubly transitive there is in it a substitution

$$s = (a_1 c_1) \ldots .$$

similar to s_1. From the preceding discussion we know that s must have a cycle new to D_{15} and hence is commutative with every substitution of D_{15}. But

$$\{s_3,\ s = a_1 c_1 \cdot a_2 c_2 \cdot b_1 d_1 \,.\, b_2 d_2 \ldots .\}$$

is not possible. Then G is simply transitive and the transitive group H_4 is of degree 21 or more. Since the group $\{t_1, s_3 t_2 s_3\}$ fixes a_1 and has a transitive set in the four letters $d_1, f_2, \zeta_1, \gamma_2$ and another in the letters c_2, d_2, f_1, ζ_2, there are two types to which s_4 reduces. One is

$$s_4 = (a_1 d_1)(\delta_1 -)(\delta_2 -)(\delta_3 -) \ldots .$$

This substitution is commutative with t_1, so that b_1 or c_1 is replaced by δ_1. But if s_4 is commutative with s_2, both b_1 and c_1 are fixed by s_1. The other type is

$$s_4 = (a_1 c_2)(\delta_1 -)(\delta_2 -)(\delta_3 -) \ldots .$$

If s_4 is commutative with neither s_1 nor t_2, s_4 fixes a_2 and b_2, whence s_4 should be commutative with t_1. Obviously s_4 is not commutative with both s_1 and t_2. If it is commutative with s_1,

$$s_4 = (a_1 c_2)(a_2 c_1)(\delta_1 -)(\delta_2 -)(\delta_3 -)(- -),$$

and displaces in four cycles $\varepsilon_1, \varepsilon_2, \zeta_1$ and ζ_2. Since s_3 and s_4 are not commutative, s_4 must displace $\varepsilon_2, \zeta_2, \gamma_1$ and γ_2 in the last four cycles. Both these conditions can not be satisfied. If s_4 is commutative with t_2,

$$s_4 = a_1 c_2 \cdot b_2 d_1 \cdot e_1 f_2 \ldots ,$$

since s_4 is commutative with s_3 because of the cycle $(b_2 d_1)$. But from $\{s_1, s_4\}$ the four letters e_1, e_2, f_1, f_2 are in four cycles of s_4.

There is still the substitution

$$s_3 = a_1 c_2 \cdot a_2 c_1 \cdot \varepsilon_1 \alpha_1 \cdot \varepsilon_2 \alpha_2 \cdot \zeta_1 \beta_1 \cdot \zeta_2 \beta_2 .$$

The substitution s_4 may be reduced to a single type. For we may apply to H_8 and s_4 all the transformations of

$$\{t_1,\ s_3 s_2 s_3 = b_1 c_2 \cdot d_1 a_2 \cdot e_1 \alpha_1 \cdot e_2 \alpha_2 \cdot f_1 \beta_1 \cdot f_2 \beta_2\}$$

and

$$\{e_1 e_2 \cdot f_1 f_2 \cdot \varepsilon_1 \varepsilon_2 \cdot \zeta_1 \zeta_2 \cdot \alpha_1 \alpha_2 \cdot \beta_1 \beta_2,\quad e_1 f_1 \cdot e_2 f_2 \cdot \varepsilon_1 \zeta_1 \cdot \varepsilon_2 \zeta_2 \cdot \alpha_1 \beta_1 \cdot \alpha_2 \beta_2\}.$$

Then

$$s_4 = (a_1 e_1) \ldots .$$

This substitution must have a cycle new to D_{15} and is in consequence commutative with every substitution of D_{15}. But such a substitution is at once seen to be out of the question.

The only non-Abelian group now remaining in our list is the D_{23} of order 6.

D_4 and D_5 may be advantageously discussed together. There are two types of substitutions to which s_3 may be reduced: $(a_1 c_1) \ldots .$ and $(a_1 e_1) \ldots .$ Now $(a_1 c_1)(a_2 c_2) \ldots .$ is not consistent with s_2; and $(a_1 e_1)(a_2 e_2) \ldots .$, since $s_2 s_3$ is of order 3, must displace one of the letters c_1, c_2 and not the other.

Consider the three groups D_i $(i = 7, 8, 9)$. A substitution s similar to s_1 can not replace one of the letters common to s_1 and s_2 by another letter of D_i. Nor can s replace one of the letters of D_i not in the common cycles by a letter new to D_i. The first remark is obviously true. In regard to the second, if s replaces α_1, say, by x, s_3 replaces one letter of each of the common cycles by a letter new to D_i, and displaces one letter from every other cycle of s_1 and s_2. Now this makes s of degree 13. It follows that substitutions of the series $s_1, s_2, \ldots .$ do not with D_i generate a transitive group.

The third group of our list can also be rejected, as will now be shown. All the substitutions of the group

$$\{a_2 b_1 \cdot c_2 d_1 \cdot e_1 \alpha_1 \cdot e_2 \alpha_2 \cdot f_1 \beta_1 \cdot f_2 \beta_2,\ c_1 c_2 \cdot d_1 d_2,\ e_1 f_1 \cdot e_2 f_2,\ \alpha_1 \beta_1 \cdot \alpha_2 \beta_2,\ e_1 e_2,\ f_1 f_2,\ \alpha_1 \alpha_2,\ c_1 d_1 \cdot c_2 d_2\}$$

transform D_3 into itself. Then s_3 is either $(a_1 c_1)$ or $(a_1 e_1) \ldots .$ First, the substitution

$$s_3 = a_1 e_1 \cdot a_2 e_2 \cdot c_1 f_1 \cdot c_2 f_2 \cdot \alpha_1 \alpha_3 \cdot \beta_1 \beta_3$$

is unique. Passing at once to s_4, we have for it four possibilities:

$$(a_1 c_1) \ldots ., \quad (a_1 d_1) \ldots ., \quad (a_1 \alpha_1) \ldots ., \quad (a_1 \alpha_2) \ldots .$$

Of these, the first gives $a_1 c_1 \cdot a_2 c_2 \cdot b_1 d_1 \cdot b_2 d_2 \cdot e_1 f_1 \cdot e_2 f_2$, which is here not possible. We conclude from this that G is not doubly transitive. The other three forms of s_4 give only

$$s_4' = a_1 d_1 \cdot a_2 d_2 \cdot b_1 c_1 \cdot b_2 c_2 \cdot \alpha_3 \alpha_4 \cdot \beta_3 \beta_4,$$
$$s_4'' = a_1 \alpha_1 \cdot b_1 \alpha_2 \cdot e_1 \alpha_3 \cdot c_1 \beta_1 \cdot d_1 \beta_2 \cdot f_1 \beta_3,$$
$$s_4''' = a_1 \alpha_2 \cdot b_1 \alpha_1 \cdot c_1 \beta_2 \cdot d_1 \beta_1 \cdot e_2 \alpha_4 \cdot f_2 \beta_4.$$

To dispose of this case it will be sufficient to show that all the above substitutions s_4', s_4'' and s_4''' lead to transitive groups of degree less than 21. The substitution s_5 to be adjoined to $H_4' = \{H_3, s_4'\}$ is

$$a_1 \alpha_1 \cdot b_1 \alpha_2 \cdot e_1 \alpha_3 \cdot c_2 \beta_1 \cdot d_2 \beta_2 \cdot f_2 \beta_3$$

or

$$a_1 \alpha_2 \cdot b_1 \alpha_1 \cdot c_2 \beta_2 \cdot d_2 \beta_1 \cdot e_2 \beta_4 \cdot f_1 \alpha_4;$$

and in both cases H' is of degree 20 and transitive. In connection with $H_4'' = \{H_3, s_4''\}$, there is only

$$a_1 d_2 \cdot a_2 d_1 \cdot b_1 c_2 \cdot b_2 c_1 \cdot \alpha_3 \alpha_4 \cdot \beta_3 \beta_4,$$

which likewise gives a transitive group of degree 20. To $H_4''' = \{H_3, s_4'''\}$ there is the single substitution

$$a_1 d_2 \cdot a_2 d_1 \cdot b_1 c_2 \cdot b_2 c_1 \cdot \alpha_3 \beta_4 \cdot \beta_3 \alpha_4,$$

making H_4''' transitive of degree 20. The present result may be stated thus:

There is no substitution s, similar to s_1, which replaces one of the letters $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ by one of the letters $e_1, e_2, f_1, f_2, \alpha_1, \alpha_2, \beta_1, \beta_2$. Such a substitution could be transformed, by a substitution which leaves D_3 invariant, into $(a_1 e_1) \ldots\ldots$

In the second place, let

$$s_3' = a_1 c_1 \cdot a_2 c_2 \cdot b_1 d_1 \cdot b_2 d_2 \cdot \gamma_1 \gamma_2 \cdot \delta_1 \delta_2.$$

Now s_4 must replace a_1 by γ_1, say. But $s_4 = (a_1 \gamma_1) \ldots\ldots$ bears the same relation to $\{s_1, s_3'\}$ as $(a_1 e_1) \ldots\ldots$ bears to $\{s_1, s_2\}$. Hence the group D_3 may also be struck from our list. There remain only D_6, D_{10} and D_{23}.

The substitutions of order 2 in D_{23} are

$$\begin{aligned} s_1 &= a_1 a_2 \cdot b_1 b_2 \cdot c_1 c_2 \cdot d_1 d_2 \cdot e_1 e_2 \cdot f_1 f_2, \\ s_2 &= a_1 a_3 \cdot b_1 b_3 \cdot c_1 c_3 \cdot d_1 d_3 \cdot e_1 e_3 \cdot f_1 f_3, \\ t_1 &= a_2 a_3 \cdot b_2 b_3 \cdot c_2 c_3 \cdot d_2 d_3 \cdot e_2 e_3 \cdot f_2 f_3. \end{aligned}$$

There is in G a substitution s_3 which is non-commutative with two of these substitutions, with s_1 and s_2, say. Now s_3 may or may not join two sets of D_{23}. First assume that sets are connected:

$$s_3 = a_2 b_3 \cdot a_3 b_2 \cdot c_1 c_4 \cdot d_1 d_4 \cdot e_1 e_4 \cdot f_1 f_4.$$

In H_3 are also present

$$\begin{aligned} t_2 &= a_1 b_3 \cdot a_3 b_1 \cdot c_2 c_4 \cdot d_2 d_4 \cdot e_2 e_4 \cdot f_2 f_4, \\ t_3 &= a_1 b_2 \cdot a_2 b_1 \cdot c_3 c_4 \cdot d_3 d_4 \cdot e_3 e_4 \cdot f_3 f_4. \end{aligned}$$

Extend H_3 by

$$s_4 = a_1 c_1 \cdot a_2 c_2 \cdot a_3 c_3 \cdot d_4 d_5 \cdot e_4 e_5 \cdot f_4 f_5,$$

or by

$$a_1 c_3 \cdot a_3 c_1 \cdot b_2 c_4 \cdot d_2 d_5 \cdot e_2 e_5 \cdot f_2 f_5.$$

Since these two substitutions are conjugate under

$$a_2 b_2 \cdot c_1 c_3 \cdot c_2 c_4 \cdot d_1 d_3 \cdot d_2 d_4 \cdot e_1 e_3 \cdot e_2 e_4 \cdot f_1 f_3 \cdot f_2 f_4,$$

which transforms s_1, s_2, t_1 into t_3, s_2, s_3 respectively, we use the first only. Note the substitutions of H_4:

$$t_4 = a_1 c_4 \cdot b_3 c_2 \cdot b_2 c_3 \cdot d_1 d_5 \cdot e_1 e_5 \cdot f_1 f_5,$$
$$t_5 = b_3 c_1 \cdot a_2 c_4 \cdot b_1 c_3 \cdot d_2 d_5 \cdot e_2 e_5 \cdot f_2 f_5,$$
$$t_6 = b_2 c_1 \cdot b_1 c_2 \cdot a_3 c_4 \cdot d_3 d_5 \cdot e_3 e_5 \cdot f_3 f_5.$$

Next we have

$$s_5 = a_1 d_3 \cdot a_3 d_1 \cdot b_2 d_4 \cdot c_2 \lambda \cdot e_2 f_5 \cdot f_2 e_5,$$

or

$$s_5' = a_1 d_1 \cdot a_2 d_2 \cdot a_3 d_3 \cdot c_4 d_5 \cdot e_4 e_6 \cdot f_4 f_6.$$

The group $H_5 = \{H_4, s_5\}$ has two constituents, the first of which is the $G_{1920}^{16,8}$, and the second is an imprimitive group, having five systems of two letters each permuted according to the $G_{5!}^5$. The head of the second constituent is obviously

$$F \equiv \{e_1 f_1 \cdot e_2 f_2, \quad e_1 f_1 \cdot e_3 f_3, \quad e_1 f_1 \cdot e_4 f_4, \quad e_1 f_1 \cdot e_5 f_5\}$$

of order 16. The intransitive subgroup

$$\Lambda \equiv \{e_1 e_2 \cdot f_1 f_2, \quad e_1 e_3 \cdot f_1 f_3, \quad e_1 e_4 \cdot f_1 f_4, \quad e_1 e_5 \cdot f_1 f_5\}$$

is maximal, so that this group certainly has a primitive representation of order 1920 on 16 letters. For the moment let the symbols (they are not substitutions)

1, 12, 13, 14, 15, 23, 24, 25, 34, 35, 45, 1234, 1235, 1245, 1345, 2345

represent the substitutions of the Abelian group F. If now the group Λ is transformed by these 16 substitutions in the order indicated, and if with these 16 subgroups the letters

$$\lambda, b_3, b_2, a_1, c_1, b_1, a_2, c_2, a_3, c_3, c_4, d_5, d_4, d_3, d_2, d_1$$

are associated, the generators

$$e_1 e_2 \cdot f_1 f_2, \quad e_1 e_3 \cdot f_1 f_3, \quad e_1 e_4 \cdot f_1 f_4, \quad e_4 e_5, \quad f_4 f_5, \quad e_2 f_5 \cdot e_5 f_2$$

permute the 16 subgroups of order 120 according to the substitutions

$$a_1 a_2 \cdot b_1 b_2 \cdot c_1 c_2 \cdot d_1 d_2, \quad a_1 a_3 \cdot b_1 b_3 \cdot c_1 c_3 \cdot d_1 d_3, \quad a_2 b_3 \cdot a_3 b_2 \cdot c_1 c_4 \cdot d_1 d_4,$$
$$a_1 c_1 \cdot a_2 c_2 \cdot a_3 c_3 \cdot d_1 d_5, \quad a_1 d_3 \cdot a_3 d_1 \cdot c_2 \lambda \cdot b_2 d_4,$$

respectively. Hence the intransitive group H_5 is exactly of order 1920.

Continuing the study of this group H_5 of order 1920, we write out for reference

$$t_7 = a_2 d_3 \cdot a_1 d_2 \cdot b_3 d_4 \cdot c_3 \lambda \cdot e_3 f_5 \cdot e_5 f_3, \quad t_8 = b_1 d_1 \cdot b_2 d_2 \cdot b_3 d_3 \cdot c_4 \lambda \cdot e_4 f_5 \cdot e_5 f_4,$$
$$t_9 = a_2 d_3 \cdot a_3 d_2 \cdot b_1 d_4 \cdot c_1 \lambda \cdot e_1 f_5 \cdot e_5 f_1, \quad t_{10} = c_1 d_2 \cdot c_2 d_1 \cdot b_3 d_5 \cdot a_3 \lambda \cdot e_3 f_4 \cdot e_4 f_3,$$
$$t_{11} = c_1 d_3 \cdot c_3 d_1 \cdot b_2 d_5 \cdot a_2 \lambda \cdot e_2 f_4 \cdot e_4 f_2, \quad t_{12} = c_2 d_3 \cdot c_3 d_2 \cdot b_1 d_5 \cdot a_1 \lambda \cdot e_1 f_4 \cdot e_4 f_1,$$
$$t_{13} = c_2 d_4 \cdot c_4 d_2 \cdot a_2 d_5 \cdot b_2 \lambda \cdot e_1 f_3 \cdot e_3 f_1, \quad t_{14} = c_1 d_4 \cdot c_4 d_1 \cdot a_1 d_5 \cdot b_1 \lambda \cdot e_2 f_3 \cdot e_3 f_2,$$
$$t_{15} = c_3 d_4 \cdot c_4 d_3 \cdot a_3 d_5 \cdot b_3 \lambda \cdot e_2 f_1 \cdot e_1 f_2.$$

The only substitution uniting the two sets of H_5 that need be considered is

$$s_6 = \lambda e_4 \cdot a_1 f_1 \cdot a_2 f_2 \cdot a_3 f_3 \cdot c_4 f_5 \cdot d_4 \mu,$$

which with H_5 generates a primitive group of degree 27. That in H_6 the subgroup leaving λ fixed is actually H_5 is a consequence of the equations

$$\begin{aligned} s_6 s_1 s_6 &= s_1, & s_6 s_3 s_6 &= s_3 s_6 s_3, \\ s_6 s_2 s_6 &= s_2, & s_6 s_4 s_6 &= s_4 s_6 s_4, \\ & & s_6 s_5 s_6 &= s_5 s_6 s_5, \end{aligned}$$

by means of which any substitution of $\{H_5, s_6\}$ can be put in the form

$$V_1 \text{ or } V_2 s_6 V_3,$$

where V_1, V_2, V_3 are substitutions of H_5.*

Can a larger group including H_6 contain other substitutions similar to s_1? H_6 does not transform such a substitution into one of its own members. Hence we have only to consider $(\lambda d_5) \ldots$, say. This gives uniquely (using $s_5 t_{13} s_6 t_{13} s_5 = (\lambda \mu)(d_1 e_1)(d_2 e_2)(d_3 e_3)(d_4 e_4)(d_5 e_5)$)

$$s_7 = \lambda d_5 \cdot a_1 b_1 \cdot a_2 b_2 \cdot a_3 b_3 \cdot e_5 \mu \,.\, f_5 \nu,$$

which does actually with H_6 generate a doubly transitive group of degree 28 and class 12. There remains only the question whether H_6 or H_7 may be contained invariantly in larger groups of degree 27 or 28.

It is enough to consider H_6. Let G_1 be the subgroup of G^{27} that leaves the letter μ fixed. It is assumed that H_6 is invariant in G, and in consequence H_5 is invariant in G_1. Then G_1 is not transitive but has the same two sets of letters as H_5. But the constituent of degree 16 in H_5 is not invariant in a larger group of degree 16.

We return to the group

$$\{H_4, s_5' = a_1 d_1 \cdot a_2 d_2 \cdot a_3 d_3 \cdot c_4 d_5 \cdot e_4 e_6 \cdot f_4 f_6\}.$$

The constituents of H_5' are the $G_{720}^{15,8}$ and the two $G_{6!}^{6}$. For use in the following steps we note

$$\begin{aligned} t_7' &= a_1 d_4 \cdot b_3 d_2 \cdot b_2 d_3 \cdot c_1 d_5 \cdot e_1 e_6 \cdot f_1 f_6, & t_8' &= b_3 d_1 \cdot a_2 d_4 \cdot b_1 d_3 \cdot c_2 d_5 \cdot e_2 e_6 \cdot f_2 f_6, \\ t_9' &= b_2 d_1 \cdot b_1 d_2 \cdot a_3 d_4 \cdot c_3 d_5 \cdot e_3 e_6 \cdot f_3 f_6, & t_{10}' &= c_1 d_1 \cdot c_2 d_2 \cdot c_3 d_3 \cdot c_4 d_4 \cdot e_5 e_6 \cdot f_5 f_6. \end{aligned}$$

Next we have

$$s_6' = a_1 e_1 \cdot a_2 e_2 \cdot a_3 e_3 \cdot c_4 e_5 \cdot d_4 e_6 \cdot f_4 f_7,$$

or

$$s = a_1 e_3 \cdot a_3 e_1 \cdot b_2 e_4 \cdot c_2 f_6 \cdot d_2 f_5 \cdot d_5 f_2.$$

---

* Cf. JORDAN, "Traité des Substitutions," 1870, p. 30.

The substitution s gives a transitive group whose generators taken in order correspond to

$$s_1,\ s_2,\ s_3,\ s_4,\ s_6,\ s_5;$$

in fact these substitutions are transformed into

$$s_1,\ s_2,\ s_3,\ s_4,\ s_5',\ s,$$

respectively, by

$$d_1e_1f_1 \cdot d_2e_2f_2 \cdot d_3e_3f_3 \cdot d_4e_4f_4 \cdot d_5e_5f_5 \cdot e_6\lambda f_6\mu.$$

Proceeding with s_6', we note

$$t_{11}' = a_1e_4 \cdot b_3e_2 \cdot b_2e_3 \cdot c_1c_5 \cdot d_1e_6 \cdot f_1f_7, \quad t_{12}' = b_3e_1 \cdot a_2e_4 \cdot b_1e_3 \cdot c_2e_5 \cdot d_2e_6 \cdot f_2f_7,$$
$$t_{13}' = b_2e_1 \cdot b_1e_2 \cdot a_3e_4 \cdot c_3e_5 \cdot d_3e_6 \cdot f_3f_7, \quad t_{14}' = c_1e_1 \cdot c_2e_2 \cdot c_3e_3 \cdot c_4e_4 \cdot d_5e_6 \cdot f_5f_7,$$
$$t_{15}' = d_1e_1 \cdot d_2e_2 \cdot d_3e_3 \cdot d_4e_4 \cdot d_5e_5 \cdot f_6f_7.$$

Now we have

$$s_7' = a_1f_1 \cdot a_2f_2 \cdot a_3f_3 \cdot c_4f_5 \cdot d_4f_6 \cdot e_4f_7$$

or

$$s_7'' = a_1f_3 \cdot a_3f_1 \cdot b_2f_4 \cdot c_2e_6 \cdot d_2e_5 \cdot e_2d_5.$$

Since $d_1f_1 \cdot d_2f_2 \cdot d_3f_3 \cdot d_4f_4 \cdot d_5f_5 \cdot \lambda e_6 \cdot \mu f_6$ is commutative with s_1, s_2, s_3, s_4, and transforms s_5 and s_6 into s_7'' and s_5' respectively, only s_7' is to be considered. The group $\{H_5', s_6's_7's_6' = e_1f_1 \cdot e_2f_2 \cdot e_3f_3 \cdot e_4f_4 \cdot e_5f_5 \cdot e_6f_6\}$ has two transitive constituents, the primitive $G_{720}^{15,8}$ and the imprimitive $G_{2(6!)}^{10}$. The relations

$$s_7's_1s_7' = s_1, \quad s_7's_3s_7' = s_3s_7's_3, \quad s_7's_5s_7' = s_5's_7's_5',$$
$$s_7's_2s_7' = s_2, \quad s_7's_4s_7' = s_4s_7's_4, \quad s_7'(s_6's_7's_6')s_7' = s_6's_7's_6's_7's_6's_7's_6'$$

show that the primitive group H_7' really has $\{H_5', s_6's_7's_6'\}$ as its subgroup leaving f_7 fixed. This group H_7' is simply isomorphic to the $G_{8!}^{8}$. It is not invariant in a larger group of the same degree. Then any larger group containing it has a substitution similar to s_1 new to H_7'. After transformation by the substitutions of $\{H_5', s_6's_7's_6'\}$, there are two types of substitutions involving f_7 which offer a chance of extending H_7': $(f_7e_4)\ldots.$ and $(f_7c_2)\ldots.$. The first, $f_7e_4 \cdot a_1f_1\ldots.$, is already in H_7', and the second,

$$s_8' = f_7c_2 \cdot e_2f_5 \cdot e_5f_2 \cdot a_1d_3 \cdot b_2d_4 \cdot d_1a_3,$$

is the substitution s_5 already discussed.

We return to D_{23}, and assume that s_3 does not join sets of D_{23}. Then

$$s_3 = a_1a_4 \cdot b_1b_4 \cdot c_1c_4 \cdot d_1d_4 \cdot e_1e_4 \cdot f_1f_4,$$
$$t_2 = a_2a_4 \cdot b_2b_4 \cdot c_2c_4 \cdot d_2d_4 \cdot e_2e_4 \cdot f_2f_4,$$
$$t_3 = a_3a_4 \cdot b_3b_4 \cdot c_3c_4 \cdot d_3d_4 \cdot e_3e_4 \cdot f_3f_4.$$

There exists a substitution s_4 which connects the above letters with new letters, and since we have discussed the case where s_4 connects sets of such a group as $\{s_1, s_2\}$ and is not commutative with all its substitutions, we have uniquely

$$s_4 = a_1 a_5 \cdot b_1 b_5 \cdot c_1 c_5 \cdot d_1 d_5 \cdot e_1 e_5 \cdot f_1 f_5,$$

and similarly,

$$s_5 = a_1 a_6 \cdot b_1 b_6 \cdot c_1 c_6 \cdot d_1 d_6 \cdot e_1 e_6 \cdot f_1 f_6.$$

Now there exists a substitution s_6 that connects two sets of H_5 and has no cycle of new letters:

$$s_6 = a_1 b_1 \cdot a_2 b_2 \cdot a_3 b_3 \cdot a_4 b_4 \cdot a_5 b_5 \cdot a_6 b_6,$$

uniquely, and so on,

$$\begin{aligned} s_7 &= a_1 c_1 \cdot a_2 c_2 \cdot a_3 c_3 \cdot a_4 c_4 \cdot a_5 c_5 \cdot a_6 c_6, \\ s_8 &= a_1 d_1 \cdot a_2 d_2 \cdot a_3 d_3 \cdot a_4 d_4 \cdot a_5 d_5 \cdot a_6 d_6, \\ s_9 &= a_1 e_1 \cdot a_2 e_2 \cdot a_3 e_3 \cdot a_4 e_4 \cdot a_5 e_5 \cdot a_6 e_6, \\ s_{10} &= a_1 f_1 \cdot a_2 f_2 \cdot a_3 f_3 \cdot a_4 f_4 \cdot a_5 f_5 \cdot a_6 f_6. \end{aligned}$$

The transitive group H_{10} is imprimitive of degree 36 and order $(6!)^2$. It is contained in one and only one primitive group $G^{36,12}_{2(6!)^2}$.

TURIN, ITALY, *February*, 1911.

The Cartesian Oval and the Elliptic Functions p and σ.

By Clara Latimer Bacon.

The object of this paper is to show that the leading properties of the Cartesian oval, as developed in such texts as Williamson's "Differential Calculus," Salmon's "Higher Plane Curves" or Loria's "Spezielle Algebraische und Transcendente Ebene Kurven," may be readily deduced from the Weierstrass elliptic functions p and σ, and that they give a geometric interpretation to the standard formulæ of these functions.

Greenhill has used these functions in connection with the Cartesian oval (*Proc. London Math. Soc.*, t. XVII, 1886, and also his treatise on "Elliptic Functions," Arts. 236, 248 and 249) and has deduced the relations between the focal radii, though we have given a more symmetric form to these relations by the introduction of the triple focus.

Professor Frank Morley published a note on the subject some years ago in the Haverford Studies. (See also Harkness and Morley, "Treatise on the Theory of Functions," p. 336.)

When the u-plane is mapped on the x-plane by means of the equation

$$du=\frac{dx}{\sqrt{Q}},$$

where Q is a quartic in x, to lines in the u-plane parallel to the real or imaginary axis, correspond in the x-plane bicircular quartics whose real foci are the zeros of Q, provided these zeros are concyclic or anticyclic. (Greenhill, *Camb. Phil. Proc.*, t. IV; Franklin, American Journal of Mathematics, Vol. XI, 3 and Vol. XII, 4.)

The problem of reducing this form to the Weierstrass form which we use, where one focus is at infinity, is analogous to the reduction of the bicircular quartic to the Cartesian. (Salmon, "Higher Plane Curves," Art. 280.)

1. *Preliminary Examples in Mapping.*

We let $x=pu$. Then

$$\frac{dx}{du}=p'u=-\sqrt{4p^3u-g_2pu-g_3}=-\sqrt{pu-e_1\cdot pu-e_2\cdot pu-e_3}.$$

(Harkness and Morley, "Introduction to Analytic Functions," p. 259, (16).)

Hence

$$du=-\frac{dx}{\sqrt{x-e_1\cdot x-e_2\cdot x-e_3}}.$$

We may suppose g_2 and g_3 to be real and

$$\Delta=g_2^3-2g_3^2>0.$$

Then e_1, e_2 and e_3 are real.

Let $e_1>e_2>e_3$. Since, when $x<e_3$, $\frac{du}{dx}$ is imaginary and, when $x>e_1$, $\frac{du}{dx}$ is real, we have a real period $2\omega_1$ and a purely imaginary period $2\omega_3$. Hence the parallelogram of periods is a rectangle. We write for symmetry

$$\omega_1+\omega_2+\omega_3=0.$$

Let u describe some path in the u-plane and trace the corresponding path of x in the x-plane.

For example, let u move from 0 to ω_1 along the real axis. Then pu or x is real and decreases from infinity to e_1. If u bends to the left to avoid the critical point ω_1 in a quarter circle about it, x turns to the left and describes a semicircle about e_1, as the isogonality breaks down at this point and the angle which u makes about ω_λ is one-half that which x makes about e_λ. And so, as u describes the perimeter of the parallelogram $0, \omega_1, -\omega_2, \omega_3$, x traverses the real axis from plus infinity to minus infinity. (Burkhardt, "Functionen Theorie," Vol. II, § 86, IX.)

As u goes from 0 to $2\omega_1$, passing to the left in a semicircle about ω_1, x moves from plus infinity to e_1, then in a circle to the left about e_1 and back to plus infinity.

Any point within the parallelogram $0, \omega_1, -\omega_2, \omega_3$ is mapped into a point on the negative half of the x-plane. The map of the whole period parallelogram doubly covers the x-plane, since pu is an elliptic function of the second order.

2. *The Map of a Vertical Line and the Equation of the Curve.*

Suppose u to describe a line parallel to the axis of imaginaries and at a distance α from it. Let

$$u=\alpha+i\beta$$

and

$$\bar{u}=\alpha-i\beta.$$

Then the equation of the line is

$$u+\bar{u}=2\alpha, \tag{1}$$

u and $\bar{u}$ being conjugate and α real.

Since

$$pu=\frac{1}{u^2}+\frac{g_2u^2}{2^2\cdot 5}+\frac{g_3u^6}{2^2\cdot 7}+\cdots$$

(Tannery and Molk, "Théorie des Fonctions Elliptiques," IX, 3) and g_2 and g_3 are real by hypothesis,

$$\left.\begin{aligned} x&=p(\alpha+i\beta)=pu\\ \text{and}\qquad \bar{x}&=p(\alpha-i\beta)=p\bar{u}\end{aligned}\right\}\quad(2)$$

are conjugate complex numbers. (Burkhardt, Vol. II, § 85; Vol. I, § 71.) To find the curve in x which corresponds to the vertical through α traced by u, eliminate β between equations (1) and (2) by means of an addition theorem which gives the relation between $p(u+v)$, pu and pv.

To obtain this theorem we have

$$p'^2u=4p^3u-g_2pu-g_3.$$

Consider

$$p'u-cpu-c',$$

where c and c' are constants. It is an elliptic function with the same periods as pu. It has an infinity of the third order at $u=0$. Therefore it has three zeros u, v and w, such that

$$u+v+w=0.$$

For each of these points

$$(cpu+c')^2=(p'u)^2=4p^3u-g_2pu-g_3,$$

or

$$4p^3u-c^2p^2u-(2cc'+g_2)pu-(c'^2+g_3)=0.$$

Then

$$S_1\equiv pu+pv+pw=\frac{1}{4}c^2,$$

$$S_2\equiv pu\,pv+pu\,pw+pv\,pw=-\frac{1}{4}(2cc'+g_2),$$

$$S_3\equiv pu\,pv\,pw=\frac{1}{4}(c'^2+g_3).$$

Therefore

$$S_1\left(S_3-\frac{1}{4}g_3\right)=\frac{1}{16}c^2c'^2=\frac{1}{4}\left(S_2+\frac{1}{4}g_2\right)^2,$$

from which we get an addition theorem:

$$\left(S_2+\frac{1}{4}g_2\right)^2=S_1(4S_3-g_3).$$

In this theorem substitute

$$u=\alpha+i\beta,$$
$$v=\alpha-i\beta,$$

whence

$$w=-2\alpha.$$

For the sake of brevity, write A for $p2\alpha$ or $p(-2\alpha)$. Then

$$S_1=x+\bar{x}+A,$$
$$S_2=x\bar{x}+A(x+\bar{x}),$$
$$S_3=Ax\bar{x}.$$

Therefore

$$\left(x\bar{x}+A[x+\bar{x}]+\frac{1}{4}g_2\right)^2=(x+\bar{x}+A)(4Ax\bar{x}-g_2). \tag{3}$$

The coefficient of x^2 is

$$(\bar{x}+A)^2-4A\bar{x}\equiv(\bar{x}-A)^2.$$

Speaking projectively, this gives the tangents at the circular point I or J and the equation is that of a bicircular quartic, with cusps at I and J; that is, a Cartesian oval.

The cusp tangents at I and J,

$$x-A=0$$

and

$$\bar{x}-A=0,$$

meet at the triple focus A corresponding to the point $u=2\alpha$.

To obtain the single foci, we have the two equations:

$$x\text{-discriminant}=0,$$

and

$$\bar{x}\text{-discriminant}=0.$$

Rearranging the equation of the curve according to the powers of $\bar{x}$, we have:

$$\bar{x}^2(x-A)^2+\bar{x}\left(\frac{1}{2}g_2x+\frac{1}{2}g_2A-2Ax^2-2A^2x+g_3\right)$$
$$+A^2x^2+\frac{1}{2}g_2Ax+\frac{1}{16}g_2^2+g_3x+g_3A=0.$$

Therefore the $\bar{x}$-discriminant is

$$*(4A^3-g_2A-g_3)(4x^3-g_2x-g_3)=0.$$

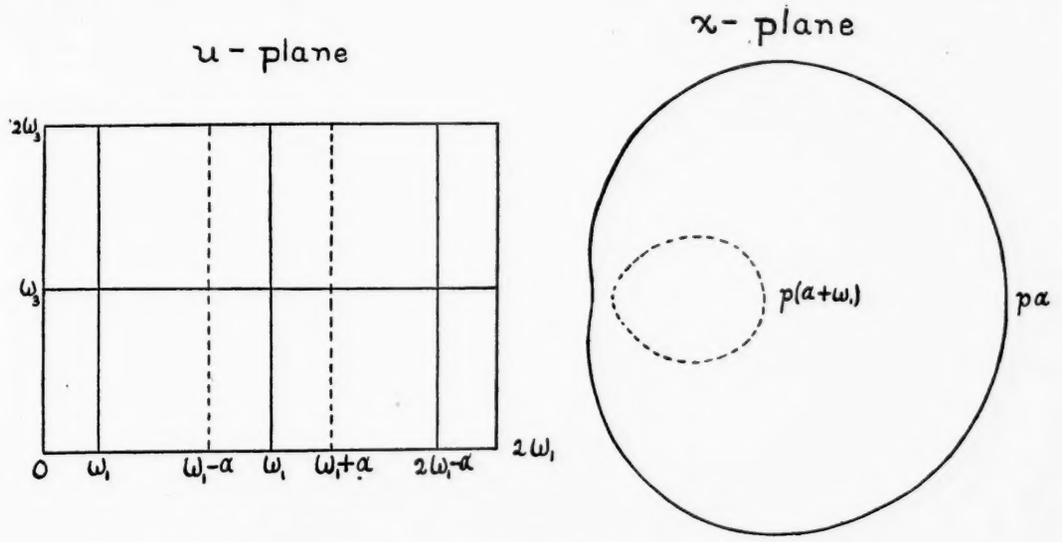
Hence the single foci are e_1, e_2 and e_3.

The curve depends on the single constant $p2\alpha$ or A. It will therefore include the representation of four lines in the rectangle; namely, those which

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* If $\alpha=\frac{1}{2}\omega_1$, and so $p2\alpha$ or A causes the first factor of this equation to vanish, the Cartesian oval becomes a circle counted twice (see p. 266) and the foci accordingly are arbitrary.

meet the real axis at the four points α, $\omega_1-\alpha$, $\omega_1+\alpha$, $2\omega_1-\alpha$. If u is any one of these points, $p2u=p2\alpha$.



Let $i\beta=v$. Since $p(\alpha+v)=p(2\omega_1-[\alpha+v])=p(2\omega_1-\alpha-v)$, the outer oval corresponds to the first and fourth lines, the p-function moving in clockwise direction from the real axis as v moves up from α and in anti-clockwise direction as it moves up from $2\omega_1-\alpha$.

Similarly the inner oval, taken twice in opposite directions, corresponds to the second and third lines.

In the same way, to lines parallel to the real axis correspond Cartesian ovals with the same foci, as may be seen by interchanging α and v in this discussion.

Since at every point $u=(\alpha+v)$ in the u-plane there is a line parallel to the axis of reals and one parallel to the axis of imaginaries, so through every point in the x-plane pass two orthogonal Cartesian ovals corresponding to these two lines.

3. *The Inversion of the Curve into Itself.*

Among the formulæ for the p-function is

$$(p[u+\omega_1]-e_1)\,(pu-e_1)=e_2-e_1\cdot e_3-e_1.$$

(Harkness and Morley, "Introduction to Analytic Functions," p. 261, (20).) Therefore, if by quasi-inversion we mean ordinary inversion followed by reflection in the real axis, $\alpha+v+\omega_1$ is the quasi-inverse of $\alpha+v$, and the inverse of $\alpha-v$ since the curve is symmetric with respect to the real axis. Hence the curve is its own inverse with respect to a circle with center e_1 and radius $\sqrt{e_2-e_1}\ \sqrt{e_3-e_1}$ (a circle with radius real).

As α approaches $\frac{1}{2}\omega_1$, the two verticals through α and $\omega_1-\alpha$ approach coincidence. Therefore, if $\alpha=\frac{1}{2}\omega_1$, the two ovals coincide and each is the above-mentioned circle.

An algebraic proof of this statement may be added. Substituting in equation (3), p. 264,

$$2\alpha=\omega_1$$

and therefore

$$p2\alpha=e_1,$$

and remembering that

$$-\frac{1}{4}g_2=\Sigma e_1e_2$$

and

$$g_3=4e_1e_2e_3,$$

we get

$$[x\bar{x}+(x+\bar{x})e_1-\Sigma e_1e_2]^2=(x+\bar{x}+e_1)(4e_1x\bar{x}-4e_1e_2e_3),$$

or

$$(x-e_1)(\bar{x}-e_1)-(e_2-e_1)(e_3-e_1)=0,$$

which is a circle with center e_1 and radius $\sqrt{e_2-e_1}\cdot\sqrt{e_3-e_1}$.

Likewise

$$[p(u+\omega_2)-e_2][pu-e_2]=(e_1-e_2)(e_3-e_2),$$

or, more generally,

$$[p(u+\omega_\lambda)-e_\lambda][pu-e_\lambda]=(e_\mu-e_\lambda)(e_\nu-e_\lambda),$$

where

$$\lambda,\ \mu,\ \nu=1,\ 2,\ 3.$$

Therefore the curve has three inversions into itself, one about each focus as center. (But the circle with e_2 as center is nullipartite, since its radius, $\sqrt{e_1-e_2}\cdot\sqrt{e_3-e_2}$, is imaginary.)

These three circles are orthogonal, since

$$(e_\mu-e_\lambda)(e_\nu-e_\lambda)+(e_\lambda-e_\nu)(e_\mu-e_\nu)=(e_\nu-e_\lambda)^2.$$

But the bicircular quartic in general inverts into itself from any one of its four foci, and the four circles of inversion are all orthogonal. The fourth focus of the Cartesian oval is at infinity and the real axis is the circle of infinite radius orthogonal to the other three circles of inversion. Hence inversion with respect to the fourth focus amounts to reflexion in the real axis. (S. Roberts, *Proc. London Math. Soc.*, t. III, p. 108.)

At first thought, it might appear possible to obtain any number of points on the curve from a given point by means of repeated quasi-inversions. But in fact we get a set of only four points. For, if

$$[p(u+\omega_\lambda)-e_\lambda][pu-e_\lambda]=(e_\mu-e_\lambda)(e_\nu-e_\lambda)$$

and

$$[p(u+\omega_\mu)-e_\mu][pu-e_\mu]=(e_\nu-e_\mu)(e_\lambda-e_\mu)$$

and

$$[p(u+\omega_\nu)-e_\nu][pu-e_\nu]=(e_\mu-e_\nu)(e_\lambda-e_\nu),$$

we get, by eliminating pu from the second and third equations,

$$[p(u+\omega_\mu)-e_\lambda][p(u+\omega_\nu)-e_\lambda]=(e_\mu-e_\lambda)(e_\nu-e_\lambda).$$

Hence, if we say that pu and $p(u+\omega_\lambda)$ are quasi-inverse as to e_λ, then $p(u+\omega_\mu)$ and $p(u+\omega_\nu)$ are also quasi-inverse as to e_λ.

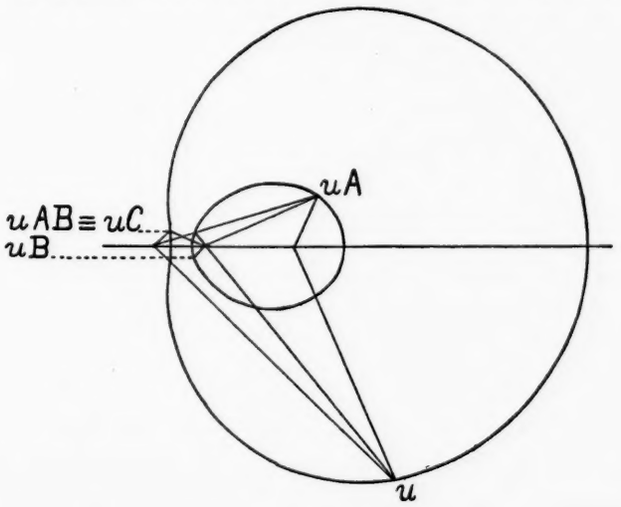
If we call inversion as to e_λ I_λ ($\lambda=1, 2, 3$) and reflexion in the real axis I_4 and let

$$A=I_1I_4=I_4I_1=I_2I_3=I_3I_2,$$
$$B=I_2I_4=I_4I_2=I_1I_3=I_3I_1,$$
$$C=I_3I_4=I_4I_3=I_1I_2=I_2I_1,$$

where A, B and C are evidently the three quasi-inversions, then 1, A, B, $C\equiv AB$ form a "four group,"

$$1=A^2=B^2=(AB)^2.$$

(Klein, "Ikosaeder," pp. 5, 13; Cayley, "Mathematical Papers," Vol. II, p. 125.)



If we reflect these four points in the real axis, we get four other points, which, together with these, are the eight points of intersection of this curve with the confocal orthogonal Cartesian oval which passes through the point $\alpha+v$; that is, the curve which is the map of the line in the u-plane through $\alpha+v$ parallel to the real axis. (For convenience we denote a point on the curve by its parameter.)

4. *Focal Radii.*

If ρ_1 and ρ_2 are the focal radii of the curve, $k\rho_1+\rho_2=$constant is a well-known property of the curve, which may be derived in the following manner.

A focal vector of any point on the curve is

$$r=p(\alpha+v)-e_1,$$

and of the conjugate point is

$$\bar{r}=p(\alpha-v)-e_1.$$

Then the focal radius is

$$\rho_1=\sqrt{r\bar{r}}=\sqrt{p(\alpha+v)-e_1}\ \sqrt{p(\alpha-v)-e_1}.$$

But

$$\sqrt{pu-e_1}=\frac{\sigma_1 u}{\sigma u}.$$

(Harkness and Morley, "Treatise on the Theory of Functions," p. 307.) Hence

$$\rho_1=\frac{\sigma_1(\alpha+v)\cdot\sigma_1(\alpha-v)}{\sigma(\alpha+v)\cdot\sigma(\alpha-v)}.$$

Therefore ρ_1 is an elliptic function with the same periods $2\omega_1$, $2\omega_2$ and $2\omega_3$ of the second order, with infinities at $\pm\alpha$ and residues $\mp\frac{\sigma_1 2\alpha}{\sigma 2\alpha}$. Likewise

$$\rho_2=\frac{\sigma_2(\alpha+v)\cdot\sigma_2(\alpha-v)}{\sigma(\alpha+v)\cdot\sigma(\alpha-v)}$$

is an elliptic function with infinities at $\pm\alpha$ and residues $\mp\frac{\sigma_2 2\alpha}{\sigma 2\alpha}$. Let the distance to the triple focus $p2\alpha$ from a single focus e_1 be λ_1. Therefore

$$p2\alpha-e_1=\lambda_1,$$

$$\sqrt{p2\alpha-e_1}=\sqrt{\lambda_1}=\frac{\sigma_1 2\alpha}{\sigma 2\alpha}.$$

Hence the residues of ρ_1 at $\pm\alpha$ are $\mp\sqrt{\lambda_1}$, of ρ_2 at $\pm\alpha$ are $\mp\sqrt{\lambda_2}$. Now if two homoperiodic elliptic functions, f_1u and f_2u, of the second order have the same infinitudes, c and c', but different residues, r_1 and r_2, at an infinity c, then

$$\begin{vmatrix} f_1u & r_1 \\ f_2u & r_2 \end{vmatrix}=\text{constant}.$$

(Harkness and Morley, "Introduction to Analytic Functions," p. 259, II.) Therefore

$$\begin{vmatrix} \rho_1 & -\sqrt{\lambda_1} \\ \rho_2 & -\sqrt{\lambda_2} \end{vmatrix}=k;$$

that is, there is a linear relation between the focal radii. We proceed to find k.

$$k = \sqrt{\lambda_1}\sqrt{p(\alpha+v)-e_2}\sqrt{p(\alpha-v)-e_2} - \sqrt{\lambda_2}\sqrt{p(\alpha+v)-e_1}\sqrt{p(\alpha-v)-e_1}.$$

Or expanding,

$$k = \sqrt{\lambda_1}\sqrt{p2\alpha-e_2+p'2\alpha(v-\alpha)\ldots.}\sqrt{\frac{1}{(\alpha-v)^2}\ldots.-e_2}$$

$$-\sqrt{\lambda_2}\sqrt{p2\alpha-e_1+p'2\alpha(v-\alpha)\ldots.}\sqrt{\frac{1}{(\alpha-v)^2}\ldots.-e_1}$$

$$=\frac{\sqrt{\lambda_1}\sqrt{\lambda_2}}{(\alpha-v)}\left[1+\frac{1}{2}p'2\alpha(v-\alpha)+P(v-\alpha)\ldots.\right]$$

$$-\frac{\sqrt{\lambda_2}\sqrt{\lambda_1}}{(\alpha-v)}\left[1+\frac{1}{2}p'2\alpha(v-\alpha)+P(v-\alpha)\ldots.\right].$$

Therefore as v approaches α

$$k=\frac{1}{2}\sqrt{\lambda_1}\sqrt{\lambda_2}\left[\frac{p'2\alpha}{\lambda_1}-\frac{p'2\alpha}{\lambda_2}\right],$$

or since $p'2\alpha=-2\sqrt{\lambda_1\lambda_2\lambda_3}$,

$$k=\lambda_1\sqrt{\lambda_3}-\lambda_2\sqrt{\lambda_3}.$$

Hence

$$\sqrt{\lambda_1}\rho_2-\sqrt{\lambda_2}\rho_1=\sqrt{\lambda_3}(\lambda_1-\lambda_2),$$

or, more generally,

$$\sqrt{\lambda_\nu}\rho_\mu-\sqrt{\lambda_\mu}\rho_\nu=\sqrt{\lambda_\lambda}(\lambda_\nu-\lambda_\mu).$$

Two of three such equations give

$$\sqrt{\lambda_1}\sqrt{\lambda_2}\rho_3-\lambda_1\lambda_2=\sqrt{\lambda_2}\sqrt{\lambda_3}\rho_1-\lambda_2\lambda_3=\sqrt{\lambda_3}\sqrt{\lambda_1}\rho_2-\lambda_3\lambda_1.$$

(Compare with Greenhill's result, "Elliptic Functions," Arts. 248 and 249.) From these equations, too, we obtain Panton's form (Math. Quest. 2622, *Ed. Times*, t. XI, p. 56).

$$\sqrt{\lambda_1}(\lambda_2-\lambda_3)\rho_1+\sqrt{\lambda_2}(\lambda_3-\lambda_1)\rho_2+\sqrt{\lambda_3}(\lambda_1-\lambda_2)\rho_3=0.$$

The bifocal equation of the Cartesian oval, as ordinarily given, is

$$l\rho_1\pm m\rho_2=n$$

(Williamson, "Differential Calculus," p. 375), in which the upper sign belongs to the equation of the inner oval and the lower to that of the outer oval.

Since for the outer oval, which maps $\alpha+v$, v varying from 0 to $2\omega_3$, each of the quantities $\rho_1, \rho_2, \sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$ is positive, our form

$$\sqrt{\lambda_2}\rho_1-\sqrt{\lambda_1}\rho_2=\sqrt{\lambda_3}(\lambda_2-\lambda_1)$$

is consistent with that given above. That it holds also for the inner oval may be explained in either of two ways:

(a) We may consider the inner oval as the map of $\alpha'+v$, where $\alpha'=\alpha+\omega_1$. In that case ρ_1, ρ_2 and $\sqrt{\lambda_2}=\frac{\sigma_2 2\alpha}{\sigma 2\alpha}$ remain positive, but $\sqrt{\lambda_1}=\frac{\sigma_1 2\alpha}{\sigma 2\alpha}$ has passed a zero (ω_1) of $\sigma_1 2\alpha$ and is therefore negative. Hence the equation agrees in sign with the one usually given.

(b) We may consider α as fixed and take as equations of the inner oval

$$x=p(\alpha+\omega_1+v)$$

and

$$\bar{x}=p(\alpha-\omega_1-v),$$

in which case $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$ are unchanged and must be taken positively. To study the signs of ρ_1 and ρ_2, let us look at the limiting case when v approaches zero. Then

$$\rho_1=\frac{\sigma_1(\alpha+\omega_1)\cdot\sigma_1(\alpha-\omega_1)}{\sigma(\alpha+\omega_1)\cdot\sigma(\alpha-\omega_1)}$$

is positive, since $\sigma_1(\alpha+\omega_1)$ is negative, $\sigma_1(\alpha-\omega_1)$ positive, $\sigma(\alpha+\omega_1)$ positive and $\sigma(\alpha-\omega_1)$ negative. (Tannery and Molk, "Théorie des Fonctions Elliptiques," t. 1, p. 201.) But

$$\rho_2=\frac{\sigma_2(\alpha+\omega_1)\sigma_2(\alpha-\omega_1)}{\sigma(\alpha+\omega_1)\sigma(\alpha-\omega_1)}$$

is negative, as $\sigma_2(\alpha+\omega_1)$ and $\sigma_2(\alpha-\omega_1)$ are both positive. Therefore our form of the equation holds equally well for the inner oval.

Similarly it may be shown that each of the expressions

$$\sqrt{\lambda_\lambda}\sqrt{\lambda_\mu}\,\rho_\nu-\lambda_\lambda\lambda_\mu=\frac{1}{2}[\rho^2-(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)],$$

where ρ is the distance of any point on the curve from the triple focus. For

$$\rho^2=[p(\alpha+v)-p2\alpha][p(\alpha-v)-p2\alpha],$$

which is an elliptic function of the second order with infinities at $\pm\alpha$ and residues $\pm p'2\alpha$. Hence

$$\begin{vmatrix} \rho_1 & -\sqrt{\lambda_1} \\ \rho^2 & p'2\alpha \end{vmatrix}=k,$$

or

$$\rho_1 p'2\alpha+\rho^2\sqrt{\lambda_1}=k,$$

or

$$k=p'2\alpha\sqrt{p(\alpha+v)-e_1}\,\sqrt{p(\alpha-v)-e_1}+\sqrt{\lambda_1}[p(\alpha+v)-p2\alpha\cdot p(\alpha-v)-p2\alpha].$$

Taking, as before, the limiting value as v approaches α, we find

$$k=-\frac{\sqrt{\lambda_1}}{2}\frac{p'^2 2\alpha}{\lambda_1}+\sqrt{\lambda_1}\frac{p''2\alpha}{2}.$$

But since $p'^2 2\alpha = 4(p2\alpha - e_1)(p2\alpha - e_2)(p2\alpha - e_3)$,

$$2p'2\alpha \cdot p''2\alpha = 4p'2\alpha[(p2\alpha - e_1)(p2\alpha - e_2) + (p2\alpha - e_2)(p2\alpha - e_3) + (p2\alpha - e_3)(p2\alpha - e_1)].$$

Therefore

$$p''2\alpha = 2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)$$

and

$$k = -2\sqrt{\lambda_1}\,\lambda_2\lambda_3 + \sqrt{\lambda_1}(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1).$$

Hence

$$\rho_1\sqrt{\lambda_2\lambda_3} - \lambda_2\lambda_3 = \frac{1}{2}[\rho^2 - (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)].$$

Similarly we may obtain a relation between the distances of any point on the curve from the double tangent and from a single focus. The equation of the double tangent is

$$x + \bar{x} + A = 0,$$

as may be seen from equation (3), p. 264 (where $A = p2\alpha = p(-2\alpha)$). If we denote by T the distance to a point of the curve from the double tangent, then

$$2T = p(\alpha + v) + p(\alpha - v) + A.$$

But if $u + v + w = 0$,

$$pu + pv + pw = \frac{1}{4}\left(\frac{p'u - p'v}{pu - pv}\right)^2 = (\zeta u + \zeta v + \zeta w)^2.$$

(Harkness and Morley, "Treatise on the Theory of Functions," pp. 300, 304). Therefore

$$\sqrt{2T} = \zeta(\alpha + v) + \zeta(\alpha - v) - \zeta 2\alpha = -\frac{1}{2}\left[\frac{p'(\alpha + v) - p'(\alpha - v)}{p(\alpha + v) - p(\alpha - v)}\right],$$

which is an elliptic function of the second order with infinities at $\pm\alpha$ and residues at ∓ 1. Hence

$$\begin{vmatrix} \sqrt{2T} & 1 \\ \rho_1 & \sqrt{\lambda_1} \end{vmatrix} = k,$$

or

$$\sqrt{\lambda_1}[\zeta(\alpha + v) + \zeta(\alpha - v) - \zeta 2\alpha] - \sqrt{p(\alpha + v) - e_1}\,\sqrt{p(\alpha - v) - e_1} = k.$$

Taking, again, the limit as v approaches α, we find

$$k = \frac{1}{2}\frac{p'2\alpha}{\sqrt{\lambda_1}} = -\frac{\sqrt{\lambda_1\lambda_2\lambda_3}}{\sqrt{\lambda_1}} = -\sqrt{\lambda_2\lambda_3}.$$

Hence

$$\sqrt{\lambda_1}\sqrt{2T} = \rho_1 - \sqrt{\lambda_2\lambda_3},$$

or

$$\sqrt{\lambda_\lambda}\sqrt{2T} = \rho_\lambda - \sqrt{\lambda_\mu\lambda_\nu}.$$

From these we could have obtained the formulæ derived previously on p. 269,

$$\sqrt{\lambda_1}\sqrt{\lambda_2}\,\rho_3 - \lambda_1\lambda_2 = \sqrt{\lambda_2}\sqrt{\lambda_3}\,\rho_1 - \lambda_2\lambda_3 = \sqrt{\lambda_3}\sqrt{\lambda_1}\,\rho_2 - \lambda_3\lambda_1.$$

5. *Another Definition of the Curve.*

The equation of the tangent to the curve at any point $p(\alpha+v)$ is

$$\begin{vmatrix} x & \bar{x} & 1 \\ p(\alpha+v) & p(\alpha-v) & 1 \\ p'(\alpha+v) & -p'(\alpha-v) & 0 \end{vmatrix} = 0.$$

Hence the equation of the normal at the same point is

$$\begin{vmatrix} x & \bar{x} & 1 \\ p(\alpha+v) & p(\alpha-v) & 1 \\ p'(\alpha+v) & p'(\alpha-v) & 0 \end{vmatrix} = 0.$$

To find the point N where the normal cuts the real axis, let

$$x=\bar{x}=N.$$

Hence

$$N=p(\alpha+v)-p'(\alpha+v)\left[\frac{p(\alpha+v)-p(\alpha-v)}{p'(\alpha+v)-p'(\alpha-v)}\right].$$

But, since

$$\begin{vmatrix} 1 & p(\alpha+v) & p'(\alpha+v) \\ 1 & p(\alpha-v) & p'(\alpha-v) \\ 1 & p2\alpha & -p'2\alpha \end{vmatrix} = 0$$

(Burkhardt, "Functionen Theorie," II, p. 62), or

$$p(\alpha+v)=p2\alpha+[p'2\alpha+p'(\alpha+v)]\left[\frac{p(\alpha+v)-p(\alpha-v)}{p'(\alpha+v)-p'(\alpha-v)}\right],$$

$$N=p2\alpha+p'2\alpha\left[\frac{p(\alpha+v)-p(\alpha-v)}{p'(\alpha+v)-p'(\alpha-v)}\right].$$

But from p. 271,

$$\sqrt{2T}=-\frac{1}{2}\left[\frac{p'(\alpha+v)-p'(\alpha-v)}{p(\alpha+v)-p(\alpha-v)}\right].$$

Hence

$$N=p2\alpha-\frac{p'2\alpha}{2\sqrt{2T}}=p2\alpha+\frac{\sqrt{\lambda_1\lambda_2\lambda_3}}{\sqrt{2T}},$$

or

$$N-e_1=\lambda_1+\frac{\sqrt{\lambda_1\lambda_2\lambda_3}}{\sqrt{2T}}=\frac{\sqrt{\lambda_1}(\sqrt{2\lambda_1 T}+\sqrt{\lambda_2\lambda_3})}{\sqrt{2T}}.$$

Therefore

$$N-e_1=\frac{\sqrt{\lambda_1}}{\sqrt{2T}}\rho_1$$

(see p. 271) and

$$N-p2\alpha=\frac{\sqrt{\lambda_1\lambda_2\lambda_3}}{\sqrt{2T}}.$$

Therefore

$$\frac{N-e_1}{N-p2\alpha}=\frac{\sqrt{\lambda_2\lambda_3}}{\rho_1}.$$

If we draw from $p2\alpha$ a line perpendicular to the tangent at x, and therefore parallel to the normal, cutting ρ_1 at τ_1, then

$$\frac{x\tau_1}{\rho_1}=\frac{N-p2\alpha}{N-e_1}=\frac{\sqrt{\lambda_2\lambda_3}}{\rho_1}.$$

Therefore

$$x\tau_1=\sqrt{\lambda_2\lambda_3}.$$

Similarly

$$x\tau_2=\sqrt{\lambda_3\lambda_1},$$

and

$$x\tau_3=\sqrt{\lambda_1\lambda_2}.$$

Hence we deduce the theorem:

The perpendicular from the triple focus on the tangent at a point x of a Cartesian oval meets a focal radius ρ_λ at a point τ_λ such that $| x-\tau_\lambda | =$ a constant.

Conversely, if a line, revolving about a fixed point $p2\alpha$, cut off on two other straight lines, revolving about two points e_1 and e_2 collinear with $p2\alpha$, constant distances $x\tau_1$ and $x\tau_2$, then x lies on a Cartesian oval. For by the theorem of Menelaus

$$x\tau_2\cdot\lambda_2\cdot\rho_1-x\tau_1=\rho_2-x\tau_2\cdot v\tau_1\cdot\lambda_1,$$

or, if we take the constants $x\tau_2$ and $x\tau_1$ to be $\sqrt{\lambda_3\lambda_1}$ and $\sqrt{\lambda_2\lambda_3}$, respectively,

$$\sqrt{\lambda_1\lambda_3}\cdot\lambda_2\cdot\rho_1-\sqrt{\lambda_2\lambda_3}=\rho_2-\sqrt{\lambda_1\lambda_3}\cdot\sqrt{\lambda_2\lambda_3}\cdot\lambda_1.$$

Hence we have

$$\sqrt{\lambda_2}\rho_1-\sqrt{\lambda_1}\rho_2=\sqrt{\lambda_3}(\lambda_2-\lambda_1).$$

Therefore x lies on the Cartesian oval. Therefore we may define a Cartesian oval as the locus of the vertex of a triangle with two of its sides of fixed lengths, and passing always through two fixed points, while the base revolves about a fixed point collinear with the other two.

If we had measured the constant distances from e_1 and e_2 along the focal radii, instead of from the point x, we should have had Newton's definition of the curve; namely, the locus of a point whose distances from the circumferences of two circles (with centers at e_1 and e_2) are in a constant ratio.

6. *Intersections of Circles with the Curve.*

If

$$x=p(\alpha+v)$$

and

$$\bar{x}=p(\alpha-v)$$

are the parametric equations of the Cartesian oval, and the equation of any

other curve is

$$R(x, \bar{x}) = 0,$$

the intersections of the two curves are given by the zeros v_i of

$$f(v) = R[p(\alpha+v),\ p(\alpha-v)],$$

whose infinities are $\pm\alpha$. Hence

$$\Sigma v_i \equiv 0, \tag{1}$$

from Abel's theorem for integrals of the first kind. (Picard, t. II, p. 396.)

Let $z=\alpha+v$. Then

$$\phi(z) - k = R[pz, p(2\alpha - z)] = A\Pi \frac{\sigma(z-z_\nu)}{\sigma(z-c_\nu)},$$

where z_ν is a zero of $\phi(z)-k$ and c_ν is an infinity, or pole, of $\phi(z)-k$. Accordingly

$$\mathcal{D} \log R = \Sigma\zeta(z-z_\nu) - \Sigma\zeta(z-c_\nu).$$

But

$$\mathcal{D} \log R = \frac{\phi'(z)}{\phi(z)-k}.$$

Hence, when $z=0$,

$$\frac{\phi'(0)}{\phi(0)-k} = -\Sigma\zeta z_\nu + \Sigma\zeta c_\nu, \tag{2}$$

which is a form of Abel's theorem for elliptic integrals of the second class (Harkness and Morley, "Treatise on the Theory of Functions," p. 458) since

$$\zeta u = -\int pu\,du = \int \frac{pu\,d(pu)}{\sqrt{4p^3u - g_2 pu - g_3}}.$$

Let the curve be a circle with center c. Its equation is

$$x\bar{x} - \bar{c}x - c\bar{x} + \lambda = 0,$$

and its intersections with our curve are given by the zeros of

$$p(\alpha+v)p(\alpha-v) - \bar{c}p(\alpha+v) - cp(\alpha-v) + \lambda = 0,$$

an elliptic function of the fourth order with double infinities at $\pm\alpha$. Therefore

$$v_1 + v_2 + v_3 + v_4 = 0,$$

or the circle meets the curve in four points, the sum of whose parameters is zero.

We proceed to apply equation (2) to find a second relation,

$$\phi(z) - k = pz \cdot p(2\alpha - z) - \bar{c}pz - cp(2\alpha - z) + \lambda - k = A\Pi \frac{\sigma(z-z_\nu)}{\sigma^2 z \sigma^2 (2\alpha - z)}.$$

Therefore

$$\frac{\phi' z}{\phi(z)-k} = \Sigma\zeta(z-z_\nu) - 2\zeta z - 2\zeta(z-2\alpha),$$

$$pz = \frac{1}{z^2} + 3S_4 z^2 + \ldots.,$$

$$p(z-2\alpha) = p2\alpha - zp'2\alpha + \frac{z^2}{2}p''2\alpha - \ldots.,$$

$$\phi(z)-k = \frac{1}{z^2}p2\alpha - \frac{1}{z}p'2\alpha + \frac{1}{2}p''2\alpha - \ldots. - \bar{c}\left(\frac{1}{z^2}+\ldots.\right) - cp2\alpha \ldots. + \lambda - k,$$

$$\frac{\phi' z}{\phi(z)-k} = \frac{-\frac{2}{z^3}(p2\alpha-\bar{c}) + \frac{1}{z^2}p'2\alpha + P_0(z)\ldots.}{\frac{1}{z^2}(p2\alpha-\bar{c}) - \frac{1}{z}p'2\alpha + \frac{1}{2}p''2\alpha - cp2\alpha + \lambda - k\ldots.}$$

$$= \left[-\frac{2}{z} + \frac{p'2\alpha}{p2\alpha-\bar{c}} + \ldots.\right]\left[1 + \frac{zp'2\alpha}{p2\alpha-\bar{c}} + \ldots.\right].$$

Therefore

$$\Sigma\zeta(z-z_\nu) - \frac{2}{z}\ldots. - 2\zeta(z-2\alpha) = -\frac{2}{z} - \frac{p'2\alpha}{p2\alpha-\bar{c}} - P(z),$$

whence, if $z=0$,

$$\overset{4}{\Sigma}\zeta z_\nu - 2\zeta 2\alpha = \frac{p'2\alpha}{p2\alpha-\bar{c}},$$

or, substituting for z its value $\alpha+v$,

$$\overset{4}{\Sigma}\zeta(\alpha+v) = 2\zeta 2\alpha + \frac{p'2\alpha}{p2\alpha-\bar{c}}.$$

Similarly

$$\overset{4}{\Sigma}\zeta(\alpha-v) = 2\zeta 2\alpha + \frac{p'2\alpha}{p2\alpha-c}.$$

These equations give the center of the circle. If c is at infinity, the circle is a line and we have, for the intersections of the line with a Cartesian oval,

$$\overset{4}{\Sigma}\zeta(\alpha+v) = 2\zeta 2\alpha$$

and

$$\overset{4}{\Sigma}\zeta(\alpha-v) = 2\zeta 2\alpha.$$

Therefore

$$\Sigma[\zeta(\alpha+v) + \zeta(\alpha-v) - \zeta 2\alpha] = 0.$$

But from p. 271 we have

$$\zeta(\alpha+v) + \zeta(\alpha-v) - 2\alpha = \sqrt{2T},$$

where T is the distance from any point of the curve to the double tangent.

Therefore, if a straight line be drawn, cutting a Cartesian oval in four points, the sum of the square roots of twice the distances of these points from the double tangent is zero.

But we found on p. 271 that

$$\sqrt{2T} = \frac{\rho_1 - \sqrt{\lambda_2\lambda_3}}{\lambda_1},$$

whence, if we denote the distance of any of the points $\alpha+v_\nu$ from the focus e_1 by $\rho_{1\nu}$, we have

$$\Sigma \frac{\rho_{1\nu} - \sqrt{\lambda_2\lambda_3}}{\sqrt{\lambda_1}} = 0,$$

or

$$\Sigma\rho_{1\nu} = 4\sqrt{\lambda_2\lambda_3}.$$

Therefore the sum of the distances of the points in which a straight line cuts a Cartesian oval from any focus is constant, a well-known property.

It follows as a corollary to this that the points of contact of the double tangent lie on a circle with center e_1 and $\sqrt{\lambda_2\lambda_3}$.

7. *Bitangent Circles.*

In the case of bitangent circles the equation

$$\Sigma v_i \equiv 0$$

reduces to

$$2v_1 + 2v_2 = 0.$$

Four cases are to be distinguished, when $v_2 = -v_1$, and when $v_2 = -v_1 + \omega_\lambda$ and the coincident point $v_2 = -v - \omega_\lambda$ $(\lambda = 1, 2, 3)$. Hence there are four systems of bitangent circles.

In the first case,

$$\Sigma\zeta(\alpha+v) = 2\zeta 2\alpha + \frac{p'2\alpha}{p2\alpha - \bar{c}}$$

becomes

$$\zeta(\alpha+v) + \zeta(\alpha-v) - \zeta 2\alpha = \frac{\frac{1}{2}p'2\alpha}{p2\alpha - \bar{c}},$$

and its conjugate

$$\zeta(\alpha-v) + \zeta(\alpha+v) - \zeta 2\alpha = \frac{\frac{1}{2}p'2\alpha}{p2\alpha - c}.$$

Therefore $c = \bar{c}$, or the center lies on the real axis, as is evident geometrically from the symmetry of the curve.

*Among the circles of this system are the bitangent line, two circles with fourfold contact with the inner oval, one at $\alpha+\omega_1$ and one at $\alpha+\omega_2$, and two with the outer oval, one at α and the other at $\alpha+\omega_3$, and three point circles with imaginary contact, the three single foci. The fourth point circle is at infinity.

In the other case we have

$$2\zeta(\alpha+v)+\zeta(\alpha-v+\omega_\lambda)+\zeta(\alpha-v-\omega_\lambda)-2\zeta 2\alpha=\frac{p'2\alpha}{p2\alpha-\bar{c}},$$

or

$$2\zeta(\alpha+v)+\zeta(\alpha-v+\omega_\lambda)+\zeta(\alpha-v+\omega_\lambda-2\omega_\lambda)-2\zeta 2\alpha=\frac{p'2\alpha}{p2\alpha-\bar{c}},$$

or

$$\zeta(\alpha+v)+\zeta(\alpha-v+\omega_\lambda)-\zeta\omega_\lambda-\zeta 2\alpha=\frac{\frac{1}{2}p'2\alpha}{p2\alpha-\bar{c}}.$$

Likewise

$$\zeta(\alpha-v)+\zeta(\alpha+v-\omega_\lambda)+\zeta\omega_\lambda-\zeta 2\alpha=\frac{\frac{1}{2}p'2\alpha}{p2\alpha-c}.$$

The following formula is one given by Professor Morley (*Proc. London Math. Soc.*, Ser. 2, Vol. IV, p. 390):

$$\zeta(a)+\zeta(b)+\zeta(c)-\zeta(a+b+c)=\frac{\sigma(b+c)\sigma(a+c)\sigma(a+b)}{\sigma a\cdot\sigma b\cdot\sigma c\cdot\sigma(a+b+c)}.$$

To prove it, suppose a to be variable. Then

$$\zeta(a)+\zeta(b)+\zeta(c)-\zeta(a+b+c)$$

---

*The relation of these bitangent circles is well brought out in a study of sphero-conics.

Suppose a sphere to be intersected by a quadric cone with the vertex at the center. Through this intersection pass a pencil of quadrics, among which are four degenerate quadrics or cones, three besides the cone already mentioned. These three cones have their vertices at infinity, or are cylinders whose axes are mutually perpendicular. From one of the foci of the sphero-conics project the figure on the plane of the great circle of which that focus is the pole.

The two ovals of the sphero-conic project into the two ovals of the Cartesian; the great circle, on which the four foci lie, into the real axis with the three foci lying on it and the fourth at infinity. Through each element of the four cones passes a tangent plane which intersects the sphere in a circle bitangent to one of the sphero-conics (or both) with real or imaginary contacts. These project into bitangent circles of the Cartesian and we get the four systems corresponding to the four cones. The sphere and each of the other quadrics have four common tangent planes (imaginary in every case but one), and since these planes cut out the point circles on the sphere, they touch it at the four real foci of the two sphero-conics which we have already said project into the four foci of the Cartesian oval (one being at infinity). The circle cut out by the other tangent plane through the focus, from which we are making the projection, to the quadric which has real common tangent planes with the sphere, projects into a straight line, namely, the bitangent line.

The tangent planes of the cone with vertex at the center, and also of the cylinder whose elements are parallel to the axis of this cone, determine circles whose projections touch different branches of the Cartesian oval. In the former case, the circles enclose the inner oval; in the latter, they lie between the two ovals. (See Darboux, "Sur une Classe Remarquable de Courbes et de Surfaces.")

is an elliptic function with two infinities, $a=0$ and $a=-b-c$, and two zeros, $a=-b$ and $a=-c$. Therefore it is equal to

$$k\frac{\sigma(a+b)\sigma(a+c)}{\sigma a\cdot\sigma(a+b+c)}.$$

Multiplying both sides of the equation by a, and taking the limit when $a=0$, we get

$$k=\frac{\sigma(b+c)}{\sigma b\cdot\sigma c}.$$

Therefore

$$\zeta(a)+\zeta(b)+\zeta(c)-\zeta(a+b+c)=\frac{\sigma(a+b)\sigma(b+c)\sigma(c+a)}{\sigma a\cdot\sigma b\cdot\sigma c\cdot\sigma(a+b+c)}.$$

Hence

$$\zeta(\alpha+v)+\zeta(\alpha-v+\omega_\lambda)-\zeta(\omega_\lambda)-\zeta 2\alpha$$
$$=-\frac{\sigma(2\alpha+\omega_\lambda)\sigma(\alpha+v-\omega_\lambda)\sigma(\alpha-v)}{\sigma(\alpha+v)\sigma(\alpha-v+\omega_\lambda)\sigma\omega_\lambda\cdot\sigma 2\alpha)}=\frac{\frac{1}{2}p'2\alpha}{p2\alpha-\bar{c}},$$

and in the same way

$$\zeta(\alpha-v)+\zeta(\alpha+v-\omega_\lambda)+\zeta\omega_\lambda-\zeta 2\alpha$$
$$=\frac{\sigma(2\alpha-\omega_\lambda)\sigma(\alpha-v+\omega_\lambda)\sigma(\alpha+v)}{\sigma(\alpha-v)\cdot\sigma(\alpha+v-\omega_\lambda)\sigma\omega_\lambda\cdot\sigma 2\alpha}=\frac{\frac{1}{2}p'2\alpha}{p2\alpha-c}.$$

Multiplying the two equations, we get

$$-\frac{\sigma(2\alpha+\omega_\lambda)\sigma(2\alpha-\omega_\lambda)}{\sigma^2\omega_\lambda\cdot\sigma^2 2\alpha}=\frac{\frac{1}{4}p'^2 2\alpha}{p2\alpha-\bar{c}\cdot p2\alpha-c}.$$

But the left-hand member of this equation is equal to $p2\alpha-e_\lambda=\lambda_\lambda$. (Harkness and Morley, "Introduction to Analytic Functions," p. 104.) Therefore

$$\lambda_\lambda=\frac{\frac{1}{4}p'^2 2\alpha}{p2\alpha-\bar{c}\cdot p2\alpha-c}=\frac{\lambda_\lambda\lambda_\mu\lambda_\nu}{p2\alpha-\bar{c}\cdot p2\alpha-c};$$

hence

$$(\bar{c}-p2\alpha)(c-p2\alpha)=\lambda_\mu\lambda_\nu.$$

Therefore the locus of the center of each of these three systems of bitangent circles is a circle whose center is $p2\alpha$, the triple focus, and whose radius is $\sqrt{\lambda_\mu\lambda_\nu}$. For example, the locus of the center of the bitangent circle touching the curve at $\alpha+v$ and $\alpha-v+\omega_1$ is a circle with a triple focus as center and $\sqrt{\lambda_2\lambda_3}$ as radius.

That this set of circles is orthogonal to a circle, with radius equal to

$$\sqrt{e_2-e_1}\cdot\sqrt{e_3-e_1}$$

and center e_1, follows from the fact developed on p. 265, that $p(\alpha+v)$ and $p(\alpha-v+\omega_1)$ are inverse points as to this circle, and the theorem of elementary

geometry, that the product of a secant to a circle from a point without and its external segment is equal to the square of the tangent from the same point.

Hence we get the theorem of Casey in the *Transactions of the Royal Irish Academy*, 1669 (Williamson, "Differential Calculus," p. 383): "If a circle cut a given circle orthogonally, while its center moves along another given circle, its envelope is a Cartesian oval."

Among the circles of this bitangent system, orthogonal to the circle with e_1 as center, are the circle touching the oval at α and $\alpha+\omega_1$ and the one touching it at $\alpha+\omega_2$ and $\alpha+\omega_3$. The center of the first circle is $\frac{1}{2}(\alpha+\alpha+\omega_1)$ and of the second is $\frac{1}{2}(\alpha+\omega_2+\alpha+\omega_3)$. Therefore the center of the circle of centers is

$$\frac{1}{4}(\alpha+\alpha+\omega_1+\alpha+\omega_2+\alpha+\omega_3).$$

But we have just proved that this point is the triple focus. Hence the triple focus is the centroid of the four points in which the real axis cuts the Cartesian oval, which gives a simple method for the construction of the triple focus.

As a check for this statement, we proved that, if a straight line cuts the curve in points whose parameters are v_r,

$$\Sigma\zeta(\alpha+v_r)=2\zeta 2\alpha.$$

If the line is the real axis we have

$$\zeta\alpha+\zeta(\alpha+\omega_1)+\zeta(\alpha+\omega_2)+\zeta(\alpha+\omega_3)=2\zeta 2\alpha.$$

Hence, by differentiation,

$$p\alpha+p(\alpha+\omega_1)+p(\alpha+\omega_2)+p(\alpha+\omega_3)=4p2\alpha,$$

or $p2\alpha$ is the centroid of these four points.

Among the circles of the system, with centers on the circle with radius $\sqrt{\lambda_1\lambda_2}$ orthogonal to the circle with center e_3 and radius $\sqrt{e_2-e_3}\cdot\sqrt{e_1-e_3}$, is the circle touching the curve at α and $\alpha+\omega_3$ and one touching at $\alpha+\omega_1$ and $\alpha+\omega_2$, also two with fourfold contact to the outer oval at the points $\alpha\pm\frac{1}{2}\omega_3$, and two to the inner oval at $\alpha+\omega_1\pm\frac{1}{2}\omega_3$. These are the points of contact of the tangents from e_3 to the curve; for since $p(\alpha-v+\omega_3)$ is the inverse of $p(\alpha+v)$ as to the circle with center e_3, mentioned above, if $v=\frac{1}{2}\omega_3$ we have

$$p\left(\alpha-\tfrac{1}{2}\omega_3+\omega_3\right)=p\left(\alpha+\tfrac{1}{2}\omega_3\right)$$

the inverse of $p\left(\alpha+\frac{1}{2}\omega_3\right)$; that is, the inverse of itself. Hence the points of contact of the bitangent circle come together where the orthogonal circle with center e_3 cuts the curve, and the focal vector e_3 is tangent at this point. A similar proof holds for the points $\alpha+\omega_1\pm\frac{1}{2}\omega_3$. The circle with e_3 as center cuts both ovals orthogonally. It, counted twice, is the Cartesian oval corresponding to $\alpha=\frac{1}{2}\omega_1$ in our system. Hence it cuts every oval of the system orthogonally.

The last system of bitangent circles is the one with centers on the circle with center $p2\alpha$ and radius $\sqrt{\lambda_1\lambda_3}$. Here we have one circle touching the outer oval at α and the inner at $\alpha+\omega_2$, and another touching the outer at $\alpha+\omega_3$ and the inner at $\alpha+\omega_1$.

8. *Relation of Tangents to Bitangent Circles.*

If $x\bar{x}-\bar{c}x-c\bar{x}+\lambda=0$ is the equation of a circle, and x_1 and $\bar{x}_1$ are the coordinates of a point, and the tangents are drawn from the point to the circle, the square of the distance from the point to the circle measured along the tangent is $x_1\bar{x}_1-\bar{c}x_1-c\bar{x}_1+\lambda$. Hence, if L be the distance from a point on the curve to a circle,

$$L=\sqrt{p(\alpha+v)\cdot p(\alpha-v)-\bar{c}p(\alpha+v)-cp(\alpha-v)+\lambda}=\sqrt{A^2\Pi\frac{\sigma(\alpha+v-v_\nu)}{\sigma^2(\alpha+v)\sigma^2(\alpha-v)}}.$$

If the circle is a bitangent circle,

$$L=\sqrt{A^2\frac{\sigma^2(\alpha+v+v_1)\sigma^2(\alpha+v-v_1)}{\sigma^2(\alpha+v)\sigma^2(\alpha-v)}}=A\frac{\sigma(\alpha+v+v_1)\sigma(\alpha+v-v_1)}{\sigma(\alpha+v)\sigma(\alpha-v)}.$$

Similarly, if L' is the distance from the same point to a second bitangent circle of the same system,

$$L'=A'\frac{\sigma(\alpha+v+v_1')\sigma(\alpha+v-v_1')}{\sigma(\alpha+v)\sigma(\alpha-v)}.$$

Therefore L and L' are elliptic functions of the second order and the same poles, and if r_1 is the residue of L at $v=\alpha$ and r_1' is the residue of L' at the same pole, then

$$r_1'L-r_1L'=k.$$

Therefore, between tangents to any two bitangent circles of the same system of a Cartesian oval there is a linear relation,

$$r_1'L-r_1L'-k=0.$$

BALTIMORE, *February*, 1911.

The Indices of Permutations and the Derivation therefrom of Functions of a Single Variable Associated with the Permutations of any Assemblage of Objects.

By Major P. A. MacMahon.

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Introduction.

This paper arises out of the postscript to the "Memoir on the Theory of the Partitions of Numbers," Part V. *Phil. Trans. R. S.*, Vol. CCXI, A 473, January, 1911. In that paper I derived algebraic functions of a single variable from *certain* permutations of any assemblage of objects. The permutations were restricted to be such as fulfilled certain conditions, and by reason of their intimate association with lattices or gratings of nodes complete or incomplete they were termed lattice-permutations. The discovery of the functions led to the solution of the problem of two-demensional partition at the points of a Ferrers-Sylvester graph such as is figured below,



where we have a system of nodes constituting the graph of a partition of a number. In two-dimensional partition we have a *certain* generalization of ordinary or one-dimensional partition. In the latter the partition is constituted by numbers placed at the nodes of a single row in descending order of magnitude, the sum of the numbers being equal to the number which is partitioned. In the former numbers are placed at the nodes of the two-dimensional system in such wise that the numbers are in descending order of

magnitude alike in *each row and in each column* of the system. The system of nodes is supposed to be given, and when the sum of the numbers placed at the nodes is equal to the partitioned number, the system of numbers in two dimensions is a two-dimensional partition of the given number *quâ* the given system of nodes. It will be observed that the system of nodes is itself defined by a one-dimensional partition of some number and the enumerative problem that was solved may be stated as the determination of the number of partitions of any number *quâ* a system of nodes defined by any one-dimensional partition of any number. The generalization that was made was brought to a conclusion in a subsequent paper, "Memoir on the Theory of the Partitions of Numbers," Part VI. *Phil. Trans. R. S.*, Vol. CCXI, A 479. It depends upon the relative position of the parts and may be made to depend upon linear Diophantine analysis. The problem is certainly very simply stated in Diophantine terms, but the actual solution reached was obtained from generating functions derived from *permutations* in association with the given system of nodes. The permutations were made to lead to algebraic functions through the agency of numbers derived from the permutations. In the postscript alluded to there was an adumbration of a similar theory connected with the entire set of permutations of any assemblage of objects. The present paper opens the discussion of this question and formally introduces various kinds of indices of permutations. So much of the matter that the postscript discussed as involved a limit to the magnitude of the parts is not broached here except in one place in an incidental manner. The theory of the greater and lesser indices is complete. That of the equal index is incomplete so far as the algebraic functions are concerned, but the theory of the average values is complete in this case also. As regards the theory of the major and minor indices, the discussion is complete when the assemblage only contains three *kinds* of objects and the *crude* form of generating function has been obtained for the most general assemblage. The theory of average values is complete in this case also. I hope, if permitted, to continue the discussion on a future occasion.

A communication on the subject-matter of this paper was made to the Mathematical Subsection of Section A of the British Association for the Advancement of Science at the Dundee meeting, September, 1912. P. A. M.

§ 1. *Indices of the First Genus.*

1. The assemblage of objects I take to be

$$\alpha^i\beta^j\gamma^k\ldots. \quad \text{or} \quad \alpha_1^{i_1}\alpha_2^{i_2}\alpha_3^{i_3}\ldots.$$

indifferently.

A contact in a permutation of these letters, which is (say) $\alpha_s\alpha_t$, is called a major contact, an equal contact or a minor contact, according as $s>$, $=$, or $<t$.

The first definitions have reference to the major contacts of a permutation.

2. *The Greater Index of a Permutation.* Of $\alpha^1\beta^2\gamma^3$ let any permutation be (say)

$$\beta\alpha\alpha\alpha\gamma\gamma\beta\alpha\gamma.$$

Whenever a letter is the left-hand member of a major contact, I write under it a number which shows how many places it is from the left of the permutation. Thus if the s-th letter stands before a letter prior to it in alphabetical (or numerical) order, I place under it the number s. In this way we obtain

$$\begin{array}{l}\beta\alpha\alpha\alpha\gamma\gamma\beta\alpha\gamma.\\ 1\quad\ 6\,7\end{array}$$

We add up the numbers so placed and obtain $1+6+7=14$, a number which I call the "greater index" of the permutation. Similarly, if, in any permutation of any assemblage of letters or of ordered objects, the p_1-th, p_2-th, p_3-th, ... letters are the left-hand members of major contacts, we have the definition

$$\text{Greater index} = p_1+p_2+p_3+\ldots.=p.$$

If, in so forming the number p, we have to add m numbers or, in other words, if the permutation possesses m major contacts, I speak of the permutation as being of "Class m" *quâ* major contacts.

3. *The Equal Index of a Permutation.* If, in any permutation, the q_1-th, q_2-th, q_3-th, ... letters immediately precede letters identical with themselves, so that such letters are the left-hand members of equal contacts, I make the definition

$$\text{Equal index} = q_1+q_2+q_3+\ldots.=q.$$

Thus from

$$\begin{array}{l}\beta\alpha\alpha\alpha\gamma\gamma\beta\alpha\gamma\\ 2\,3\ \ 5\end{array}$$

we obtain the equal index $2+3+5=10$.

If, in so forming the number q, we have to add m numbers, or if, in other words, the permutation possesses m equal contacts, I speak of the permutation as being of "Class m" *quâ* equal contacts.

4. *The Lesser Index of a Permutation.* If, in any permutation, the r_1-th, r_2-th, r_3-th, ... letters immediately precede letters which are later in alphabetical order, so that such letters are the left-hand members of minor contacts, I make the definition

$$\text{Lesser index} = r_1+r_2+r_3+\ldots.=r.$$

If in so forming the number r we have to add m numbers, or, in other words, if the permutation possesses m minor contacts, I speak of the permutation as being of "Class m" *quâ* minor contacts.

5. *The Greater and Equal Index.* This refers to letters which immediately precede letters which are *not later* in alphabetical order and is equal to the sum of the greater and equal indices, or to $p+q$.

In the same manner we may have to consider other combinations of the indices p, q, r.

6. *Evaluation of Σx^p for Permutations of Any Given Assemblage.* In the case of the assemblage $\alpha^i\beta^j\gamma^k\ldots$. I wish to give a formal proof of the result

$$\Sigma x^p = \frac{(1-x)(1-x^2)\ldots(1-x^{i+j+k+\ldots})}{(1-x)(1-x^2)\ldots(1-x^i)\cdot(1-x)(1-x^2)\ldots(1-x^j)\cdot(1-x)(1-x^2)\ldots(1-x^k)\cdot\ldots}$$

of which an indication only was given in the postscript to the paper, to which reference has been made in the preamble above.

In Cayley's notation I write

$$1-x^s=(\mathbf{s}),$$

so that the result to be established is written

$$\Sigma x^p = \frac{(\mathbf{1})(\mathbf{2})\ldots(\mathbf{i+j+k+}\ldots)}{(\mathbf{1})(\mathbf{2})\ldots(\mathbf{i})\cdot(\mathbf{1})(\mathbf{2})\ldots(\mathbf{j})\cdot(\mathbf{1})(\mathbf{2})\ldots(\mathbf{k})\cdot\ldots}.$$

Consider a system of rows of nodes, containing i, j, k, ... nodes respectively and at these nodes numbers to be placed in such wise that there is a descending order of magnitude in each row. The number zero is not excluded from being placed at a node and there is no condition whatever in regard to the numbers which appear in the columns of nodes. If it be asked,—In how many ways can numbers be so placed that their sum is w?—it is readily seen that the answer is supplied by the coefficient of x^w in the ascending expansion of the algebraic fraction

$$\frac{1}{(\mathbf{1})(\mathbf{2})\ldots(\mathbf{i})\cdot(\mathbf{1})(\mathbf{2})\ldots(\mathbf{j})\cdot(\mathbf{1})(\mathbf{2})\ldots(\mathbf{k})\cdot\ldots};$$

because if w_i, w_j, w_k, ... be any numbers such that $w_i+w_j+w_k+\ldots=w$, the members w_i, w_j, w_k, ... can be partitioned in the 1st, 2d, 3d, ... rows in numbers of ways indicated by the generating functions

$$\frac{1}{(\mathbf{1})(\mathbf{2})\ldots(\mathbf{i})}, \quad \frac{1}{(\mathbf{1})(\mathbf{2})\ldots(\mathbf{j})}, \quad \frac{1}{(\mathbf{1})(\mathbf{2})\ldots(\mathbf{k})}, \quad \ldots.$$

respectively.

The present proof consists in showing that the same generating function

can be exhibited as the quotient of Σx^p, p being the greater index of a permutation of $\alpha^i \beta^j \gamma^k \ldots$, by the function

$$(1)\,(2)\ldots.(i+j+k+\ldots).$$

Consider a simple particular case, $i=3$, $j=2$, to which belongs the system of nodes

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ & \cdot & \cdot \end{array}$$

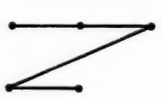
and suppose numbers to be placed at the nodes, viz.,

$$\begin{array}{ccc} a_1 & a_2 & a_3 \\ & b_1 & b_2 \end{array}$$

in such wise that $a_1 \geq a_2 \geq a_3$, $b_1 \geq b_2$. Place also the quantities α, β of the assemblage $\alpha^3 \beta^2$ at the nodes in the manner

$$\begin{array}{ccc} \alpha & \alpha & \alpha \\ & \beta & \beta \end{array}$$

We are about to enquire into the number of ways of placing the numbers at the nodes by arranging them in all possible descending orders, and then enumerating each order separately. We must determine the lines of route through the system of nodes which may give a descending order of magnitude of the numbers a_1, a_2, a_3, b_1, b_2 collectively. These numbers may be in any permutation so long as the two orders a_1, a_2, a_3; b_1, b_2 are in evidence. Thus through the system of nodes we may have the lines of route


$\alpha\alpha\alpha\beta\beta$


$\alpha\alpha\beta\alpha\beta$


$\alpha\beta\alpha\alpha\beta$


$\beta\alpha\alpha\alpha\beta$

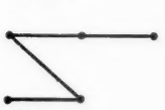

$\alpha\alpha\beta\beta\alpha$


$\alpha\beta\alpha\beta\alpha$


$\beta\alpha\alpha\beta\alpha$


$\alpha\beta\beta\alpha\alpha$


$\beta\alpha\beta\alpha\alpha$


$\beta\beta\alpha\alpha\alpha$

Beneath each line of route is placed the corresponding permutation of $\alpha^3 \beta^2$. There is obviously one line of route for each permutation.

If we take any one of these lines of route, we may lay out five numbers in descending order of magnitude along it. In the general case each line of route is $i+j+k+\ldots$ nodes long, and we may lay out $i+j+k+\ldots$ numbers in descending order along the one selected. The number of ways in which this can be done is given by the coefficient of x^w in

$$\{(1)\,(2)\ldots.(i+j+k+\ldots)\}^{-1};$$

and if we so proceeded, in the case of each permutation of $\alpha^i\beta^j\gamma^k\ldots$, we would obtain this coefficient multiplied by the number of permutations of $\alpha^i\beta^j\gamma^k\ldots$. We would not, however, thus obtain the required generating function, because, in general, many systems of numbers may be laid out along more than one line of route. For example, the arrangement

$$\begin{matrix} 4\,2\,2 \\ 3\,2 \end{matrix}$$

occurs along each of the lines of route



For the enumeration before us we must arrange that each line of route is utilized without overlapping. This is accomplished by imposing certain conditions upon the successions of numbers which appertain to the various lines of route.

Observe the subjoined scheme of descending orders for the particular case under examination:

1.	$\alpha\alpha\alpha\beta\beta,$	$a_1 \geq a_2 \geq a_3 \geq b_1 \geq b_2,$
2.	$\alpha\alpha\beta\alpha\beta,$	$a_1 \geq a_2 \geq b_1 > a_3 \geq b_2,$
3.	$\alpha\beta\alpha\alpha\beta,$	$a_1 \geq b_1 > a_2 \geq a_3 \geq b_2,$
4.	$\beta\alpha\alpha\alpha\beta,$	$b_1 > a_1 \geq a_2 \geq a_3 \geq b_2,$
5.	$\alpha\alpha\beta\beta\alpha,$	$a_1 \geq a_2 \geq b_1 \geq b_2 > a_3,$
6.	$\alpha\beta\alpha\beta\alpha,$	$a_1 \geq b_1 > a_2 \geq b_2 > a_3,$
7.	$\beta\alpha\alpha\beta\alpha,$	$b_1 > a_1 \geq a_2 \geq b_2 > a_3,$
8.	$\alpha\beta\beta\alpha\alpha,$	$a_1 \geq b_1 \geq b_2 > a_2 \geq a_3,$
9.	$\beta\alpha\beta\alpha\alpha,$	$b_1 > a_1 \geq b_2 > a_2 \geq a_3,$
10.	$\beta\beta\alpha\alpha\alpha,$	$b_1 \geq b_2 > a_1 \geq a_2 \geq a_3,$

where the descending order on the right appertains to the permutation on the left and therefore to the line of route which the permutation denotes. Wherever in the permutation there is either an equal contact or a minor contact, the symbol $\geq$ is placed in the corresponding place in the descending order; but wherever there is a major contact, the symbol employed is $>$. The successions of numbers thus defined are all distinct. We obtain ten sets of numbers in descending order without overlapping. It will be noticed that the symbol $>$ invariably occurs in correspondence with one of the numbers which, when added together, become the greater index of the permutation. We have now to form the ten generating functions which enumerate the ten sets of numbers.

To enumerate set No. 6, put

$$\begin{aligned} b_2 &= a_3+A+1, \\ a_2 &= a_3+A+B+1, \\ b_1 &= a_3+A+B+C+1+1, \\ a_1 &= a_3+A+B+C+D+1+1, \end{aligned}$$

where A, B, C, D are arbitrary positive (including zero) integers; then

$$\Sigma x^{a_1+a_2+a_3+b_1+b_2} = \Sigma x^{5a_3+4A+3B+2C+D+2+4},$$

where the two numbers 2, 4, which present themselves in the exponent of x, are necessarily the two numbers which are added together to form the greater index of the associated permutation $\alpha\beta\alpha\beta\alpha$. This generating function is therefore

$$\frac{x^6}{(1)\,(2)\,(3)\,(4)\,(5)};$$

and it is readily seen that the sum of the ten generating functions must be

$$\frac{\Sigma x^p}{(1)\,(2)\,(3)\,(4)\,(5)},$$

the sum being in respect of the greater indices of the ten permutations. Hence

$$\frac{\Sigma x^p}{(1)\,(2)\,(3)\,(4)\,(5)} = \frac{1}{(1)\,(2)\,(3)\cdot(1)\,(2)},$$

or

$$\Sigma x^p = \frac{(1)\,(2)\,(3)\,(4)\,(5)}{(1)\,(2)\,(3)\cdot(1)\,(2)}.$$

The above reasoning is of general application. When the system of nodes has rows containing $i, j, k, \ldots$ nodes and we consider the number of ways of placing numbers in descending orders in each row so that the sum of all the numbers may be w, we have to consider

$$\frac{(i+j+k+\ldots)\,!}{i\,!\,j\,!\,k\,!\ldots}$$

conditioned descending orders corresponding to the permutations of $\alpha^i\beta^j\gamma^k\ldots$; whenever the contact is major we employ the symbol $>$, and $\geq$ in other cases, as conditions of the descending order, and we find in each case that x^p is the numerator of the generating function, where p is the greater index of the permutation. The reader will note that when the p_1-th interval between the letters $a_1, a_2, a_3, \ldots; b_1, b_2, \ldots; c_1, c_2, c_3, \ldots; \ldots$, in appropriate order, is occupied by the symbol $>$, one outstanding number, in the exponent of x which first arises, is p_1; and that the whole outstanding number must be p.

Hence in general

$$\Sigma x^p = \frac{(1)\,(2)\,\ldots\,(i+j+k+\ldots)}{(1)\,(2)\,\ldots\,(i)\cdot(1)\,(2)\,\ldots\,(j)\cdot(1)\,(2)\,\ldots\,(k)\cdot\ldots}.$$

7. When x is put equal to unity the generating function naturally reduces to the number which enumerates the permutations of the assemblage.

8. *Important Property of* Σx^p. From the symmetry of the expression just obtained we gather the information that the expression of Σx^p is unaltered by any substitution impressed upon the numbers $i, j, k, \ldots\ldots$

Ex. gr., Σx^p is the same for $\alpha^3\beta^2$ as for $\alpha^2\beta^3$.

9. *Proof that* $\Sigma x^r = \Sigma x^p$. The last remark enables us to establish that in general $\Sigma x^r = \Sigma x^p$. Consider a permutation of any assemblage, viz.

$$\beta\alpha\alpha\alpha\gamma\gamma\beta\alpha\gamma,$$

and transfrom it by the substitution

$$\begin{pmatrix}\alpha\beta\gamma\\ \gamma\beta\alpha\end{pmatrix}$$

(equivalent to changing the assemblage $\alpha^4\beta^2\gamma^3$ into $\alpha^3\beta^2\gamma^4$). The permutation becomes

$$\beta\gamma\gamma\gamma\alpha\alpha\beta\gamma\alpha,$$

and its lesser index is necessarily equal to the greater index of the original permutation. Hence Σx^r for the transformed is equal to Σx^p for the original assemblage; and since, as we have seen above, Σx^p remains unchanged by the transformation, it follows at once that for every assemblage

$$\Sigma x^r = \Sigma x^p;$$

for we have merely to make the substitution

$$\begin{pmatrix}\alpha_1\alpha_2 & \alpha_3 & \ldots\ldots\alpha_s\\ \alpha_s\alpha_{s-1}\alpha_{s-2} & \ldots\ldots & \alpha_1\end{pmatrix},$$

and employ the same reasoning.

10. *The Expression of* Σx^{p+q}. Of course p, q are the greater and equal indices of the same permutation. We have

$$p+q+r = 1+2+\ldots\ldots+(\Sigma i-1) = \binom{\Sigma i}{2},$$

p, q, r referring to the same permutation, and $\binom{\Sigma i}{2}$ denoting $\frac{1}{2}\Sigma i(\Sigma i-1)$. Hence

$$\Sigma x^{p+q} = x^{p+q+r}\Sigma x^{-r}$$

$$= x^{\binom{\Sigma i}{2}}\,\frac{\left(1-\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)\ldots\left(1-\frac{1}{x^{i_1+i_2+\ldots}}\right)}{\left(1-\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)\ldots\left(1-\frac{1}{x^{i_1}}\right)\left(1-\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)\ldots\left(1-\frac{1}{x^{i_2}}\right)\ldots}$$

$$= x^{\binom{\Sigma i}{2}+\binom{i_1+1}{2}+\binom{i_2+1}{2}+\ldots-\binom{\Sigma i+1}{2}}\,\frac{(1)\,(2)\,\ldots\,(i_1+i_2+\ldots)}{(1)\,(2)\,\ldots\,(i_1)\cdot(1)\,(2)\,\ldots\,(i_2)\cdot\ldots};$$

and thence, on reduction,

$$\Sigma x^{p+q} = x^{\binom{i}{2}+\binom{j}{2}+\binom{k}{2}+\cdots}\Sigma x^p,$$

a noteworthy result, the truth of which can also be seen *à priori*.

We have merely to recur to the method by which the expression of Σx^p was investigated. In the descending orders of numbers associated with the permutations we must clearly have the symbol $>$ *both* when there are major and when there are equal contacts. This will happen if we impress a condition upon the numbers in each row. The condition is that in any row of numbers, placed at the nodes of a row of nodes, there must be no two numbers equal. This is secured by adding to every row of numbers

$$a_1 a_2 a_3 \ldots . a_i$$

the numbers $i-1, i-2, i-3, \ldots, 1, 0$ respectively to each number in succession from left to right, so that the row becomes

$$a_1+i-1,\ a_2+i-2,\ a_3+i-3,\ \ldots,\ a_{i-1}+1,\ a_i.$$

The result of this is to multiply Σx^p by $x^{1+2+\cdots+i-1}$ or by $x^{\binom{i}{2}}$ in respect of each row. Thence at once

$$\Sigma x^{p+q} = x^{\binom{i}{2}+\binom{j}{2}+\binom{k}{2}+\cdots}\Sigma x^p.$$

11. *Difference Equation for* Σx^p. Taking the assemblage to be $\alpha_1^{i_1}\alpha_2^{i_2}\ldots.\alpha_s^{i_s}$ write

$$\Sigma x^p = u(i_1, i_2, \ldots, i_s);$$

and if Σx^p is restricted to those permutations which terminate with a letter α_m, write

$$\underset{m}{\Sigma} x^p = \underset{m}{u}(i_1, i_2, \ldots, i_s), \quad \text{and} \quad \Sigma i = S;$$

so that

$$\Sigma x^p = \sum_{1}^{s} \underset{m}{u}(i_1, i_2, \ldots, i_s).$$

Obviously

$$\underset{s}{u}(i_1 i_2 \ldots . i_s) = u(i_1, i_2, \ldots, i_{s-1}, i_s-1).$$

The function $\underset{s-1}{u}(i_1 i_2 \ldots . i_s)$ refers to permutations that may end with $\alpha_s \alpha_{s-1}$. These give rise to the function $x^{S-1}\underset{s}{u}(i_1, i_2, \ldots, i_{s-1}-1, i_s)$. The remaining permutations yield the function

$$u(i_1, i_2, \ldots, i_{s-1}-1, i_s) - \underset{s}{u}(i_1, i_2, \ldots, i_{s-1}-1, i_s).$$

Hence

$$\underset{s-1}{u}(i_1, i_2, \ldots, i_s) = u(i_1, i_2, \ldots, i_{s-1}-1, i_s) - (1-x^{S-1})\,u(i_1, i_2, \ldots, i_{s-1}-1, i_s-1).$$

Similarly it will be found that

$$\begin{aligned}\underset{s-2}{u}(i_1, i_2, \ldots, i_s) = {} & u(i_1, i_2, \ldots, i_{s-2}-1, i_{s-1}, i_s) \\ & -(1-x^{s-1})u(i_1, i_2, \ldots, i_{s-2}-1, i_{s-1}-1, i_s) \\ & -(1-x^{s-1})u(i_1, i_2, \ldots, i_{s-2}-1, i_{s-1}, i_s-1) \\ & +(1-x^{s-1})(1-x^{s-2})u(i_1, i_2, \ldots, i_{s-2}-1, i_{s-1}-1, i_s-1),\end{aligned}$$

and so forth. Thence

$$\begin{aligned}u(i_1, i_2, \ldots, i_s) = {} & u(i_1, i_2, \ldots, i_{s-1}, i_s-1) + \{u(i_1, i_2, \ldots, i_{s-1}-1, i_s) \\ & -(\mathbf{S}-\mathbf{1})u(i_1, i_2, \ldots, i_{s-1}-1, i_s-1)\} \\ & +\{u(i_1, i_2, \ldots, i_{s-2}-1, i_{s-1}, i_s) \\ & -(\mathbf{S}-\mathbf{1})u(i_1, i_2, \ldots, i_{s-2}-1, i_{s-1}-1, i_s) \\ & -(\mathbf{S}-\mathbf{1})u(i_1, i_2, \ldots, i_{s-2}-1, i_{s-1}, i_s-1) \\ & +(\mathbf{S}-\mathbf{1})(\mathbf{S}-\mathbf{2})u(i_1, i_2, \ldots, i_{s-2}-1, i_{s-1}-1, i_s-1)\} + \ldots,\end{aligned}$$

and we obtain without difficulty the general equation of differences

$$\begin{aligned}u(i_1, i_2, \ldots, i_s) = {} & \sum_l u(i_1, \ldots, i_l-1, \ldots, i_s) \\ & -(\mathbf{S}-\mathbf{1})\sum_{l,m} u(i_1, \ldots, i_l-1, \ldots, i_m-1, \ldots, i_s) \\ & +(\mathbf{S}-\mathbf{1})(\mathbf{S}-\mathbf{2})\sum_{l,m,n} u(i_1, \ldots, i_l-1, \ldots, i_m-1, \ldots, i_n-1, \ldots, i_s) \\ & -\ldots.\end{aligned}$$

This relation will be of service in the sequel.

12. *The Particular Case of the Assemblage $\alpha^i\beta^j$.* Merely considering the main result for the permutations of $\alpha^i\beta^j$, viz.

$$\Sigma x^p = \frac{(\mathbf{1})(\mathbf{2})\ldots.(\mathbf{i}+\mathbf{j})}{(\mathbf{1})(\mathbf{2})\ldots.(\mathbf{i}) \cdot (\mathbf{1})(\mathbf{2})\ldots.(\mathbf{j})},$$

we notice that Σx^p is equal to the generating function which enumerates the partitions of all numbers into parts limited in magnitude by i and in number by j. This indicates that the number of permutations of $\alpha^i\beta^j$ which have p for greater index is equal to the number of partitions of p, the parts being restricted as above stated by the numbers i, j.

If the permutation is of class m (see Art. 2) the greater index is obtained as the sum of m numbers. If we restrict ourselves to such permutations, I showed, in the postscript of the paper above referred to, that

$$\overset{m}{\Sigma} x^p = x^{m^2} \frac{(\mathbf{i}-\mathbf{m}+\mathbf{1})(\mathbf{i}-\mathbf{m}+\mathbf{2})\ldots.(\mathbf{i})}{(\mathbf{1})(\mathbf{2})\ldots.(\mathbf{m})} \cdot \frac{(\mathbf{j}-\mathbf{m}+\mathbf{1})(\mathbf{j}-\mathbf{m}+\mathbf{2})\ldots.(\mathbf{j})}{(\mathbf{1})(\mathbf{2})\ldots.(\mathbf{m})},$$

a result which yields the well-known expansion

$$\frac{(\mathbf{1})(\mathbf{2})\ldots.(\mathbf{i}+\mathbf{j})}{(\mathbf{1})(\mathbf{2})\ldots.(\mathbf{i}) \cdot (\mathbf{1})(\mathbf{2})\ldots.(\mathbf{j})} = 1 + x\frac{(\mathbf{i})(\mathbf{j})}{(\mathbf{1}) \cdot (\mathbf{1})} + x^4\frac{(\mathbf{i}-\mathbf{1})(\mathbf{i}) \cdot (\mathbf{j}-\mathbf{1})(\mathbf{j})}{(\mathbf{1})(\mathbf{2}) \cdot (\mathbf{1})(\mathbf{2})} + \ldots.$$

This fact is referred to here in order that I may make the observation that the expansion is identical with that which Sylvester obtained from Durfee's dissection of the graph of a partition into a square of nodes with lateral and subjacent appendages. (Compare *Collected Mathematical Papers*, Vol. IV, pp. 1 *et seq.*).

13. For the assemblage $\alpha^i\beta^j$ we can find the equal index function Σx^q. It has first to be shown that the number $p-r$ is equal to $-i$ or $+j$ according as the permutation terminates with β or α.

Assume the law to obtain for $\alpha^i\beta^j$ and denote by $(\alpha^i\beta^j)_\alpha$, $(\alpha^i\beta^j)_\beta$ any permutations terminating with α, β respectively.

Then for	$(\alpha^i\beta^j)_\alpha\alpha$,	$p-r=+j$;
for	$(\alpha^i\beta^j)_\alpha\beta$,	$p-r=+j-(i+j)=-i$;
for	$(\alpha^i\beta^j)_\beta\alpha$,	$p-r=-i+i+j=+j$;
for	$(\alpha^i\beta^j)_\beta\beta$,	$p-r=+j$;
for	$(\alpha^i\beta^j)_\alpha\alpha\beta$,	$p-r=+j-(i+j+1)=-i-1$;
for	$(\alpha^i\beta^j)_\alpha\beta\alpha$,	$p-r=+j-(i+j)+(i+j+1)=+j+1$;
for	$(\alpha^i\beta^j)_\beta\alpha\beta$,	$p-r=-i+(i+j)-(i+j+1)=-i-1$;
for	$(\alpha^i\beta^j)_\beta\beta\alpha$,	$p-r=-i+(i+j+1)=+j+1$.

These results establish the theorem.

Hence, for the permutations ending with β,

$$\Sigma x^{p-r}=\binom{i+j-1}{i}x^{-i};$$

and, for those ending with α,

$$\Sigma x^{p-r}=\binom{i+j-1}{j}x^{j}.$$

Since, for all permutations,

$$\Sigma x^{p+q+r}=\binom{i+j}{i}x^{\binom{i+j}{2}},$$

it follows that

$$\Sigma x^{2p+q}=\binom{i+j-1}{i}x^{\binom{i+j}{2}-i}+\binom{i+j-1}{j}x^{\binom{i+j}{2}+j}.$$

Writing Σx^p, for the whole of the permutation, equal to $F_{i,j}(x)$, we see that, for those ending with β, α,

$$\Sigma x^p=F_{i,j-1}(x) \quad \text{and} \quad x^jF_{i-1,j}(x)$$

respectively. (Compare Art. 11.) Hence

$$\Sigma x^q=\Sigma x^{2p+q-2p}=x^{\binom{i+j}{2}-i}F_{i,j-1}\left(\frac{1}{x^2}\right)+x^{\binom{i+j}{2}+j-2j}F_{i-1,j}\left(\frac{1}{x^2}\right);$$

easily reducing to the formula

$$\Sigma x^q = x^{\left(\frac{i-j+1}{2}\right)} F_{i,j-1}(x^2) + x^{\left(\frac{j-i+1}{2}\right)} F_{i-1,j}(x^2).$$

It will be noted that the index q is *always* even when $i-j \equiv 0 \bmod 4$ and *always* uneven when $i-j \equiv 2 \bmod 4$.

14. We may obtain the expression of Σx^{p+r}; for this is

$$\Sigma x^{p-r+2r};$$

and, since Σx^r is $x^i F_{i,j-1}(x)$ or $F_{i-1,j}(x)$ according as the permutation ends with β or α,

$$\begin{aligned}\Sigma x^{p+r} &= x^{-i} \cdot x^{2i} F_{i,j-1}(x^2) + x^j F_{i-1,j}(x^2), \\ &= x^i F_{i,j-1}(x^2) + x^j F_{i-1,j}(x^2).\end{aligned}$$

The reader will have no difficulty in obtaining the expression of $\Sigma x^{Ap+Bq+Cr}$ for this assemblage.

15. Resuming the consideration of the assemblage $\alpha^i \beta^j \gamma^k \ldots$., it is to be remarked that $p-r$ has no longer any simple relation to the permutations when the number of Greek letters exceeds 2. It is not possible to obtain expressions for Σx^q and Σx^{p+r} in the foregoing manner. Similar theorems, however, will be reached when major and minor indices come under discussion later in the paper. Before proceeding to those matters I give some interesting properties of the indices that have been already under discussion.

16. *The Parity of the Greater Index.* The permutations of $\alpha^i \beta^j \gamma^k \ldots$. are such that the number with an even index is equal to the number with an uneven index whenever more than one of the numbers $i, j, k, \ldots$. are uneven. For consider the function

$$\frac{(1)(2)\ldots.(i+j+k+\ldots.)}{(1)(2)\ldots.(i)\cdot(1)(2)\ldots.(j)\cdot(1)(2)\ldots.(k)\cdot\ldots.};$$

if we put therein $x=-1$ and the function happens to vanish it must be because the permutations with even index are equal in number to those with uneven index. The factor (m) becomes 2 for $x=-1$ whenever m is uneven. Eliminating these factors we are left with a function which may be an indeterminate form and may be evaluated when $x=-1$. This happens when numerator and denominator contain the same number of factors. If this be not the case, the numerator must contain *more* factors than the denominator and the function must therefore vanish for $x=-1$. The numerator cannot contain less factors than the denominator, for in that case the function would become infinite for $x=-1$ and this result, from the nature of the function, is absurd. We have then to find out the circumstances under which the numerator has more factors than the

denominator after the elimination of factors referred to above. If $E\sigma$ denotes the greatest integer in σ, the difference in number between the numerator and and denominator factors is

$$E\frac{1}{2}(i+j+k+\ldots.)-E\frac{i}{2}-E\frac{j}{2}-E\frac{k}{2}-\ldots.;$$

and this is greater than zero whenever more than one of the numbers $i, j, k, \ldots.$ are uneven. This establishes the theorem.

17. There is an analogous result in the case of every *prime* modulus μ. For, putting x equal to ωx, where ω is an imaginary μ-th root of unity, we set apart all factors which involve ω^t, where t is not equal to μ or $\equiv 0 \bmod \mu$. We are left with a number of factors free from ω, and the function involving them is seen to vanish when $x=\omega$, whenever

$$\Sigma\frac{1}{\mu}(i+j+k+\ldots.)-E\frac{i}{\mu}-E\frac{j}{\mu}-E\frac{k}{\mu}-\ldots.>0.$$

This happens if $i\equiv i' \bmod \mu$, $j\equiv j' \bmod \mu$, $k\equiv k' \bmod \mu$, etc., whenever

$$i'+j'+k'+\ldots.>\mu-1.$$

Under these circumstances the number of permutations, which have indices $\equiv \sigma \bmod \mu$, does not vary by variation of σ.

Ex. gr., take the assemblage $\alpha^2\beta\gamma$:

	$p=$		$p=$
$\alpha\alpha\beta\gamma$	0	$\beta\alpha\alpha\gamma$	1
$\alpha\alpha\gamma\beta$	3	$\gamma\alpha\alpha\beta$	1
$\alpha\beta\alpha\gamma$	2	$\beta\alpha\gamma\alpha$	4
$\alpha\gamma\alpha\beta$	2	$\gamma\alpha\beta\alpha$	4
$\alpha\beta\gamma\alpha$	3	$\beta\gamma\alpha\alpha$	2
$\alpha\gamma\beta\alpha$	5	$\gamma\beta\alpha\alpha$	3

The reader will verify that the even and uneven indices are equal in number; also, since 2, 1, 1 are $\equiv$ 2, 1, 1 mod 3 respectively and $2+1+1>3$, that the indices $\equiv$ 0, 1, 2 mod 3 are equal in number.

18. *Average Values of the Indices.* The expression obtained for Σx^p enables us to find the average values of the different indices of a permutation.

Write $\Sigma x^p=\Sigma C_p x^p$, so that C_p permutations have the greater index equal to p. The average value of p is the quotient of $\Sigma C_p p$ by

$$\frac{(i+j+k+\ldots.)!}{i!\,j!\,k!\ldots.}.$$

To find $\Sigma C_p p$ we may differentiate Σx^p with regard to x and then put x

equal to unity. This is not the only, or the most obvious, way of finding $\Sigma C_p p$, but it is the best way for present purposes, as will appear later. To evaluate

$$\left\{\partial_x \frac{(1)\,(2)\,\ldots\,(i+j)}{(1)\,(2)\,\ldots\,(i)\cdot(1)\,(2)\,\ldots\,(j)}\right\}_{x=1},$$

we require the value of

$$\left\{\partial_x \frac{(i+\sigma)}{(\sigma)}\right\}_{x=1}.$$

This is readily found to be

$$\frac{i(i+\sigma)}{2\sigma},$$

so that

$$\left\{\partial_x \frac{(i+1)\,\ldots\,(i+j)}{(1)\,\ldots\,(j)}\right\}_{x=1} = \frac{(i+j)\,!}{i!\,j!}\sum_{\sigma=1}^{\sigma=j}\frac{\frac{i(i+\sigma)}{2\sigma}}{\frac{i+\sigma}{\sigma}} = \frac{(i+j)\,!}{i!\,j!}\cdot\frac{1}{2}ij;$$

establishing that the average value of the greater (or lesser) index of the permutation of $\alpha^i\beta^j$ is

$$\frac{1}{2}ij.$$

Thence

$$\begin{aligned}
&\left\{\partial_x \frac{(1)\,(2)\,\ldots\,(i+j+k)}{(1)\,(2)\,\ldots\,(i)\cdot(1)\,(2)\,\ldots\,(j)\cdot(1)\,(2)\,\ldots\,(k)\cdot\ldots}\right\}_{x=1}\\
&\quad=\left\{\frac{(1)\,(2)\,\ldots\,(i+j+k)}{(1)\,(2)\,\ldots\,(i)\cdot(1)\,(2)\,\ldots\,(j+k)}\right\}_{x=1}\left\{\partial_x \frac{(1)\,(2)\,\ldots\,(j+k)}{(1)\,(2)\,\ldots\,(j)\cdot(1)\,(2)\,\ldots\,(k)}\right\}_{x=1}\\
&\quad+\left\{\frac{(1)\,(2)\,\ldots\,(j+k)}{(1)\,(2)\,\ldots\,(j)\cdot(1)\,(2)\,\ldots\,(k)}\right\}_{x=1}\left\{\partial_x \frac{(1)\,(2)\,\ldots\,(i+j+k)}{(1)\,(2)\,\ldots\,(i)\cdot(1)\,(2)\,\ldots\,(j+k)}\right\}_{x=1}\\
&\quad=\frac{(i+j+k)\,!}{i!\,j!\,k!}\left\{\frac{1}{2}jk+\frac{1}{2}i(j+k)\right\};
\end{aligned}$$

establishing that the average value of the greater (or lesser) index of the permutations of $\alpha^i\beta^j\gamma^k$ is

$$\frac{1}{2}(ij+ik+jk).$$

From the above it is quite clear that, for the permutations of $\alpha^i\beta^j\gamma^k\ldots.$, the average value of the greater (or lesser) index is

$\frac{1}{2}$ (sum of the products two and two together of the numbers $i, j, k, \ldots.$).

19. *Average Value of the Equal Index.* From the general formula

$$\Sigma x^{p+q} = x^{\binom{i}{2}+\binom{j}{2}+\binom{k}{2}+\cdots}\Sigma x^p,$$

it is seen that the average value of the equal index is

$$\binom{i}{2}+\binom{j}{2}+\binom{k}{2}+\ldots;$$

or from the circumstance that the average value of $p+r$ is

$$ij+ik+jk+\ldots,$$

and that the sum $p+q+r$ is invariably

$$\binom{i+j+k+\ldots}{2},$$

the average value of q is

$$\binom{i+j+k+\ldots}{2}-(ij+ik+jk+\ldots),$$

which is

$$\binom{i}{2}+\binom{j}{2}+\binom{k}{2}+\ldots.$$

20. *Average Value of the Square of the Greater Index.* The value in question is derived from the average value of $p(p-1)$, which is clearly the quotient of

$$\left\{\partial_x^2 \frac{(1)\,(2)\ldots.(i+j+k+\ldots.)}{(1)\,(2)\ldots.(i)\cdot(1)\,(2)\ldots.(j)\cdot(1)\,(2)\ldots.(k)\cdot\ldots.}\right\}_{x=1}$$

by

$$\frac{(i+j+k+\ldots.)\,!}{i!\,j!\,k!\ldots.}.$$

To evaluate this we require the value of

$$\left\{\partial_x^2 \frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1}$$

when $\lambda>\mu$. This I find to be

$$\frac{1}{6}\frac{\lambda}{\mu}(\lambda-\mu)\,(2\lambda-\mu-3).$$

There is theoretically no difficulty in determining the average value of any given positive integral power of the greater index. It depends upon finding the value of

$$\left\{\partial_x^s \frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1}$$

when $\lambda>\mu$.

The simplest cases, corresponding to $s=1$ and $s=2$, may be written

$$\left\{\partial_x \frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1}=\mu\left\{\frac{1}{2}\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-1\right)\right\},$$

$$\left\{\partial_x^2 \frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1}=\mu^2\left\{\frac{1}{6}\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-1\right)\left(2\frac{\lambda}{\mu}-1\right)\right\}-\mu\left\{\frac{1}{2}\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-1\right)\right\},$$

and the general result, at which I have arrived, may be stated in the following manner: Let

$$\mu(\mu-1)(\mu-2)\ldots.(\mu-s+1)=\mu^s+A_{s-1}\mu^{s-1}+A_{s-2}\mu^{s-2}+\ldots.+A_1\mu,$$

and

$$1^\nu+2^\nu+\ldots.+n^\nu={}_nS_\nu,$$

then

$$\left\{\partial_x^s\frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1}=\{\mu^s{}_nS_s+A_{s-1}\mu^{s-1}{}_nS_{s-1}+A_{s-2}\mu^{s-2}{}_nS_{s-2}+\ldots.+A_1\mu{}_nS_1\}_{n=\frac{\lambda}{\mu}-1}.$$

I do not interrupt the present investigation by giving a proof of this elegant theorem. I write at length the two cases which follow those given above.

$$\left\{\partial_x^3\frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1}=\mu^3\left\{\frac{1}{4}\frac{\lambda^2}{\mu^2}\left(\frac{\lambda}{\mu}-1\right)^2\right\}-3\mu^2\left\{\frac{1}{6}\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-1\right)\left(2\frac{\lambda}{\mu}-1\right)\right\}+2\mu\left\{\frac{1}{2}\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-1\right)\right\},$$

$$\left\{\partial_x^4\frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1}=\mu^4\left\{\frac{1}{30}\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-1\right)\left(2\frac{\lambda}{\mu}-1\right)\left(3\frac{\lambda^2}{\mu^2}-3\frac{\lambda}{\mu}-1\right)\right\}-6\mu^3\left\{\frac{1}{4}\frac{\lambda^2}{\mu^2}\left(\frac{\lambda}{\mu}-1\right)^2\right\}$$
$$+11\mu^2\left\{\frac{1}{6}\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-1\right)\left(2\frac{\lambda}{\mu}-1\right)\right\}-6\mu\left\{\frac{1}{2}\frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}-1\right)\right\}.$$

These formulæ may be often simplified. Thus I find that

$$\left\{\partial_x^3\frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1}=\frac{1}{4}\frac{\lambda}{\mu}(\lambda-\mu)(\lambda-\mu-2)(\lambda-2).$$

The result depends upon the sums of the powers of the natural numbers and therefore ultimately upon the numbers of Bernouilli. I notice that

$${}_{-n}S_\nu=(-)^{\nu+1}{}_{n-1}S_\nu,$$

a useful relation. Writing

$$\left\{\partial_x^s\frac{1-x^\lambda}{1-x^\mu}\right\}_{x=1}=K_s,\quad\text{and}\quad({}_nS_\nu)_{n=\frac{\lambda}{\mu}-1}=S'_\nu,$$

we have

$K_1=\mu S'_1,$	$\mu S'_1=K_1,$
$K_2=\mu^2S'_2-\mu S'_1,$	$\mu^2S'_2=K_1+K_2,$
$K_3=\mu^3S'_3-3\mu^2S'_2+2\mu S'_1,$	$\mu^3S'_3=K_1+3K_2+K_3,$
$K_4=\mu^4S'_4-6\mu^3S'_3+11\mu^2S'_2-6\mu S'_1,$	$\mu^4S'_4=K_1+7K_2+6K_3+K_4,$
.....,	.....,

where the coefficients in the expression of K_s are derived from the product $x(x-1)\ldots.(x-s+1)$, and those in the expression of $\mu^sS'_s$ from the expression of x^s as a linear function of x, $x(x-1)$, $x(x-1)(x-2)$, ...

To apply these results I write

$$(X-i)(X-j)(X-k)\ldots.=X^s-\{1\}X^{s-1}+\{11\}X^{s-2}-\ldots..$$

I find that the average value of $p(p-1)$ is

$$\frac{1}{12}[3\{22\}+6\{211\}+18\{1111\}+\{21\}+2\{111\}-5\{11\}],$$

where, in the notation of symmetric functions, $\{22\}$ denotes $\Sigma i^2 j^2$, etc. Since we have established that the average value of p is $\frac{1}{2}\{11\}$, we see that the average value of p^2 is

$$\frac{1}{12}[3\{22\}+6\{211\}+18\{1111\}+\{21\}+2\{111\}+\{11\}],$$

which may be written

$$\frac{1}{4}\{11\}^2+\frac{1}{12}[\{21\}+2\{111\}+\{11\}],$$

wherein it will be noticed that the first term is the square of the average value of p.

As a verification observe that, for the permutations of $\alpha\beta\gamma\delta$,

$$\Sigma x^p = 1+3x+5x^2+6x^3+5x^4+3x^5+x^6;$$

so that the sum of the squares of the indices is

$$3\cdot 1^2+5\cdot 2^2+6\cdot 3^2+5\cdot 4^2+3\cdot 5^2+1\cdot 6^2 = 268,$$

and the average value of the square $= \frac{268}{24} = \frac{67}{6}$.

The formula gives $\frac{1}{12}(3\cdot 6+6\cdot 12+18\cdot 1+12+2\cdot 4+6) = \frac{76}{6}$.

For the case $\alpha^i\beta^j$ the average value of the square of p is

$$\frac{1}{12}ij(3ij+i+j+1).$$

21. *Average Value of the Cube of the Greater Index.* Similarly the reader will find with little difficulty that the average value of $p(p-1)(p-2)$ for the assemblage $\alpha^i\beta^j$ is

$$\frac{1}{8}ij(i^2j^2+i^2j+ij^2-5ij-2i-2j+6);$$

and, since $p^3 = p+3p(p-1)+p(p-1)(p-2)$, we find the average value of p^3 by adding to the above

$$\frac{1}{2}ij+\frac{1}{4}ij(3ij+i+j-5);$$

thence the average value of p^3 is found to be

$$\frac{1}{8}i^2j^2(i+1)(j+1).$$

22. For the assemblage $\alpha^i\beta^j\gamma^k$ if we write

$$\frac{(1)\,(2)\,\ldots\,(i+j+k)}{(1)\,(2)\,\ldots\,(i)\cdot(1)\,(2)\,\ldots\,(j)\cdot(1)\,(2)\,\ldots\,(k)}$$

$$=\frac{(1)\,(2)\,\ldots\,(i+j+k)}{(1)\,(2)\,\ldots\,(i)\cdot(1)\,(2)\,\ldots\,(j+k)}\cdot\frac{(1)\,(2)\,\ldots\,(j+k)}{(1)\,(2)\,\ldots\,(j)\cdot(1)\,(2)\,\ldots\,(k)}$$

$$=F_{i,\,j+k}\cdot F_{j,\,k},$$

since

$$\partial_x^3 F_{i,j,k}=F_{i,j+k}\partial_x^3 F_{j,k}+3\,\partial_x F_{i,j+k}\partial_x^2 F_{j,k}+3\,\partial_x^2 F_{i,j+k}\partial_x F_{j,k}+\partial_x^3 F_{i,j+k}\cdot F_{j,k},$$

the average value of $p(p-1)(p-2)$ is seen to be

$$\frac{1}{8}jk(j^2k^2+j^2k+jk^2-5jk-2j-2k+6)+\frac{3}{2}i(j+k)\cdot\frac{1}{12}jk(3jk+j+k-5)$$

$$+3\cdot\frac{1}{12}i(j+k)\{3i(j+k)+i+j+k-5\}\cdot\frac{1}{2}jk$$

$$+\frac{1}{8}i(j+k)\{i^2(j+k)^2+i^2(j+k)+i(j+k)^2-5i(j+k)-2i-2(j+k)+6\},$$

which is

$$\frac{1}{8}\{\Sigma i^3j^3+3\Sigma i^3j^2k+6\Sigma i^2j^2k^2+\Sigma i^3j^2+2\Sigma i^3jk+4\Sigma i^2j^2k$$
$$-5\Sigma i^2j^2-10\Sigma i^2jk-2\Sigma i^2j-4ijk+6\Sigma ij\}.$$

Recalling that, for the same assemblage,

Average value of p is $\frac{1}{2}\Sigma ij$,

and of $p(p-1)$ is $\frac{1}{12}(3\Sigma i^2j^2+6\Sigma i^2jk+\Sigma i^2j+2ijk-5\Sigma ij)$,

we find that the average value of p^3 is

$$\frac{1}{8}\{\Sigma i^3j^3+3\Sigma i^3j^2k+6i^2j^2k^2+\Sigma i^3j^2+2\Sigma i^3jk+4\Sigma i^2j^2k+2\Sigma i^2jk+\Sigma i^2j^2\}.$$

23. With small labor I find that for the general assemblage the average value of p^3 is

$$\frac{1}{8}[6\{3111\}+15\{2211\}+\{33\}+3\{321\}+6\{222\}+9\{2111\}$$
$$+\{32\}+2\{311\}+4\{221\}+\{22\}+2\{211\}+6\{1111\}].$$

24. *Average Values from Another Standpoint.* For the assemblage $\alpha_1^{i_1}\alpha_2^{i_2}\ldots\alpha_s^{i_s}$ let us inquire into the effect which the particular major contact $\alpha_\mu\alpha_\lambda$ $(\mu>\lambda)$ has upon the sum of the greater indices Σp. When the contact has σ letters to the left of it, it adds $\sigma+1$ to the index of the permutation. This happens in the case of

$$\frac{(S-2)!}{i_1!\,i_2!\ldots.i_s!}\,i_\lambda i_\mu \text{ permutations, where } S=\Sigma i,$$

so that, altogether, the effect of the contact $\alpha_\mu\alpha_\lambda$ is to add to Σp the number

$$\sum_{\sigma=0}^{\sigma=S-2}\frac{(S-2)!}{i_1!\,i_2!\ldots.i_s!}\,i_\lambda i_\mu\,(\sigma+1),$$

or

$$\frac{S!}{i_1!\,i_2!\ldots.i_s!}\,\frac{1}{2}\,i_\lambda i_\mu.$$

Hence the contact $\alpha_\mu\alpha_\lambda$, on the average, adds the number $\frac{1}{2}\,i_\lambda i_\mu$ to the index of a permutation, and thence we find that the average value of p is

$$\frac{1}{2}\,\Sigma i_\lambda i_\mu,$$

verifying the conclusion of Art. 18.

25. The number p is obtained as the sum of m numbers $p_1, p_2, \ldots., p_m$,* where m is the class of the permutation *quâ* the greater index. If we had under consideration the sum

$$\Sigma(p_1^\nu+p_2^\nu+\ldots.+p_m^\nu),$$

we might inquire into the effect which the particular contact $\alpha_\mu\alpha_\lambda$ has upon such sum. The answer to this question is that the effect is to add to the sum the number

$$\sum_{\sigma=0}^{\sigma=S-2}\frac{(S-2)!}{i_1!\,i_2!\ldots.i_s!}\,i_\lambda i_\mu\,(\sigma+1)^\nu,$$

which is

$$\frac{S!}{i_1!\,i_2!\ldots.i_s!}\,\frac{1^\nu+2^\nu+\ldots.+(S-1)^\nu}{S(S-1)}\,i_\lambda i_\mu.$$

Hence the contact $\alpha_\mu\alpha_\lambda$, on the average, adds to the sum

$$p_1^\nu+p_2^\nu+\ldots.+p_m^\nu,$$

with regard to a permutation, the number

$$\frac{1^\nu+2^\nu+\ldots.+(S-1)^\nu}{S(S-1)}\,i_\lambda i_\mu.$$

From this result it follows that the average value of $p_1^\nu+p_2^\nu+\ldots.+p_m^\nu$ is

$$\frac{1^\nu+2^\nu+\ldots.+(S-1)^\nu}{S(S-1)}\,\Sigma i_\lambda i_\mu.$$

26. The average value of $p_1^2+p_2^2+\ldots.+p_m^2$ is therefore

$$\frac{1}{6}\,(2\Sigma i_1-1)\Sigma i_1 i_2;$$

---

* These numbers it is convenient to term the "components" of the index p.

and of $p_1^3+p_2^3+p_3^3+\ldots.+p_m^3$,

$$\frac{1}{4}\Sigma i_1(\Sigma i_1-1)\Sigma i_1 i_2;$$

and so forth.

To verify these results, take the assemblage $\alpha^2\beta\gamma$:

	p_1	p_2	$p_1^2+p_2^2$	$p_1^3+p_2^3$		p_1	p_2	$p_1^2+p_2^2$	$p_1^3+p_2^3$
$\alpha\alpha\beta\gamma$	0	0	0	0	$\beta\alpha\alpha\gamma$	1	0	1	1
$\alpha\alpha\gamma\beta$	3	0	9	27	$\gamma\alpha\alpha\beta$	1	0	1	1
$\alpha\beta\alpha\gamma$	2	0	4	8	$\beta\alpha\gamma\alpha$	1	3	10	28
$\alpha\gamma\alpha\beta$	2	0	4	8	$\gamma\alpha\beta\alpha$	1	3	10	28
$\alpha\beta\gamma\alpha$	3	0	9	27	$\beta\gamma\alpha\alpha$	2	0	4	8
$\alpha\gamma\beta\alpha$	2	3	13	35	$\gamma\beta\alpha\alpha$	1	2	5	9
			39	105				31	75

giving the average of $p_1^2+p_2^2$ and $p_1^3+p_2^3$ equal to $\frac{35}{6}$ and 15 respectively.

The formulæ give, since $i_1=2$, $i_2=1$, $i_3=1$, $\Sigma i_1=4$, $\Sigma i_1 i_2=5$,

$$\text{Average of } p_1^2+p_2^2=\frac{1}{6}(2\cdot 4-1)\cdot 5=\frac{35}{6},$$

$$\text{Average of } p_1^3+p_2^3=\frac{1}{4}\cdot 4\cdot 3\cdot 5=15,$$

a verification.

27. *Average Value of the Class of a Permutation quâ the Greater Index.* If in the foregoing results we put $\nu=0$, then

$$p_1^0+p_2^0+\ldots.+p_m^0=m,$$

the class of the permutation. Hence the average value of the class is

$$\frac{S-1}{S(S-1)}\Sigma i_1 i_2 \text{ or } \frac{\Sigma i_1 i_2}{\Sigma i_1}.^*$$

In the above example

$$\frac{\Sigma i_1 i_2}{\Sigma i_1}=\frac{5}{4},$$

and we observe that one permutation is of class 0, seven of class 1 and four of class 2, so that the average class is

$$\frac{0+7+8}{12}=\frac{5}{4},$$

* Another corollary, paying attention to Art. 25, is that the average value of $p_\rho^\nu (p_\rho \neq 0)$ is

$$\frac{1}{\Sigma i-1}\{1^\nu+2^\nu+\ldots.+(\Sigma i-1)^\nu\}.$$

a verification. In other words we may say that the average number of major contacts possessed by a permutation is

$$\frac{\Sigma i_1 i_2}{\Sigma i_1}.$$

28. An easy corollary is that a permutation possesses on the average

$$\frac{2\Sigma i_1 i_2}{\Sigma i_1}$$

contacts which are either major or minor.

Since, moreover, the whole number of contacts is $\Sigma i_1 - 1$, it follows that, on the average, a permutation possesses a number of equal contacts equal to

$$\Sigma i_1 - 1 - 2\frac{\Sigma i_1 i_2}{\Sigma i_1} \text{ or } 2\frac{\Sigma \binom{i_1}{2}}{\Sigma i_1},$$

an elegant result.

This last result may be obtained independently.

29. *Average Values Connected with the Equal Index.* The particular contact $\alpha_\lambda \alpha_\lambda$ has an effect upon the sum of the equal indices $\Sigma q = q_1 + q_2 + \ldots + q_m$, where m is the class of the permutation *quâ* equal contacts. When such a contact has σ letters to the left of it, it adds $\sigma+1$ to the equal index of the permutation. This happens in the case of

$$\frac{(S-2)!}{i_1!\, i_2! \ldots i_s!} \cdot 2\binom{i_\lambda}{2}$$

permutations; so that, altogether, the contact adds to Σq the number

$$\sum_{\sigma=0}^{\sigma=S-2} \frac{(S-2)!}{i_1!\, i_2! \ldots i_s!} \cdot 2\binom{i_\lambda}{2}(\sigma+1),$$

or

$$\frac{S!}{i_1!\, i_2! \ldots i_s!}\binom{i_\lambda}{2}.$$

Hence the contact $\alpha_\lambda \alpha_\lambda$, on the average, adds the number

$$\binom{i_\lambda}{2}$$

to the equal index of a permutation. Thence it appears that the average value of the equal index is

$$\Sigma\binom{i_1}{2}$$

(see Art. 19).

Moreover, the contact $\alpha_\lambda \alpha_\lambda$, on the average, adds to the sum

$$q_1^r + q_2^r + \ldots + q_m^r$$

(where $q=q_1+q_2+\ldots.+q_m$) the number

$$2\frac{1^\nu+2^\nu+\ldots.+(S-1)^\nu}{S(S-1)}\binom{i_\lambda}{2};$$

and thence the average value of $q_1^\nu+q_2^\nu+\ldots.+q_m^\nu$ is

$$2\cdot\frac{1^\nu+2^\nu+\ldots.+(S-1)^\nu}{S(S-1)}\Sigma\binom{i_1}{2}.$$

The average values of $q_1^2+q_2^2+\ldots.+q_m^2$ and $q_1^3+q_2^3+\ldots.+q_m^3$ respectively are

$$\frac{1}{3}(2\Sigma i_1-1)\Sigma\binom{i_1}{2}$$

and

$$\frac{1}{2}\Sigma i(\Sigma i-1)\Sigma\binom{i_1}{2}.$$

Moreover, putting $\nu=0$, we find for the average class, *quâ* the equal index,

$$\frac{2}{\Sigma i_1}\Sigma\binom{i_1}{2},$$

agreeing with the result of Art. 28.

An easy corollary is that the average value of q_g^ν $(q_g\neq 0)$ is

$$\frac{1}{\Sigma i-1}\{1^\nu+2^\nu+\ldots.+(\Sigma i-1)^\nu\}.$$

In fact q_g^ν and p_g^ν have the same average values in respect of those permutations for which they exist as positive non-zero numbers.

§ 2. *The Indices of the Second Genus.*

30. When we come to consider the permutations of $\alpha^i\beta^j\gamma^k$, we find that the value of $p-r$ does not follow any simple law, and moreover the only linear function of p, q and r that does so is the sum $p+q+r$. I now import a new idea into the subject by defining the major and minor indices of a permutation.

The Major Index of a Permutation. Consider two adjacent letters $\alpha_\mu\alpha_\lambda$ $(\mu>\lambda)$ of a permutation of the assemblage

$$\alpha_1^{i_1}\alpha_2^{i_2}.\ldots.\alpha_s^{i_s}.$$

If $\alpha_\mu\alpha_\lambda$ be the g-th major contact of the permutation, counting from the beginning, and α_μ be the p_g-th letter of the permutation, put

$$(\mu-\lambda)p_g=P_g.$$

Then $\sum_{g=1}^{g=m}(\mu-\lambda)p_g$ or $\sum_{g=1}^{g=m}P_g$ is the major index of the permutation and m is the class of the permutation *quâ* the major index. We put the major index equal to P, so that

$$P_1+P_2+\ldots.+P_m=P.$$

31. *The Minor Index of a Permutation.* Consider two adjacent letters $\alpha_\lambda\alpha_\mu$ $(\mu>\lambda)$ of a permutation.

If $\alpha_\lambda\alpha_\mu$ be the g-th minor contact of the permutation, counting from the beginning, and α_λ be the r_g-th letter of the permutation, put

$$(\mu-\lambda)r_g=R_g.$$

Then $\sum_{g=1}^{g=m}(\mu-\lambda)r_g$ or $\sum_{g=1}^{g=m}R_g$ is the minor index of the permutation and m is the class of the permutation *quâ* the minor index. We put the minor index equal to R, so that

$$R_1+R_2+\ldots.+R_m=R.$$

In this system of indices the equal index is zero since $(\mu-\lambda)$ vanishes for $\lambda=\mu$.

Ex. gr., for the permutation $\alpha_4\alpha_1\alpha_3\alpha_1\alpha_5$

$$\text{Major index}=3\cdot1+2\cdot3=9,$$
$$\text{Minor index}=2\cdot2+4\cdot4=20.$$

It will be at once noticed that for the particular assemblage $\alpha^i\beta^j$ the major and minor indices are the same as the greater and lesser indices respectively.

32. *Theorem in regard to P—R. We have now the important circumstance that, for the most general assemblage, the number P—R has only s values corresponding to the s different letters that may terminate the permutations.*

This will now be established. To bring something definite under the reader's eye I give the major and minor indices of the permutations of $\alpha\beta\gamma\delta$.

	P	R	$P-R$		P	R	$P-R$
$\alpha\beta\gamma\delta$	0	6	−6	$\alpha\beta\delta\gamma$	3	5	−2
$\alpha\gamma\beta\delta$	2	8	−6	$\alpha\delta\beta\gamma$	4	6	−2
$\beta\alpha\gamma\delta$	1	7	−6	$\beta\alpha\delta\gamma$	4	6	−2
$\beta\gamma\alpha\delta$	4	10	−6	$\beta\delta\alpha\gamma$	6	8	−2
$\gamma\alpha\beta\delta$	2	8	−6	$\delta\alpha\beta\gamma$	3	5	−2
$\gamma\beta\alpha\delta$	3	9	−6	$\delta\beta\alpha\gamma$	4	6	−2
	P	R	$P-R$		P	R	$P-R$
$\alpha\gamma\delta\beta$	6	4	2	$\beta\gamma\delta\alpha$	9	3	6
$\alpha\delta\gamma\beta$	5	3	2	$\beta\delta\gamma\alpha$	8	2	6
$\gamma\alpha\delta\beta$	8	6	2	$\gamma\beta\delta\alpha$	10	4	6
$\gamma\delta\alpha\beta$	6	4	2	$\gamma\delta\beta\alpha$	7	1	6
$\delta\alpha\gamma\beta$	6	4	2	$\delta\beta\gamma\alpha$	8	2	6
$\delta\gamma\alpha\beta$	5	3	2	$\delta\gamma\beta\alpha$	6	0	6

The symmetry of these numbers is striking. It gives a notion of the theorems to be established. Let

$$(\alpha_1^{i_1}\alpha_2^{i_2}\ldots.\alpha_s^{i_s})$$

denote any permutation of the assemblage $\alpha_1^{i_1}\alpha_2^{i_2}\ldots.\alpha_s^{i_s}$, and let

$$(\alpha_1^{i_1}\alpha_2^{i_2}\ldots.\alpha_s^{i_s})_l$$

denote any permutation of the same assemblage which terminates with α_l. Assume that, for every permutation

$$(\alpha_1^{i_1}\alpha_2^{i_2}\ldots.\alpha_s^{i_s})_l,$$

$P-R$ has the value $-l\Sigma i+i_1+2i_2+3i_3+\ldots.+si_s$. Now consider the value of $P-R$ when α_k is placed as terminal letter to the above permutation. We have three cases to consider.

(i) If $k<l$, P for $(\alpha_1^{i_1}\alpha_2^{i_2}\ldots.\alpha_s^{i_s})_l\,\alpha_k$ becomes $P+(l-k)\Sigma i$ and R remains unaltered, so that $P-R$ becomes

$$-k(\Sigma i+1)+i_1+2i_2+\ldots.+k(i_k+1)+\ldots.+si_s,$$

and, since $\Sigma i+1$ and i_k+1 are the new values of Σi and i_k, we see that the value of $P-R$ is independent of l and thus obeys the law assumed.

(ii) If $k=l$, P and R are both unaltered and we may write the value of $P-R$

$$-l(\Sigma i+1)+i_1+2i_2+\ldots.+l(i_l+1)+\ldots.+si_s,$$

which is in accord with the assumed law.

(iii) If $k>l$, P is unchanged but R is increased by $(k-l)\Sigma i$, so that $P-R$ is diminished by $(k-l)\Sigma i$ and thus becomes

$$-k(\Sigma i+1)+i_1+2i_2+\ldots.+k(i_k+1)+\ldots.+si_s,$$

and again the assumed law is verified.

Therefore by induction from simple cases the general law that for the permutations of $(\alpha_1^{i_1}\alpha_2^{i_2}\ldots.\alpha_s^{i_s})_l$ we have

$$P-R=-l\Sigma i+i_1+2i_2+\ldots.+si_s$$

is established.

33. The law is, however, more extensive than this, for, instead of having letters $\alpha_1, \alpha_2, \ldots., \alpha_s$ with subscripts forming the series of natural numbers, the subscripts may be any numbers whatever. Thus, take the assemblage to be

$$\alpha_{m_1}^{i_{m_1}}\alpha_{m_2}^{i_{m_2}}\ldots.\alpha_{m_s}^{i_{m_s}},$$

where $m_1, m_2, \ldots., m_s$ are *any* integers, of which one is l. Assume the law

$$P-R \text{ for } \left(\alpha_{m_1}^{i_{m_1}}\alpha_{m_2}^{i_{m_2}}\ldots.\alpha_{m_s}^{i_{m_s}}\right)_l=-l\Sigma i+m_1i_{m_1}+m_2i_{m_2}+\ldots.+li_l+\ldots.+m_si_{m_s}.$$

Now adding α_k to the permutation as before we get for the new value of $P-R$

$$-k(\Sigma i+1)+m_1 i_{m_1}+m_2 i_{m_2}+\ldots.+k(i_k+1)+\ldots.+m_s i_{m_s},$$

whether k be $>$, $=$ or $<l$ and whether k is one of the numbers $m_1, m_2, \ldots., m_s$ or not. This is independent of l and obeys the assumed law, so that the general law is established by induction. Setting forth the values of $P-R$ for the assemblages $\alpha_1^{i_1}\alpha_2^{i_2}$, $\alpha_1^{i_1}\alpha_2^{i_2}\alpha_3^{i_3}$, ... we obtain the schemes

$$\begin{array}{l|llll} \alpha_2 & -i_1 & & & \\ \alpha_1 & & +i_2, & & \end{array} \qquad \begin{array}{l|llll} \alpha_3 & -2i_1 & -i_2 & & \\ \alpha_2 & -i_1 & & +i_3 & \\ \alpha_1 & & +i_2 & +2i_3, & \end{array} \qquad \begin{array}{l|llll} \alpha_4 & -3i_1 & -2i_2 & -i_3 & \\ \alpha_3 & -2i_1 & -i_2 & & +i_4 \\ \alpha_2 & -i_1 & & +i_3 & +2i_4 \\ \alpha_1 & & i_2 & +2i_3 & +3i_4, \end{array} \qquad \text{etc.,}$$

the letters α denoting the terminal letters of the permutations.

The values for the terminal letters α_m, α_{s-m+1} are symmetrically related. Each line of a scheme is derived from the preceding line by adding Σi.

34. *General Relation between Σx^P and Σx^R.* It is not true in general that $\Sigma x^P=\Sigma x^R$, although this equality happens to obtain for the particular case of the assemblage $\alpha_1^{i_1}\alpha_2^{i_2}\alpha_3^{i_3}$, as will be seen presently. The above investigation shows that if $\sum_l x^P$ refers to the permutations which terminate with α_l,

$$\sum_l x^P = x^{-l\Sigma i+i_1+2i_2+\ldots.+si_s}\sum_l x^R.$$

This appears to be the most general relation obtainable, but, if the assemblage be unaltered by the substitution

$$\begin{pmatrix}\alpha_1\alpha_2 & \alpha_3 & \ldots.\alpha_s\\ \alpha_s\alpha_{s-1}\alpha_{s-2}\ldots.\alpha_1\end{pmatrix},$$

it is clear that the P and R of any permutation are the same as the R and P respectively of the permutation obtained by impressing the substitution. Thus in this case

$$\sum_l x^P = \sum_{s-l+1} x^R = x^{l\Sigma i-i_1-2i_2-\ldots.-si_s}\sum_{s-l+1} x^P$$

and

$$\Sigma x^P=\Sigma x^R.$$

35. *Difference Equation for Σx^P.* Let

$$\Sigma x^P = v(i_1, i_2, \ldots., i_s),$$

and

$$\sum_m x^P = \underset{m}{v}(i_1, i_2, \ldots., i_s).$$

It is clear that $\underset{s}{v}(i_1, i_2, \ldots., i_s)=v(i_1, i_2, \ldots., i_{s-1}, i_s-1)$. The function $\underset{s-1}{v}(i_1, i_2, \ldots., i_s)$ refers to permutations that may end with $\alpha_s\alpha_{s-1}$; these give rise to the function

$$x^{\Sigma i-1}\underset{s}{v}(i_1, i_2, \ldots., i_{s-1}-1, i_s);$$

the remaining permutations give rise to the function

$$v(i_1, i_2, \ldots, i_{s-1}-1, i_s) - \underset{s}{v}(i_1, i_2, \ldots, i_{s-1}-1, i_s).$$

Hence

$$\underset{s-1}{v}(i_1 i_2 \ldots i_s) = v(i_1, i_2, \ldots, i_{s-1}-1, i_s) - (\Sigma \mathbf{i}-\mathbf{1})\, v(i_1, i_2, \ldots, i_{s-1}-1, i_s-1),$$

where $(\Sigma \mathbf{i}-\mathbf{1})$ means $1-x^{\Sigma i-1}$.

By similar reasoning we obtain

$$\begin{aligned}\underset{s-2}{v}(i_1 i_2 \ldots i_s) = {} & v(i_1, i_2, \ldots, i_{s-2}-1, i_{s-1}, i_s) \\ & -(2\Sigma \mathbf{i}-\mathbf{2})\, v(i_1, i_2, \ldots, i_{s-2}-1, i_{s-1}, i_s-1) \\ & -(\Sigma \mathbf{i}-\mathbf{1})\, v(i_1, i_2, \ldots, i_{s-2}-1, i_{s-1}-1, i_s) \\ & +(\Sigma \mathbf{i}-\mathbf{1})(\Sigma \mathbf{i}-\mathbf{2})\, v(i_1, i_2, \ldots, i_{s-2}-1, i_{s-1}-1, i_s-1),\end{aligned}$$

and the general formula is easy to obtain, but we do not need it at present. An application will be made at once to a particular case.

36. *The Assemblage* $\alpha^i \beta^j \gamma^k$. Since $v(i,j,k) = \underset{3}{v}(i,j,k) + \underset{2}{v}(i,j,k) + \underset{1}{v}(i,j,k)$, we find

$$\begin{aligned}\Sigma x^P = v(i, j, k) = {} & v(i, j, k-1) + v(i, j-1, k) + v(i-1, j, k) \\ & -(\Sigma \mathbf{i}-\mathbf{1})\, v(i, j-1, k-1,) - (\Sigma \mathbf{i}-\mathbf{1})\, v(i-1, j-1, k) \\ & -(2\Sigma \mathbf{i}-\mathbf{2})\, v(i-1, j, k-1) + (\Sigma \mathbf{i}-\mathbf{1})(\Sigma \mathbf{i}-\mathbf{2})\, v(i-1, j-1, k-1).\end{aligned}$$

Also interchanging i and k,

$$\begin{aligned}v(k, j, i) = {} & v(k, j, i-1,) + v(k, j-1, i) + v(k-1, j, i) \\ & -(\Sigma \mathbf{i}-\mathbf{1})\, v(k, j-1, i-1) - (\Sigma \mathbf{i}-\mathbf{1})\, v(k-1, j-1, i) \\ & -(2\Sigma \mathbf{i}-\mathbf{2})\, v(k-1, j, i-1) + (\Sigma \mathbf{i}-\mathbf{1})(\Sigma \mathbf{i}-\mathbf{2})\, v(k-1, j-1, i-1).\end{aligned}$$

These results enable us to establish that

$$v(i, j, k) = v(k, j, i);$$

for if this law obtains in all cases when the sum of the three numbers i, j, k is less than w, it is seen that the right-hand sides of the expressions obtained for $v(i, j, k)$ and $v(k, j, i)$ are identical. If then we can show that the law obtains when $i+j+k$ is equal to any number w in all cases, the law must be universal.

We may note in the first place that

$$v(0, j, k) = \frac{(\mathbf{1})(\mathbf{2}) \ldots (\mathbf{j}+\mathbf{k})}{(\mathbf{1})(\mathbf{2}) \ldots (\mathbf{j}) \cdot (\mathbf{1})(\mathbf{2}) \ldots (\mathbf{k})} = F_{j,k}(x),$$

$$v(i, j, 0) = F_{i,j}(x), \quad v(i, 0, k) = F_{i,k}(x^2);$$

and that for these forms the law obtains. It follows that the law obtains when $w=2$, for then the forms are

$$v(2, 0, 0), \quad v(0, 2, 0), \quad v(0, 0, 2), \quad v(1, 1, 0), \quad v(1, 0, 1), \quad v(0, 1, 1).$$

Hence universally

$$v(i, j, k) = v(k, j, i).$$

The function is therefore symmetrical in i and k.

37. *Proof that for the Assemblage* $\alpha^i\beta^j\gamma^k$, $\Sigma x^P = \Sigma x^R$. Since the value of P for any permutation of the assemblage is equal to the value of R for the permutation obtained from the former by the substitution $(\alpha\gamma)$, it follows that Σx^P for the assemblage $\alpha^i\beta^j\gamma^k$ is equal to Σx^R for the assemblage $(\alpha^k\beta^j\gamma^i)$. Since, moreover, the substitution has been shown above not to alter the expression Σx^P, it follows that universally $\Sigma x^P = \Sigma x^R$.

38. *The Algebraic Expression of* Σx^P *for the Assemblage* $\alpha^i\beta^j\gamma^k$. Since it has been shown that the functions $u(0, j, k)$, $u(i, 0, k)$, $u(i, j, 0)$ are equal to $F_{j,k}(x)$, $F_{i,k}(x^2)$, $F_{i,j}(x)$ respectively, we are led to consider the function

$$\frac{(1)\,(2)\,\ldots\,(\mathbf{i+j+k})}{(1)\,(2)\,\ldots\,(\mathbf{j})\cdot(1)\,(2)\,\ldots\,(\mathbf{i+k})}\cdot\frac{(2)\,(4)\,\ldots\,(\mathbf{2i+2k})}{(2)\,(4)\,\ldots\,(\mathbf{2i})\cdot(2)\,(4)\,\ldots\,(\mathbf{2k})},$$

which possesses these properties and also becomes equal to the number of permutations of $\alpha^i\beta^j\gamma^k$ when $x=1$. It is, moreover, symmetrical in i and k, and is obviously correct in simple particular cases. *Ex. gr.*, it is $\frac{(3)\,(4)}{(1)\,(2)}$ when $i=j=k=1$.

In fact, it will be shown subsequently that

$$\Sigma x^P = F_{j,i+k}(x)\cdot F_{i,k}(x^2);$$

and it has been verified that this expression satisfies the difference equation for $v(i, j, k)$ obtained above.

Another verification of our results is obtained by showing algebraically that Σx^R has the same expression as Σx^P.

39. *Derivation of* Σx^R *from* Σx^P. Since $\Sigma x^R = \Sigma x^{R-P+P} = x^{R-P}\Sigma x^P$ for all sets of permutations for which $R-P$ has a constant value, we must have

$$\Sigma x^R = x^{2i+j}\underset{3}{v}(i, j, k) + x^{i-k}\underset{2}{v}(i, j, k) + x^{-j-2k}\underset{1}{v}(i, j, k);$$

and this will be found to be equal to the product of

$$\frac{(1)\,(2)\,\ldots\,(\mathbf{i+j+k-1})}{(1)\,(2)\,\ldots\,(\mathbf{j})\cdot(1)\,(2)\,\ldots\,(\mathbf{i+k})}\cdot\frac{(2)\,(4)\,\ldots\,(\mathbf{2i+2k-2})}{(2)\,(4)\,\ldots\,(\mathbf{2i})\cdot(2)\,(4)\,\ldots\,(\mathbf{2k})}$$

and

$$\begin{aligned}&x^{2i+j}(\mathbf{i+k})\,(\mathbf{2k}) + x^{i-k}(\mathbf{j})\,(\mathbf{2i+2k}) - x^{i-k}(\mathbf{j})\,(\mathbf{i+k})\,(\mathbf{2k})\\ &\quad + x^{-j-2k}(\mathbf{i+k})\,(\mathbf{2i}) - x^{j-2k}(\mathbf{j})\,(\mathbf{i+k})\,(\mathbf{2i}) - x^{-2k}(\mathbf{i+k})\,(\mathbf{2i})\,(\mathbf{2k}).\end{aligned}$$

The expression last written is found to be

$$(\mathbf{i+j+k})\,(\mathbf{2i+2k}),$$

establishing that

$$\Sigma x^R = \frac{(1)\,(2)\,\ldots\,(i+j+k)}{(1)\,(2)\,\ldots\,(j)\cdot(1)\,(2)\,\ldots\,(i+k)}\cdot\frac{(2)\,(4)\,\ldots\,(2i+2k)}{(2)\,(4)\,\ldots\,(2i)\cdot(2)\,(4)\,\ldots\,(2k)},$$

the expression already found for Σx^P.

40. *Evaluation of* Σx^{P+R}. We could also find the expression of $\Sigma x^{\lambda P+\mu R}$, where λ, μ are arbitrary positive or negative integers. To find Σx^{P+R} we consider

$$x^{R-P}\Sigma x^{2P}$$

in three fragments and if $\Sigma x^P = G_{i,j,k}(x)$, we find

$$\Sigma x^{P+R} = x^{2i+j}\underset{3}{G}_{i,j,k}(x^2) + x^{i-k}\underset{2}{G}_{i,j,k}(x^2) + x^{-j-2k}\underset{1}{G}_{i,j,k}(x^2),$$

where $\underset{t}{u}(i,j,k)$ is written $\underset{t}{G}_{i,j,k}(x)$.

I thence find

$$\Sigma x^{P+R} = \frac{(2)\,(4)\,\ldots\,(2i+2j+2k-2)\,(+i+j+k)}{(2)\,(4)\,\ldots\,(2j)\cdot(2)\,(4)\,\ldots\,(2i+2k-2)}$$
$$\cdot\frac{(4)\,(8)\,\ldots\,(4i+4k-4)}{(4)\,(8)\,\ldots\,(4i)\cdot(4)\,(8)\,\ldots\,(4k)}\cdot(x^{2i}+x^{2k})\,(x^j+x^{i+k}).$$

Ex. gr., for $i=2$, $j=1$, $k=1$, I thus find

$$\Sigma x^{P+R} = x^3 + 3x^5 + 4x^7 + 3x^9 + x^{11},$$

which the reader can readily verify from the permutations of $\alpha^2\beta\gamma$.

41. *Average Values of the Major and Minor Indices.* Following the method of Art. 18, we have only to differentiate the major or minor index function of the assemblage $\alpha^i\beta^j\gamma^k$, viz.

$$F_{j,i+k}(x)\,F_{i,k}(x^2),$$

with regard to x, and then to put $x=1$, to obtain the value of ΣP or ΣR, and thence the mean value of P or R.

The mean value is in fact

$$\frac{\{\partial_x F_{j,i+k}(x)\}_{x=1}}{\binom{i+j+k}{j}} + \frac{\{\partial_x F_{i,k}(x^2)\}_{x=1}}{\binom{i+k}{i}},$$

which, making use of results obtained above, is found to be

$$\frac{1}{2}j(i+k)+ik$$

or

$$\frac{1}{2}(ij+2ik+jk).$$

42. The mean value of $P(P-1)$ or $R(R-1)$ is similarly found to be

$$\frac{\{\partial_x^2 F_{j,i+k}(x)\}_{x=1}}{\binom{i+j+k}{j}} + 2\frac{\{\partial_x F_{j,i+k}(x)\}_{x=1}}{\binom{i+j+k}{j}} \cdot \frac{\{\partial_x F_{i,k}(x^2)\}_{x=1}}{\binom{i+k}{i}} + \frac{\{\partial_x^2 F_{i,k}(x^2)\}_{x=1}}{\binom{i+k}{i}};$$

and, making use of previous results, this is found to be

$$\frac{1}{12}\{12i^2k^2+4i^2k+4ik^2-8ik+j(12i^2k+12ik^2+i^2+k^2+2ik-5i-5k) \\ +j^2(3i^2+3k^2+6ik+i+k)\},$$

symmetrical in i, k, as should be the case.

Adding to the above the mean value of P, viz. $\frac{1}{2}(ij+2ik+jk)$, we find for the mean value of P^2 the expression

$$\frac{1}{12}\{4ik(3ik+i+k+1)+j(i+k)(12ik+i+k+1)+j^2(i+k)(3i+3k+1)\}.$$

There is of course no theoretical difficulty in obtaining the mean values of higher powers of P.

43. *Mean Values Discussed in the Second Manner.* Following the procedure taken with regard to the greater, equal and lesser indices, we can obtain some results which appertain to the major and minor indices of the general assemblage $\alpha_1^{i_1}\alpha_2^{i_2}\ldots.\alpha_s^{i_s}$.

I suppose that P is formed by addition in the manner

$$P=P_1+P_2+\ldots.+P_m,$$

where m is the class of the permutation *quâ* the major index.

We inquire into the effect which the particular major contact $\alpha_{\mu+\nu}\alpha_\mu$ has upon the sum of the indices ΣP. When the contact has σ letters to its left and $\Sigma i-\sigma-2$ letters to its right, it adds $\nu(\sigma+1)$ to the index of the permutation. This happens in the case of

$$\frac{(\Sigma i-2)!\, i_\mu i_{\mu+\nu}}{i_1!\, i_2!\ldots.i_s!}$$

permutations; so that, altogether, the contact adds to ΣP the number

$$\sum_{\sigma=0}^{\sigma=\Sigma i-2} \frac{(\Sigma i-2)!\, i_\mu i_{\mu+\nu}}{i_1!\, i_2!\ldots.i_s!}\nu(\sigma+1),$$

which is

$$\frac{1}{2}\nu\frac{(\Sigma i)!}{i_1!\, i_2!\ldots.i_s!} i_\mu i_{\mu+\nu}.$$

Hence, on the average, it adds the number

$$\frac{1}{2}\nu\, i_\mu i_{\mu+\nu}$$

to the major index of a permutation.

Also the $s-\nu$ contacts $\alpha_{\nu+1}\alpha_1, \alpha_{\nu+2}\alpha_2, \ldots, \alpha_s\alpha_{s-\nu}$ add, on the average, the number

$$\frac{1}{2}\nu(i_1 i_{\nu+1}+i_2 i_{\nu+2}+\ldots+i_{s-\nu} i_s)$$

to the index.

Now making ν vary from 1 to $s-1$, we find that the average value of the major index is

$$\frac{1}{2}\sum_{\mu=2}^{\mu=s}\sum_{\lambda=1}^{\lambda=\mu-1}(\mu-\lambda) i_\lambda i_\mu,$$

where $\mu>\lambda$, a result which clearly embraces and generalizes that already found for the assemblage $\alpha^i\beta^j\gamma^k$ from other principles.

44. The result is essentially the average value of $P_1+P_2+\ldots+P_m$. To find the average value of $P_1^t+P_2^t+\ldots+P_m^t$, we have merely to note that, under circumstances such as the foregoing, the contact $\alpha_{\mu+\nu}\alpha_\nu$ adds the number

$$\sum_{\sigma=0}^{\sigma=\Sigma i-2}\frac{(\Sigma i-2)!\, i_\mu i_{\mu+\nu}}{i_1! i_2!\ldots i_s!}\nu^t(\sigma+1)^t$$

to the sum ΣP_1^t. Hence we readily find that

$$\text{Average value of } \Sigma P_1^2=\frac{1}{6}(2\Sigma i-1)\sum_{\mu=2}^{\mu=s}\sum_{\lambda=1}^{\lambda=\mu-1}(\mu-\lambda)^2 i_\lambda i_\mu, \text{ where } \mu>\lambda;$$

$$\text{Average value of } \Sigma P_1^3=\frac{1}{4}(\Sigma i)(\Sigma i-1)\sum_{\mu=2}^{\mu=s}\sum_{\lambda=1}^{\lambda=\mu-1}(\mu-\lambda)^3 i_\lambda i_\mu, \text{ where } \mu>\lambda.$$

There is no difficulty in continuing the series, which ultimately, of course, involve the numbers of Bernouilli.

It will be noted that the above results enable us to find the average value of

$$\Sigma P_1 P_2.$$

Applied to the assemblage $\alpha^i\beta^j\gamma^k$, we find

$$\text{Average value of } \Sigma P_1^2=\frac{1}{6}(2i+2j+2k-1)(4ik+ij+jk),$$

$$\text{Average value of } \Sigma P_1^3=\frac{1}{4}(i+j+k)(i+j+k-1)(8ik+ij+jk),$$

. . . . . .

45. The same course of reasoning applies to the minor index

$$R=R_1+R_2+\ldots+R_m,$$

and we thus find that, for the most general assemblage,

$$\Sigma R=\Sigma P;$$

and that the average values of ΣR_1^t and ΣP_1^t are identical. This is somewhat remarkable, because the minor index function Σx^R is not equal in general to the

major index function Σx^P; nor is there in general any one-to-one equality between the values of R and the values of P for the same assemblage. The reader may test this statement by looking into the indices of the assemblage $\alpha_1^{i_1}\alpha_2^{i_2}\alpha_3^{i_3}\alpha_4^{i_4}$ when $i_1=i_2=i_4=1$ and $i_3=0$. However, there is a one-to-one equality between the numbers P_ϱ and the numbers R_ϱ for the whole of the permutations of the assemblage. Indeed, if this were not the case, we could not have

$$\Sigma P_\varrho^t = \Sigma R_\varrho^t,$$

the summation being for all the permutations, and our results absolutely necessitate this.

46. *General Connection between Σx^P and Partitions.* With the object of investigating the functions Σx^P, Σx^R, I next place them in general connection with the theory of partitions. Consider any permutation

$$\beta\alpha\gamma\gamma\alpha\gamma\beta\alpha\gamma$$

of any assemblage $\alpha^3\beta^2\gamma^4$. Associate with it three rows of numbers

$$\begin{array}{l} a_1\, a_2\, a_3 \\ b_1\, b_2 \\ c_1\, c_2\, c_3\, c_4 \end{array}$$

the numbers in each row being in descending order of magnitude from left to right. I suppose the permutation to correspond to a descending order of magnitude, subject to conditions, of these nine numbers. I regard the permutation as yielding the Diophantine relations

$$\begin{array}{ccccccccccc} \beta & & \alpha & \gamma & \gamma & \overbrace{\alpha} & & \overbrace{\alpha} & \gamma & \beta & \alpha & \gamma \\ b_1 > a_1 \geq & & c_1 \geq & c_2 > & & a_2+1\,; & & a_2 \geq & c_3 > & b_2 > & a_3 \geq & c_4\,, \end{array}$$

where it will be noted that in correspondence with the letters α, β, γ, we have the letters a, the letters b and the letters c, each in the *given* descending order. The contacts $\beta\alpha$, $\gamma\beta$ are associated with the symbol $>$; the contact $\gamma\alpha$ also with the symbol $>$; but the letter a, corresponding to α, in this case is increased by unity. So, in general, when the contact is $\alpha_{\mu+\nu}\alpha_\mu$, ν being >0, the symbol is $>$, but the letter corresponding to α_μ is increased by $\nu-1$. When the contact is $\alpha_\mu\alpha_{\mu+\nu}$, where ν is zero or positive, the symbol is $\geq$, and the letter corresponding to $\alpha_{\mu+\nu}$ is not increased or diminished.

Under these circumstances the enumerating generating function of the numbers a, b, c, which satisfy the conditions, is

$$\frac{x^{22}}{(1)\,(2)\,\ldots\,(9)},$$

where the number which is the exponent of x in the numerator is necessarily the major index of the permutation (compare Art. 6). If we go through a like process for every permutation of the given assemblage and add the generating functions, we find

$$\frac{\Sigma x^P}{(1)\,(2)\,\ldots\,(9)},$$

Σx^P being the major index function of the given assemblage $\alpha^3\beta^2\gamma^4$.

47. The question now arises: What have we enumerated by this generating function? We clearly have not counted the number of ways of arranging the numbers a, b, c in the three rows so that there may be descending order in each row and also a sum equal to a given number w, for that is what we succeeded in doing by means of the greater index function Σx^p. We have in fact eliminated certain arrangements from those under enumeration. It is easy to see that with reference to the contacts $\alpha\gamma$, $\gamma\alpha$, the former gives

$$\text{Either } a_1 \text{ or } a_2 \text{ or } a_3 \geq \text{either } c_1,\ c_2,\ c_3, \text{ or } c_4,$$

while the latter gives

$$\text{Either } c_1 \text{ or } c_2 \text{ or } c_3 \text{ or } c_4 > \text{either } a_1+1 \text{ or } a_2+1 \text{ or } a_3+1,$$

so that in no circumstances can any number in the third row be *one* greater than any number in the first row. There is no other condition, so that we have enumerated all of the arrangements that were connected with the greater index function *except* those which involve any third row number greater by one than any first row number. Having obtained the precise condition of partition corresponding to the major index function of the assemblage $\alpha^i\beta^j\gamma^k$, we are now in a position to investigate the major index function. Before doing this it is convenient to establish the like conditions that are associated with the major index function of the general assemblage $\alpha_1^{i_1}\alpha_2^{i_2}\ldots\alpha_s^{i_s}$.

48. In correspondence with the contact $\alpha_\mu\alpha_{\mu+\nu}$, we have as a condition of summation

$$\text{Some } \mu\text{-th row number} \geq \text{some } (\mu+\nu)\text{-th row number},$$

and for the contact $\alpha_{\mu+\nu}\alpha_\mu$

$$\text{Some } (\mu+\nu)\text{-th row number} > \text{some } \mu\text{-th row number} + \nu-1;$$

hence under no circumstances can a $(\mu+\nu)$-th row number be greater than a μ-th row number by any of the numbers

$$1, 2, 3, \ldots, \nu-1.$$

In regard to the rows under examination, there are thus $(\nu-1)i_\mu i_{\mu+\nu}$ conditions. Between adjacent rows there are no conditions. Altogether there are

$$\Sigma(\nu-1)i_\mu i_{\mu+\nu}$$

conditions, the summation extending to every pair of the numbers $i_1, i_2, \ldots, i_s$ whose subscripts differ, at least, by the number 2.

If we now find the function which enumerates the ways in which numbers, of given sum, can be arranged in s rows (containing respectively $i_1, i_2, \ldots, i$ numbers, zero being permissible as a number) in descending order in each row so that the above-mentioned $\Sigma(\nu-1)i_\mu i_{\mu+\nu}$ conditions are satisfied, we shall evidently obtain an expression equivalent to

$$\frac{\Sigma x^P}{(1)\,(2)\ldots.\,(\Sigma i)},$$

and thus we can arrive at the form of the major index function Σx^P.

The minor index function Σx^R presents no further difficulty, since it was shown in Art. 34 that Σx^R for the assemblage $\alpha_1^{i_1}\alpha_2^{i_2}\ldots.\alpha_s^{i_s}$ is identical with Σx^P for the assemblage $\alpha_1^{i_1}\alpha_2^{i_2-1}\ldots.\alpha_s^{i_s}$.

49. *Determination of Σx^P for the Assemblage $\alpha^i\beta^j\gamma^k$.* The associated problem in partitions is merely concerned with the first and third rows of the three rows of numbers. We have thus two rows

$$a_1a_2\ldots.a_i,$$
$$c_1c_2\ldots.c_k,$$

and we have to find $\Sigma x^{a_1+a_2+\ldots.+a_i+c_1+c_2+\ldots.+c_k}$ subject to the conditions

$$a_1 \geq a_2 \geq a_3 \geq \ldots. \geq a_i,$$
$$c_1 \geq c_2 \geq c_3 \geq \ldots. \geq c_k,$$

with the added set of conditions that no c is to be one greater than any a. Let this function be $\phi_{i,k}(x)$. Then we shall have

$$\frac{\Sigma x^P}{(1)\,(2)\ldots.\,(i+j+k)} = \frac{\phi_{i,k}(x)}{(1)\,(2)\ldots.\,(j)}.$$

We will first determine $\phi_{1,1}(x)$. We have to find Σx^{a+c} subject to the single condition that c is not to be equal to $a+1$. This is very simple, because we have merely to exclude the terms for which $c=a+1$ from the complete function $\frac{1}{(1)^2}$. Thus

$$\phi_{1,1}(x) = \frac{1}{(1)_2} - \Sigma x^{2a+1} = \frac{1}{(1)^2} - \frac{1}{(2)} = \frac{(4)}{(1)\,(2)^2}.$$

It is, however, more advantageous to proceed by the method that is of general application. For this purpose I construct the function

$$\chi\frac{\lambda}{(1-\lambda x)\left(1-\frac{x}{\lambda}\right)},$$

where the symbol χ denotes that the function is to be expanded in ascending powers of the variable; that, from such expansion, all terms are to be rejected which do not involve either a positive or a negative power of λ; and that then λ is to be put equal to unity. *Ex. gr.*, the general term is

$$\lambda(\lambda x)^a\left(\frac{x}{\lambda}\right)^c \quad \text{or} \quad \lambda^{a+1-c}x^{a+c};$$

so that the symbol χ rejects terms for which $c=a+1$, and thus the final function is Σx^{a+c}, the sum being for all positive integers a, c except those for which $c=a+1$. This is the required condition, so that

$$\varphi_{1,1}(x)=\chi\frac{\lambda}{(1-\lambda x)\left(1-\frac{x}{\lambda}\right)}.$$

Observe that the χ function is *necessarily* unaltered when we write $\frac{1}{\lambda}$ for λ, although its form is changed. We have the identity

$$\frac{\lambda}{(1-\lambda x)\left(1-\frac{x}{\lambda}\right)}=\frac{\lambda}{(2)\,(1-\lambda x)}+\frac{\frac{x^2}{\lambda}}{(2)\left(1-\frac{x}{\lambda}\right)}+\frac{x}{(2)};$$

so that

$$\chi\frac{\lambda}{(1-\lambda x)\left(1-\frac{x}{\lambda}\right)}=\frac{1}{(1)\,(2)}+\frac{x^2}{(1)\,(2)}=\frac{(4)}{(1)\,(2)^2};$$

leading of course to the result that, for the assemblage $\alpha\beta^i\gamma$, Σx^P has the expression

$$\frac{(1)\,(2)\,\ldots.\,(j+2)}{(1)\,(2)\,\ldots.\,(j)}\,\frac{(4)}{(1)\,(2)^2} \quad \text{or} \quad \frac{(j+1)\,(i+2)\cdot(4)}{(1)\,(2)\cdot(2)}.$$

50. To determine $\varphi_{2,1}(x)$ I form a function

$$\chi\cdot\Omega\cdot\frac{\lambda\mu}{(1-A\lambda x)\left(1-\frac{\mu x}{A}\right)\left(1-\frac{x}{\lambda\mu}\right)}.$$

The general term in the expansion of the function is

$$\lambda\mu(A\lambda x)^{a_1}\left(\frac{\mu x}{A}\right)^{a_2}\left(\frac{x}{\lambda\mu}\right)^{c_1} \quad \text{or} \quad A^{a_1-a_2}\lambda^{a_1+1-c_1}\mu^{a_2+1-c_1}x^{a_1+a_2+c_1}.$$

If, therefore, in the expansion, the exponent of A is zero or positive, the condition $a_1\geq a_2$ is satisfied. If also the exponents of λ and μ are neither of them zero, the conditions $a_1+1\neq c_1$, $a_2+1\neq c_1$ are satisfied. The symbol Ω is associated with the auxiliary coefficient A in such wise that its effect upon the ex-

panded function is to abolish all terms which involve A to a negative power. The symbol χ is associated with the auxiliaries λ and μ in such wise that it abolishes from the expansion all terms which do not involve λ and μ (each of them) positively or negatively. Hence, from our definitions

$$\phi_{2,1}(x)=\chi\cdot\Omega\cdot\frac{\lambda\mu}{(1-A\lambda x)\left(1-\frac{\mu x}{A}\right)\left(1-\frac{x}{\lambda\mu}\right)}.$$

The effect of Ω can be seen with little trouble, so that we may write

$$\phi_{2,1}(x)=\chi\cdot\frac{\lambda\mu}{(1-\lambda x)\,(1-\lambda\mu x^2)\left(1-\frac{x}{\lambda\mu}\right)}.$$

The reader will have little difficulty in proving that, if σ be any auxiliary or combination of auxiliaries,

$$\chi\frac{\sigma}{(1-\sigma x^s)\left(1-\frac{x^t}{\sigma}\right)}=\frac{\sigma}{(s+t)\,(1-\sigma x^s)}+\frac{\frac{1}{\sigma}x^{2t}}{(s+t)\left(1-\frac{x^t}{\sigma}\right)},\quad \sigma=1;$$

$$\chi\frac{\frac{1}{\sigma}}{(1-\sigma x^s)\left(1-\frac{x^t}{\sigma}\right)}=\frac{\sigma x^{2s}}{(s+t)\,(1-\sigma x^s)}+\frac{\frac{1}{\sigma}}{(s+t)\left(1-\frac{x^t}{\sigma}\right)},\quad \sigma=1.$$

Applying the first of these results we can eliminate the auxiliary μ. We have

$$\phi_{2,1}(x)=\chi\frac{1}{1-\lambda x}\left\{\frac{\lambda}{(3)\,(1-\lambda x^2)}+\frac{\frac{1}{\lambda}x^2}{(3)\left(1-\frac{x}{\lambda}\right)}\right\}$$

$$=\frac{1}{(1)\,(2)\,(3)}+\frac{x^2}{(3)}\,\chi\frac{\frac{1}{\lambda}}{(1-\lambda x)\left(1-\frac{x}{\lambda}\right)};$$

and, since in the χ function we can write $\frac{1}{\lambda}$ for λ and $\phi_{2,0}=\frac{1}{(1)\,(2)}$, we find

$$\phi_{2,1}(x)=\frac{1}{(3)}\{\phi_{2,0}(x)+x^2\phi_{1,1}(x)\},$$

a suggestive formula of reduction.

51. In a precisely similar manner we find

$$\phi_{3,1}(x)=\chi\cdot\Omega\cdot\frac{\lambda\mu\nu}{(1-A_1\lambda x)\left(1-\frac{A_2\mu x}{A_1}\right)\left(1-\frac{\nu x}{A_2}\right)\cdot\left(1-\frac{x}{\lambda\mu\nu}\right)},$$

where Ω abolishes the terms which involve negative powers of A_1 and A_2, and χ acts in the usual manner on the auxiliaries λ, μ, ν.

We find at once

$$\phi_{3,1}(x) = \chi \frac{\lambda\mu\nu}{(1-\lambda x)(1-\lambda\mu x^2)(1-\lambda\mu\nu x^3)\cdot\left(1-\frac{x}{\lambda\mu\nu}\right)};$$

and now, eliminating ν, we find

$$\phi_{3,1}(x) = \chi \frac{1}{(1-\lambda x)(1-\lambda\mu x^2)} \cdot \left\{ \frac{\lambda\mu}{(4)(1-\lambda\mu x^3)} + \frac{\frac{1}{\lambda\mu}x^2}{(4)\left(1-\frac{x}{\lambda\mu}\right)} \right\},$$

$$= \frac{1}{(4)}\phi_{3,0}(x) + \frac{x^2}{(4)}\chi \frac{\frac{1}{\lambda\mu}}{(1-\lambda x)(1-\lambda\mu x^2)\left(1-\frac{x}{\lambda\mu}\right)}.$$

Now

$$\chi \frac{\frac{1}{\lambda\mu}}{(1-\lambda x)(1-\lambda\mu x^2)\left(1-\frac{x}{\lambda\mu}\right)}$$

is not of the same form as the expression for $\phi_{2,1}(x)$; but it is equivalent to it, because, from the fact that $\phi_{i,k}(x) = \phi_{k,i}(x)$, established above, if we proceed to form $\phi_{1,2}(x)$ we find

$$\phi_{1,2}(x) = \chi \frac{\lambda\mu}{(1-\lambda\mu x)\left(1-\frac{x}{\lambda}\right)\left(1-\frac{x^2}{\lambda\mu}\right)},$$

and herein we may substitute $\frac{1}{\lambda}$ for λ and $\frac{1}{\mu}$ for μ.

Hence

$$\phi_{3,1}(x) = \frac{1}{(4)}\phi_{3,0}(x) + \frac{x^2}{(4)}\phi_{2,1}(x).$$

52. Examination of the foregoing shows that

$$\phi_{s,1}(x) = \frac{1}{(s+1)}\phi_{s,0}(x) + \frac{x^2}{(s+1)}\phi_{s-1,1}(x),$$

and assuming that

$$\phi_{s-1,1}(x) = \frac{(2s)}{(1)(2)^2(3)\ldots(s)}$$

we find

$$\phi_{s,1}(x) = \frac{1}{(1)(2)\ldots(s+1)} + \frac{x^2(1-x^{2s})}{(1)(2)^2(3)\ldots(s+1)} = \frac{(2s+2)}{(1)(2)^2(3)\ldots(s+1)},$$

verifying the assumption.

Hence Σx^p for the assemblage $\alpha^i\beta^j\gamma$ is

$$\frac{(1)(2)\ldots(i+j+1)}{(1)(2)\ldots(j)\cdot(1)(2)\ldots(i+1)} \cdot \frac{(2)(4)\ldots(2i+2)}{(2)(4)\ldots(2i)\cdot(2)}.$$

53. We now examine the series $\phi_{2,2}(x)$, $\phi_{3,2}(x)$, ... It will be agreed that

$$\phi_{2,2}(x) = \chi\cdot\Omega\frac{\lambda_{11}\lambda_{12}\lambda_{21}\lambda_{22}}{(1-A\lambda_{11}\lambda_{12}x)\left(1-\frac{\lambda_{21}\lambda_{22}}{A}x\right)\cdot\left(1-\frac{B}{\lambda_{11}\lambda_{21}}x\right)\left(1-\frac{x}{B\lambda_{12}\lambda_{22}}\right)},$$

where Ω has reference to A and B, and χ to the auxiliaries λ_{11}, λ_{12}, λ_{21}, λ_{22}. We immediately eliminate A and B and find

$$\phi_{2,2}(x) = \chi\cdot\frac{\lambda_{11}\lambda_{12}\lambda_{21}\lambda_{22}}{(1-\lambda_{11}\lambda_{12}x)(1-\lambda_{11}\lambda_{12}\lambda_{21}\lambda_{22}x^2)\cdot\left(1-\frac{x}{\lambda_{11}\lambda_{21}}\right)\left(1-\frac{x^2}{\lambda_{11}\lambda_{12}\lambda_{21}\lambda_{22}}\right)}.$$

Eliminating λ_{22} we find

$$\phi_{2,2}(x) = \chi\cdot\frac{1}{(1-\lambda_{11}\lambda_{12}x)\left(1-\frac{x}{\lambda_{11}\lambda_{21}}\right)(4)}\left\{\frac{\lambda_{11}\lambda_{12}\lambda_{21}}{1-\lambda_{11}\lambda_{12}\lambda_{21}x^2} + \frac{\frac{1}{\lambda_{11}\lambda_{12}\lambda_{21}}x^4}{1-\frac{x^2}{\lambda_{11}\lambda_{12}\lambda_{21}}}\right\}.$$

$\phi_{2,2}(x)$ is now the sum of two χ functions; in the first of these we may put $\lambda_{12}=1$, since λ_{12} occurs to a positive power in every term of the expansion, and similarly we may put $\lambda_{21}=1$ in the second function, for it occurs to a negative power in every term of the expansion. Hence

$$\phi_{2,2}(x) = \frac{1}{(4)}\chi\cdot\frac{\lambda_{11}\lambda_{21}}{(1-\lambda_{11}x)(1-\lambda_{11}\lambda_{21}x^2)\left(1-\frac{x}{\lambda_{11}\lambda_{21}}\right)} + \frac{x^4}{(4)}\chi\cdot\frac{\frac{1}{\lambda_{11}\lambda_{12}}}{(1-\lambda_{11}\lambda_{12}x)\left(1-\frac{x}{\lambda_{11}}\right)\left(1-\frac{x^2}{\lambda_{11}\lambda_{12}}\right)},$$

and from previous results we may write

$$\phi_{2,2}(x) = \frac{1}{(4)}\phi_{2,1}(x) + \frac{x^4}{(4)}\phi_{1,2}(x).$$

54. Similarly we find

$$\phi_{3,2}(x) = \frac{1}{(5)}\phi_{3,1}(x) + \frac{x^4}{(5)}\phi_{2,2}(x);$$

and in general

$$\phi_{i,2}(x) = \frac{1}{(i+2)}\phi_{i,1}(x) + \frac{x^4}{(i+2)}\phi_{i-1,2}(x).$$

We know the value of $\phi_{1,2}(x)$, and assuming that

$$\phi_{i-1,2}(x) = \frac{1}{(1)(2)\ldots(i+1)} \cdot \frac{(2)(4)\ldots(2i+2)}{(2)(4)\ldots(2i-2)\cdot(2)(4)},$$

we find

$$\begin{aligned}\phi_{i,2}(x) &= \frac{(2i+2)}{(1)(2)^2(3)(4)\ldots(i+2)} + \frac{x^4}{(1)(2)\ldots(i+2)} \cdot \frac{(2i)(2i+2)}{(2)(4)}, \\ &= \frac{(2i+2)}{(1)(2)^2(3)(4)^2(5)\ldots(i+2)}\{1-x^4+x^4-x^{2i+4}\}, \\ &= \frac{(2i+2)(2i+4)}{(1)(2)^2(3)(4)^2(5)\ldots(i+2)}, \\ &= \frac{1}{(1)(2)\ldots(i+2)} \cdot \frac{(2)(4)\ldots(2i+4)}{(2)(4)\ldots(2i)\cdot(2)(4)};\end{aligned}$$

establishing the assumed law.

Hence Σx^P for the assemblage $\alpha^i\beta^j\gamma^2$ is

$$\frac{(1)(2)\ldots(i+j+2)}{(1)(2)\ldots(j)\cdot(1)(2)\ldots(i+2)} \cdot \frac{(2)(4)\ldots(2i+4)}{(2)(4)\ldots(2i)\cdot(2)(4)}.$$

55. On precisely the same reasoning we find

$$\phi_{i,k}(x) = \frac{1}{(i+k)}\phi_{i,k-1}(x) + \frac{x^{2k}}{(i+k)}\phi_{i-1,k}(x);$$

and assuming that

$$\phi_{i,k-1}(x) = \frac{1}{(1)(2)\ldots(i+k-1)} \cdot \frac{(2)(4)\ldots(2i+2k-2)}{(2)(4)\ldots(2i)\cdot(2)(4)\ldots(2k-2)},$$

$$\phi_{i-1,k}(x) = \frac{1}{(1)(2)\ldots(i+k-1)} \cdot \frac{(2)(4)\ldots(2i+2k-2)}{(2)(4)\ldots(2i-2)\cdot(2)(4)\ldots(2k)},$$

we find

$$\phi_{i,k}(x) = \frac{1}{(1)(2)\ldots(i+k)} \cdot \frac{(2)(4)\ldots(2i+2k-2)}{(2)(4)\ldots(2i)\cdot(2)(4)\ldots(2k)}\{1-x^{2k} + x^{2k}(1-x^{2i})\},$$

or

$$\phi_{i,k}(x) = \frac{1}{(1)(2)\ldots(i+k)} \cdot \frac{(2)(4)\ldots(2i+2k)}{(2)(4)\ldots(2i)\cdot(2)(4)\ldots(2k)},$$

the final result.

Hence Σx^P for the assemblage $\alpha^i\beta^j\gamma^k$ is

$$\frac{(1)\,(2)\,\ldots\,(i+j+k)}{(1)\,(2)\,\ldots\,(j)\cdot(1)\,(2)\,\ldots\,(i+k)}\cdot\frac{(2)\,(4)\,\ldots\,(2i+2k)}{(2)\,(4)\,\ldots\,(2i)\cdot(2)\,(4)\,\ldots\,(2k)},$$

or

$$F_{j,\,i+k}(x)\cdot F_{i,\,k}(x^2),$$

as previously conjectured.

56. *The Expression of Σx^P for the Assemblage $\alpha^i\beta^j\gamma^k\delta^l$.* The actual determination of this expression is reserved for a future occasion, but a few remarks upon the problem will not be out of place. For the assemblage $\alpha^i\beta^j\gamma^k$ the second row of numbers was not involved in the partition question, so that we had only two rows to deal with. In the present instance we have all four rows essentially involved in the partition question. We may write

$$\frac{\Sigma x^P}{(1)\,(2)\,\ldots\,(i+j+k+l)}=\phi_{i,\,j,\,k,\,l}(x),$$

and the question in partition is the determination of $\phi_{i,\,j,\,k,\,l}(x)$ by considering the four rows of numbers

$$\begin{aligned}&a_1\,a_2\,a_3\,\ldots\,a_i,\\ &b_1\,b_2\,b_3\,\ldots\,b_j,\\ &c_1\,c_2\,c_3\,\ldots\,c_k,\\ &d_1\,d_2\,d_3\,\ldots\,d_l.\end{aligned}$$

We have to find $\Sigma x^{\Sigma a+\Sigma b+\Sigma c+\Sigma d}$ from the conditions

$$\begin{aligned}&a_1\geq a_2\geq a_3\geq\ldots\geq a_i,\\ &b_1\geq b_2\geq b_3\geq\ldots\geq b_j,\\ &c_1\geq c_2\geq c_3\geq\ldots\geq c_k,\\ &d_1\geq d_2\geq d_3\geq\ldots\geq d_l,\end{aligned}$$

with the added conditions:

(i) No c is to be one greater than any a.

(ii) No d is to be one greater than any b.

(iii) No d is to be one greater or two greater than any a.

There is no difficulty in writing down the crude generating function involving the symbols χ and Ω and the associated auxiliary coefficients. There is, however, considerable difficulty in handling the function in question so as to eliminate all the auxiliaries and arrive at the desired formula.

The function Σx^P or $(1)\,(2)\,\ldots\,(i+j+k+l)\,\phi_{i,\,j,\,k,\,l}(x)$ has the following properties:

(i) For $x=1$ it must reduce to the number which enumerates the permutation of the assemblage $\alpha^i\beta^j\gamma^k\delta^l$.

(ii) If x be put equal to unity after differentiation with regard to x, we must obtain the value of ΣP that has been set forth in Art. 43.

(iii) For $j=k=0$ it must reduce to

$$F_{i,l}(x^3).$$

(iv) For $l=0$ and $i=0$ it must reduce to

$$F_{j,i+k}(x)\cdot F_{i,k}(x^2) \quad \text{and} \quad F_{k,j+l}(x)\cdot F_{j,l}(x^2)$$

respectively.

If we put $k=0$ we must get the expression of Σx^P for the assemblage $\alpha^i\beta^j\delta^l$; if we put $j=0$ we must get the expression of Σx^P for the assemblage $\alpha^i\gamma^k\delta^l$; and it would seem to be imperative to investigate these expressions in the first instance. In fact, when we come to the general assemblage $\alpha_1^{i_1}\alpha_2^{i_2}\ldots\alpha_s^{i_s}$, we may put any $s-3$ of the repetitional exponents equal to zero, and we would thus realize the expression of Σx^P for the assemblage

$$\alpha_{s_1}^{i_{s_1}}\alpha_{s_2}^{i_{s_2}}\alpha_{s_3}^{i_{s_3}},$$

where s_1, s_2, s_3 are any three different numbers in ascending order of magnitude. It would thus appear to be imperative ultimately to investigate this case also.

It is of course obvious that Σx^P for the assemblage $\alpha_{s_1}^{i_{s_1}}\alpha_{s_2}^{i_{s_2}}$ is

$$F_{i_{s_1}}, F_{i_{s_2}}(x^{s_2-s_1}),$$

and we thus obtain three indications to help us in the investigation of the assemblage

$$\alpha_{s_1}^{i_{s_1}}\alpha_{s_2}^{i_{s_2}}\alpha_{s_3}^{i_{s_3}}.$$

To these we may add the known results of putting $x=1$ before and after differentiation.

57. *Crude Expression for Σx^P for the Assemblage $\alpha\beta\gamma\delta$.* The actual expression of Σx^P can be obtained from the permutation set forth in Art. 32. It is

$$1+x+2x^2+3x^3+4x^4+2x^5+5x^6+x^7+3x^8+x^9+x^{10}.$$

Its unsymmetrical character will be noted. It shows that we can not expect the expression of Σx^P for the assemblage $\alpha^i\beta^j\gamma^k\delta^l$ to be in the form of a single algebraic fraction involving factors all of the form (s).

For the assemblage $\alpha\beta\gamma\delta$ I construct the function

$$\chi \cdot \frac{\lambda_{13}\lambda_{14}\lambda_{24}\mu_{14}^2}{(1-\lambda_{13}\lambda_{14}\mu_{14}x)(1-\lambda_{24}x)\left(1-\frac{x}{\lambda_{13}}\right)\left(1-\frac{x}{\lambda_{14}\lambda_{24}\mu_{14}}\right)}$$

and note that the general term in the expansion is

$$\lambda_{13}\lambda_{14}\lambda_{24}\mu_{14}^2\,(\lambda_{13}\lambda_{14}\mu_{14}x)^a(\lambda_{24}x)^b\left(\frac{x}{\lambda_{13}}\right)^c\left(\frac{x}{\lambda_{14}\lambda_{24}\mu_{14}}\right)^d$$

or

$$\lambda_{13}^{a+1-c}\lambda_{14}^{a+1-d}\lambda_{24}^{b+1-d}\mu_{14}^{a+2-d}x^{a+b+c+d}.$$

The operation of χ excludes zero powers of each of the four auxiliaries, so that the required conditions are satisfied, and the constructed function is therefore the crude expression of $\phi_{1,1,1,1}(x)$. Then

$$\Sigma x^P = (1)(2)(3)(4)\,\phi_{1,1,1,1}(x).$$

I have actually verified that

$$\phi_{1,1,1,1}(x) = \frac{1+x+2x^2+3x^3+4x^4+2x^5+5x^6+x^7+3x^8+x^9+x^{10}}{(1)(2)(3)(4)},$$

but as I have not yet succeeded in carrying out the reduction in a satisfactory manner, so as to show how the general case should be handled, I withhold the algebra.

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Conjugate Line Congruences of the Third Order Defined by a Family of Quadrics.

By Helen Brewster Owens.

Introduction.

The first systematic study of algebraic congruences was made by Kummer, who derived most of the forms of those of order 2. His work is both synthetic and analytic, and is a model of elegance and directness. All of these congruences can be arranged on a system of quadric surfaces in such a way that only one system of generators comes in.

No other congruences have been studied from this point of view; the present paper aims to do for congruences of order 3 what Kummer did for those of order 2.*

I wish to express my appreciation and thanks to Professor Virgil Snyder for advice and assistance in the preparation of this paper.

1. The totality of all lines of space satisfying two conditions is called a *congruence.* The *order* of a *congruence* is the number of lines of the congruence which pass through any point of space. The *class* of a *congruence* is the number of lines of the congruence which lie in any plane in space. A *singular point* is a point through which pass an infinite number of lines of the congruence. A *singular plane* is a plane in which lie an infinite number of lines of the congruence. A *basis point* of a system of quadrics is a point common to every quadric of the system.

2. Congruences of certain types may be arranged on a system of quadric surfaces. Such a congruence arises if the two conditions which determine it are of the form:

$$\Sigma a_i x+\sigma\Sigma b_i x=0,$$
$$\Sigma \bar{a}_i x+\sigma\Sigma \bar{b}_i x=0,$$

where a_i, $\bar{a}_i$, b_i, $\bar{b}_i$ are functions of a parameter λ, and the summation sign

* Since this paper was written, R. Baldus, in an Erlangen dissertation, 1910, entitled "Ueber Strahlensysteme, welche unendlich viele Regelflächen zweiten Grades enthalten," has contributed some interesting information which supplements certain parts of this paper.

indicates that the a_i, $\bar{a}_i$, b_i, $\bar{b}_i$ are to be taken with the homogeneous point coordinates x, y, z, w. Solving each equation for σ:

$$\frac{\Sigma a_i x}{\Sigma b_i x} = \frac{\Sigma \bar{a}_i x}{\Sigma \bar{b}_i x} = \sigma. \tag{1}$$

In this form it is evident that the lines for all values of σ define the generators of one system of the family of quadrics:

$$H(\lambda) \equiv (\Sigma a_i x)(\Sigma \bar{b}_i x) - (\Sigma \bar{a}_i x)(\Sigma b_i x) = 0. \tag{2}$$

If $\Sigma \bar{a}_i x$ and $\Sigma b_i x$ be interchanged in the equations, there results a new congruence,

$$\frac{\Sigma a_i x}{\Sigma \bar{a}_i x} = \frac{\Sigma b_i x}{\Sigma \bar{b}_i x} = \tau, \tag{3}$$

the lines of which define the generators of the second system in the same family of quadrics:

$$H(\lambda) = 0. \tag{2}$$

Two such congruences will be referred to as the σ and τ congruences respectively, and will be called *conjugate congruences*. The envelop of the system of quadrics $H(\lambda)$ is the *focal surface* of the conjugate congruences σ and τ.

3. In this paper will be discussed the congruences defined by the equations

$$\text{(I)} \begin{cases} \dfrac{a+b\lambda+c\lambda^2}{x+y\lambda} = \dfrac{a'+b'\lambda+c'\lambda^2}{z+w\lambda} = \sigma, & (4) \\ \dfrac{a+b\lambda+c\lambda^2}{a'+b'\lambda+c'\lambda^2} = \dfrac{x+y\lambda}{z+w\lambda} = \tau, & (5) \end{cases}$$

$$\text{(II)} \begin{cases} \dfrac{a+b\lambda+c\lambda^2+d\lambda^3}{x+y\lambda+z\lambda^2} = \dfrac{a'+b'\lambda}{w} = \sigma, & (6) \\ \dfrac{a+b\lambda+c\lambda^2+d\lambda^3}{a'+b'\lambda} = \dfrac{x+y\lambda+z\lambda^2}{w} = \tau, & (7) \end{cases}$$

in which a, b, c, d, a', b', c' are linear functions of the variables x, y, z, w; and λ, σ and τ are variable parameters. The congruences will be discussed both in case the linear functions are general and in case they satisfy certain given conditions. It is to be observed that both parameters, λ, σ (τ), enter rationally and one of them, σ (τ), linearly in the equation. The conjugate congruences (4) and (5) can be arranged on the family of quadrics:

$$H(\lambda) \equiv (a+b\lambda+c\lambda^2)(z+w\lambda) - (a'+b'\lambda+c'\lambda^2)(x+y\lambda) = 0. \tag{8}$$

Each congruence is composed of the generators of one system on the family of quadrics. Similarly, the conjugate congruences (6) and (7) are composed of the two sets of generators of the family:

$$H'(\lambda) \equiv w(a+b\lambda+c\lambda^2+d\lambda^3) - (a'+b'\lambda)(x+y\lambda+z\lambda^2) = 0. \tag{9}$$

The two systems of generators of each family of quadrics are rationally separable.

4. If the coordinates of a fixed point be substituted in equation (8) [or (9)], it is evident that there exist three values of λ satisfying the equation; *i. e.*, through every point of space pass three quadrics of the family.

Through every point of space pass three lines of the σ and τ conqruences, respectively; hence the congruence (σ or τ) is of order 3.

5. To determine the class of a congruence, impose the condition that an arbitrary plane,

$$A_1x+A_2y+A_3z+A_4w=0,$$

shall be tangent to a quadric of $H(\lambda)=0$. This will give an equation in λ satisfied by nine values:

$$\begin{vmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x\partial y} & \frac{\partial^2 H}{\partial x\partial z} & \frac{\partial^2 H}{\partial x\partial w} & A_1 \\ \frac{\partial^2 H}{\partial x\partial y} & \frac{\partial^2 H}{\partial y^2} & \frac{\partial^2 H}{\partial y\partial z} & \frac{\partial^2 H}{\partial y\partial w} & A_2 \\ \frac{\partial^2 H}{\partial x\partial z} & \frac{\partial^2 H}{\partial y\partial z} & \frac{\partial^2 H}{\partial z^2} & \frac{\partial^2 H}{\partial z\partial w} & A_3 \\ \frac{\partial^2 H}{\partial x\partial w} & \frac{\partial^2 H}{\partial y\partial w} & \frac{\partial^2 H}{\partial z\partial w} & \frac{\partial^2 H}{\partial w^2} & A_4 \\ A_1 & A_2 & A_3 & A_4 & 0 \end{vmatrix} \equiv f_9(\lambda)=0. \qquad (10)$$

In any plane of space lie three lines of the σ and τ congruences respectively; hence the congruence (σ or τ) is of class 9.

6. Corresponding to those values of λ for which the discriminant of the quadric H vanishes, the quadric becomes a cone. This discriminant is of order 12 in λ, $D_{12}(\lambda)$.

The necessary and sufficient condition* that the systems of generators depending rationally on a parameter λ shall be separable into two conjugate congruences is that the discriminant shall be a perfect square in λ. Hence we have

$$D_{12}(\lambda)=[f_6(\lambda)]^2; \qquad (11)$$

thus the discriminant possesses six double roots and there exists in H a cone for each such root.

The general congruences σ and τ are arranged on a family of quadrics which includes six quadric cones. The lines of the cones belong to both the σ and τ congruences.

* Snyder: "Conjugate Line Congruences Contained in a Bundle of Quadric Surfaces," *Transactions Amer. Math. Soc.*, Vol. XI (1910), pp. 371–387.

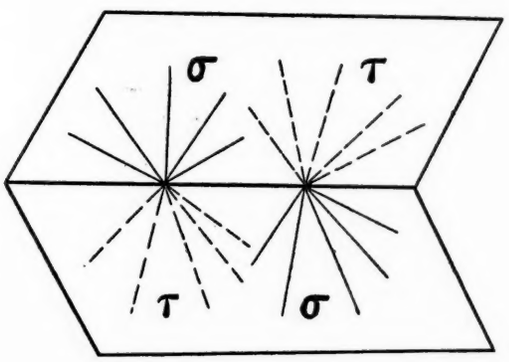
7. A singular plane may arise either from a basis plane or from the breaking into planes of one of the cones included in H, while a singular point may arise from a basis point or the vertex of a cone in H.

Imposing the condition that one of the cones of the system shall degenerate to a pair of planes, *i. e.*, making the minors of the Hessian of $H(\lambda)=0$ vanish for one value λ_1 of λ, the $f_9(\lambda)=0$ which determines the class of the congruence will have that value λ_1 as a factor, and the remaining function will be of order 8, $F_8(\lambda)=0$. This shows that the class of the congruence is lowered by one unit by the presence of a degenerate cone in the system. The argument may be repeated for each additional cone.

The class of the congruence σ (τ) is diminished by unity by the degeneracy of each of the six quadric cones into a pair of planes.

8. Suppose in equation (1) that $\Sigma a_i x \equiv (\lambda-\lambda_1)\Sigma A_i x$, but that no other Σ contains $(\lambda-\lambda_1)$ as a factor. Then, for $\lambda=\lambda_1$, the equation becomes:

$$\frac{0\cdot\Sigma A_i x}{\Sigma \overline{A}_i x}=\frac{\Sigma B_i x}{\Sigma \overline{B}_i x}=\tau, \tag{12}$$



where $A_i \equiv a_i$ for $\lambda=\lambda_1$, etc. Since τ is a parameter, taking all values, it follows that $(\Sigma \overline{A}_i x)\tau=0\cdot\Sigma A_i x$ requires that $\Sigma \overline{A}_i x=0$ contain a line of the τ congruence for every value of τ. Hence $\Sigma \overline{A}_i x=0$ is a singular plane. Further, any line of the τ congruence in this plane passes through the point $\Sigma B_i x=0$, $\Sigma \overline{B}_i x=0$, $\Sigma \overline{A}_i x=0$. This point will be called the "τ vertex" in the plane $\Sigma \overline{A}_i x=0$. The plane $\Sigma A_i x=0$ is likewise a singular plane for the σ congruence, all lines of which in the plane pass through the point $\Sigma \overline{A}_i x=0$, $\Sigma A_i x=0$, $\Sigma B_i x=0$. This is the "σ vertex" in the plane $\Sigma \overline{A}_i x=0$. Similarly, for $\lambda=\lambda_1$, $\Sigma B_i x_i=0$ is a singular plane with τ vertex at $\Sigma \overline{A}_i x=0$, $\Sigma B_i x=0$, $\Sigma A_i x=0$, and the σ vertex at the point $\Sigma \overline{A}_i x=0$, $\Sigma B_i x=0$, $\Sigma \overline{B}_i x=0$. Since λ_1 is any root of all the first minors of the Hessian, the above argument holds for any cone included in H which breaks into two planes. Hence the following theorem holds:

When a quadric of $H(\lambda)$ breaks up into a pair of planes, the lines of each system of generators become pencils of lines, one pencil in each plane at each of two vertices, which are in general distinct, lying on the line of intersection of the planes. The σ vertex in one is the τ vertex in the other, and vice versa.

9. For any basis point P_1 of a cubic congruence all the lines are given by the equations

$$H(\lambda) \equiv H_0\lambda^3 + H_1\lambda^2 + H_2\lambda + H_3 = 0,$$

and the tangent plane

$$\Sigma x \frac{\partial H_0}{\partial x_1}\lambda^3 + \ldots . = 0.$$

The λ eliminant is therefore of degree 9; *i. e.*, the lines of both σ and τ congruences form a cone of order 9. If the σ lines and τ lines form distinct cones, the sum of the orders of such cones is 9. Further, if the class of the congruence be decreased by unity by the breaking up of a cone into a pair of planes, there will appear a factor common to the two equations, so that the degree of the eliminant is decreased by one.

The sum of the orders of the singular cones of all σ and τ lines through a basis point is always equal to the class of each congruence.

All of the above statements hold for either the conjugate congruences (4) and (5) or the conjugate congruences (6) and (7). Hereafter the congruence determined from (4) and (5) by imposing various conditions will be referred to as Type I; those determined from (6) and (7) as Type II. In general the σ congruence will be discussed, since the properties of the conjugate τ are immediately deducible from those of the σ. When there is danger of ambiguity, both congruences are taken into consideration. The notation $\mathrm{I}(3,k)^{a,\,b}_{c,\,d}$ stands for a congruence of Type I, order 3, class k, with a basis points, with b basis planes, with c singular points which are vertices of cones or pencils in degenerate cones, with d singular planes arising from a cone breaking into pairs of planes.

10. Since the focal surface of a congruence has been defined as the envelop of the family of quadrics $H(\lambda)=0$, the focal surface of the rational congruence (3, 9) is of order 8. The vertices of the six cones of the family will be double points on it, since they result from the double roots of the discriminant of $H(\lambda)=0$. Each quadric of the family H will intersect the focal surface in a curve C_8 and the curve of contact C_4, defined by the equations

$$H(\lambda)=0, \quad \frac{\partial H(\lambda)}{\partial \lambda} = 0.$$

Since there exists a parameter λ in these equations, the focal surface contains

∞^1 twisted quartics of the first kind. The six cones of the family H will each touch the focal surface in a quartic curve, having a double point at the vertex of the cone. Further, the equation for

$$H(\lambda)\equiv H_0\lambda^3+3H_1\lambda^2+3H_2\lambda+H_3=0$$

will have a triple root if

$$\frac{H_0}{H_1}=\frac{H_1}{H_2}=\frac{H_2}{H_3}.$$

Hence the focal surface has the intersection of these two quartic surfaces as a cuspidal edge. This cuspidal edge is of order 12, since the intersection of $H_1=0$, $H_2=0$ does not lie on $H_0H_3=H_1H_2$.

Hence, the focal surface of the general σ (τ) congruence contains ∞^1 twisted quartics of the first kind, and six quartic curves with one double point each. The double points of these six curves are the only double points on the surface. The surface contains in addition to these curves a cuspidal edge of order 12.

11. Every line of the congruence will cut in two points the quartic curve of contact of its corresponding quadric and the focal surface F_8. Hence, every line of the congruence is a bitangent of the focal surface. The transformation S, defined by interchanging the two points of contact of a σ line (bitangent) of the focal surface, is an involutoric, non-linear, birational transformation under which the focal surface F_8 is invariant. Under this transformation, a point has its image always on the C_4 of contact which passes through this point. Similarly, the same surface F_8 is the focal surface of the conjugate congruence τ. The operation T of interchanging the points of contact of a τ line is also of order 2. The group generated by S and T is generally of infinite order. In all the congruences discussed in this paper, the order of the focal surface is the same as in the case of the general congruence, *i. e.*, it is of order 8.

TYPE I.

12. The general congruence

$$\frac{a+b\lambda+c\lambda^2}{x+y\lambda}=\frac{a'+b'\lambda+c'\lambda^2}{z+w\lambda}=\sigma \tag{4}$$

has no basis planes and no basis points. Hence it has no singular planes and only six singular points, namely, the vertices of the quadric cones in H.

The symbol for the conjugate congruences (4) *and* (5) *is* $\mathrm{I}(3, 9)_6$.

13. If the condition $a=0$ be imposed on (4) and (5), there result:

$$\frac{b\lambda+c\lambda^2}{x+y\lambda}=\frac{a'+b'\lambda+c'\lambda^2}{z+w\lambda}=\sigma, \quad \frac{b\lambda+c\lambda^2}{a'+b'\lambda+c'\lambda^2}=\frac{x+y\lambda}{z+w\lambda}=\tau. \tag{13}$$

For the value $\lambda=0$, the quadric of H is $a'x=0$, *i.e.*, a pair of planes $a'=0$, $x=0$. From Arts. 4, 7, 8 results: The congruence (13) has the singular planes $a'=0$, $x=0$, seven singular points, the vertices of five cones, and the σ vertices in the singular planes:

$$x=0,\ x=a'=z=0;\quad a'=0,\ x=a'=b=0.$$

The symbol for this congruence is $\mathrm{I}(3,\ 8)_{7,2}$.

14. In a manner similar to the above, if on (13) be imposed the additional condition $c'=0$, a second cone breaks into plane pencils. The table for these planes and points includes those of Art. 13 and in addition:

$$c=0,\ c=w=b'=0;\quad w=0,\ w=c=y=0.$$

The equations for the congruence are

$$\frac{b\lambda+c\lambda^2}{x+y\lambda}=\frac{a'+b'\lambda}{z+w\lambda}=\sigma,\qquad \frac{b\lambda+c\lambda^2}{a'+b'\lambda}=\frac{x+y\lambda}{z+w\lambda}=\tau. \tag{14}$$

The symbol for the congruence (14) *is* $\mathrm{I}(3,\ 7)_{8,4}$.

15. Assuming* that $\lambda=1$ is to be a third value for which a cone degenerates, the necessary and sufficient condition to be imposed in addition to the preceding is:

$$b+c-k(a'+b')=p[(x+y)-k(z+w)].$$

Then

$$c=p[x+y-k(z+w)]-b+k(a'+b')=\bar{c}.$$

Substituting for c its value $\bar{c}$ and factoring:

$$\frac{(1-\lambda)[b\lambda-px(1+\lambda)-py\lambda+kpz(1+\lambda)+pkw\lambda-ka'\lambda+1-kb'\lambda]}{x+y\lambda-k(z+w\lambda)}$$
$$=\frac{a'+b'\lambda-p(z+w\lambda)}{z+w\lambda}=\sigma. \tag{15}$$

Applying Arts. 1, 7, 8, the complete statement of planes and vertices includes the planes and points of Art. 14 and in addition:

$$x+y-k(z+w)=0,\qquad x+y=a'+b'=z+w=0;$$
$$a'+b'-p(z+w)=0,\quad x+y-k(z+w)=a'+b'-p(z+w)$$
$$=b-p(2x+y-kz)-ka'=0.$$

The congruence (15) *has the symbol* $\mathrm{I}(3,\ 6)_{9,6}$.

16. In order that a fourth cone for $\lambda=\lambda_1$ shall break into planes, the necessary and sufficient condition to be imposed on (15) is:

$$b\lambda_1+c\lambda_1^2-l(\bar{a}'+b'\lambda_1)=r[x+y\lambda_1-l(z+w)\lambda_1],$$

*It is no restriction to impose these three values of λ for the parameters of three composite cones.

whence

$$b=\frac{r[x+y\lambda_1-l(z+w\lambda_1)]-\bar{c}\lambda_1^2+l(a'+b'\lambda)}{\lambda_1}=\bar{b}.$$

Substituting, subtracting s from each member and factoring:

$$\frac{(\lambda-\lambda_1)\{rx-lsz+\bar{c}\lambda\lambda_1+la'\}}{\lambda_1[x+y\lambda-l(z+w\lambda)]}=\frac{a'+b'\lambda-r(z+w\lambda)}{z+w\lambda}=\sigma. \tag{16}$$

From Arts. 7, 8, the complete statement of planes and points includes those of Art. 15, and in addition:

$$(x+y\lambda_1)-l(z+w\lambda_1)=0, \qquad a'+b'\lambda_1=z+w\lambda_1=x+y\lambda_1=0;$$
$$(a'+b'\lambda_1)-r(z+w\lambda_1)=0, \quad x+y\lambda_1-l(z+w\lambda_1)=a'+b'\lambda_1-r(z+w\lambda_1)$$
$$=r(x-lz)+\bar{c}\lambda_1^2+la'=0.$$

The symbol for the congruence (16) *is* $\mathrm{I}(3,5)_{10,8}$.

17. That the fifth cone degenerate, the necessary and sufficient condition on (16) is:

$$\bar{b}\lambda_2+\bar{c}\lambda_2^2-m(a'+b'\lambda_2)=s[x+y\lambda_2-m(z+w\lambda_2)],$$

whence

$$a'=\frac{-s[x+y\lambda_2-m(z+w\lambda_2)]-b'\lambda_2 m-\bar{b}\lambda_2-\bar{c}\lambda_2^2}{m}=\bar{a}'.$$

Substituting, subtracting s from each member and factoring:

$$\frac{(\lambda-\lambda_2)[\bar{b}+\bar{c}(\lambda+\lambda_2)-sy-smw-b'm]}{x+y\lambda-m(z+w\lambda)}=\frac{\bar{a}'+b'\lambda-s(z+w\lambda)}{z+w\lambda}=\sigma. \tag{17}$$

The complete statement of planes and points includes those of Art. 16 and in addition:

$$x+y\lambda_2-m(z+w\lambda_2)=0, \qquad \bar{a}'+b'\lambda_2=x+y\lambda_2=z+w\lambda_2=0;$$
$$\bar{a}'+b'\lambda_2-s(z+w\lambda_2)=0, \quad x+y\lambda_2-m(z+w\lambda_2)=\bar{a}'+b'\lambda_2-s(z+w\lambda_2)$$
$$=\bar{b}+2\bar{c}\lambda_2-sy+smw-b'm=0.$$

The symbol for the congruence (17) *is* $\mathrm{I}(3,4)_{11,10}$.

18. In order that all six of the quadric cones of H may be composed of pairs of planes, the following six conditions must be simultaneously imposed on $H(\lambda)=0$.

$$a=0, \quad c'=0,$$
$$b+c-k(a'+b')=p[x+y-k(z+w)],$$
$$b\lambda_1+c\lambda_1^2-l(a'+b'\lambda_1)=r[x+y\lambda_1-l(z+w\lambda_1)],$$
$$b\lambda_2+c\lambda_2^2-m(a'+b'\lambda_2)=s[x+y\lambda_2-m(z+w\lambda_2)],$$
$$b\lambda_3+c\lambda_3^2-n(a'+b'\lambda_3)=t[x+y\lambda_3-n(z+w\lambda_3)].$$

Imposing these conditions on $H(\lambda)=0$ gives

$$\begin{vmatrix} 1 & 1 & k & k & p[x+y-k(z+w)] \\ \lambda_1 & \lambda_1^2 & l & l\lambda_1 & r[x+y\lambda_1-l(z+w\lambda_1)] \\ \lambda_2 & \lambda_2^2 & m & m\lambda_2 & s[x+y\lambda_2-m(z+w\lambda_2)] \\ \lambda_3 & \lambda_3^2 & n & n\lambda_3 & t[x+y\lambda_3-n(z+w\lambda_3)] \\ z+w\lambda & \lambda(z+w\lambda) & x+y\lambda & \lambda(x+y\lambda) & 0 \end{vmatrix} = 0. \quad (18)$$

Reducing this determinant, we have:

$$\frac{\lambda \begin{vmatrix} 1-\lambda & k & k & p[x+y-k(z+w)] \\ \lambda_1(\lambda_1-\lambda) & l & l\lambda_1 & r[x+y\lambda_1-l(z+w\lambda_1)] \\ \lambda_2(\lambda_2-\lambda) & m & m\lambda_2 & s[x+y\lambda_2-m(z+w\lambda_2)] \\ \lambda_3(\lambda_3-\lambda) & n & n\lambda_3 & t[x+y\lambda_3-n(z+w\lambda_3)] \end{vmatrix}}{x+y\lambda}$$

$$= \frac{-\begin{vmatrix} 1 & 1 & k(1-\lambda) & p[x+y-k(z+w)] \\ \lambda_1 & \lambda_1^2 & l(\lambda_1-\lambda) & r[x+y\lambda_1-l(z+w\lambda_1)] \\ \lambda_2 & \lambda_2^2 & m(\lambda_2-\lambda) & s[x+y\lambda_2-m(z+w\lambda_2)] \\ \lambda_3 & \lambda_3^2 & n(\lambda_3-\lambda) & t[x+y\lambda_3-n(z+w\lambda_3)] \end{vmatrix}}{z+w\lambda} = \sigma$$

$$\equiv \frac{S}{x+y\lambda} = \frac{T}{z+w\lambda} = \sigma. \quad (19)$$

This equation is in the exact form of the general equation for congruences of Type I. By reference to the conditions imposed and Art. 8 it is possible to write out the complete table of singular planes and points for the six values of λ: $0, \infty, 1, \lambda_1, \lambda_2, \lambda_3$. This table is as follows. The notation $(1, a)$ indicates the first singular plane for value $\lambda = a$. $(\sigma, 1, a)$ indicates the σ vertex in the plane $(1, a)$.

$$(1, 0) \equiv 0 = x; \quad (\sigma, 1, 0) : x = z = \begin{vmatrix} 1 & k & k & p(y-wk) \\ \lambda_1^2 & l & l\lambda_1 & r\lambda_1(y-wl) \\ \lambda_2^2 & m & m\lambda_2 & s\lambda_2(y-wm) \\ \lambda_3^2 & n & n\lambda_3 & t\lambda_3(y-wn) \end{vmatrix} = 0.$$

Let A, B, C, D indicate the $1, \ldots, 4$ terms of the last column of the determinant in (19).

$$(2, 0) \equiv \begin{vmatrix} 1 & 1 & k & A \\ \lambda_1^2 & \lambda_1 & l\lambda_1 & B \\ \lambda_2^2 & \lambda_2 & m\lambda_2 & C \\ \lambda_3^2 & \lambda_3 & n\lambda_3 & D \end{vmatrix} = 0; \quad (\sigma, 2, 0,) : (1, 0) = (2, 0) = \begin{vmatrix} 1 & k & k & A \\ \lambda_1^2 & l & l\lambda_1 & B \\ \lambda_2^2 & m & m\lambda_2 & C \\ \lambda_3^2 & n & n\lambda_3 & D \end{vmatrix} = 0.$$

$$(1, \infty) \equiv w = 0; \quad (\sigma, 1, \infty) : (1, \infty) = (2, \infty) = y = 0.$$

$$(2,\infty)\equiv\begin{vmatrix}1 & k & k & A\\ \lambda_1 & l & l\lambda_1 & B\\ \lambda_2 & m & m\lambda_2 & C\\ \lambda_3 & n & n\lambda_3 & D\end{vmatrix}=0;\quad (\sigma,2,\infty):(1,\infty)=(2,\infty)=\begin{vmatrix}1 & 1 & k & A\\ \lambda_1 & \lambda_1^2 & l & B\\ \lambda_2 & \lambda_2^2 & m & C\\ \lambda_3 & \lambda_3^2 & n & D\end{vmatrix}=0.$$

$$(1,1)\equiv x+y-k(z+w)=0;\quad (\sigma,1,1):(1,1)=(1,2)=z+w=0.$$

$$(2,1)\equiv\begin{vmatrix}1 & 1 & 0 & A\\ \lambda_1 & \lambda_1^2 & l(\lambda_1-1) & B\\ \lambda_2 & \lambda_2^2 & m(\lambda_2-1) & C\\ \lambda_3 & \lambda_3^2 & n(\lambda_3-1) & D\end{vmatrix}=0;$$

$$(\sigma,2,1,):(1,1)=(1,2)=\frac{S-kT-p[x+y-kz-kw]}{1-\lambda}\Big]_{\lambda\doteq 1}=0.$$

$$(1,\lambda_1)\equiv x+y\lambda_1-l(z+w\lambda_1)=0;\quad (\sigma,1,\lambda_1):x+y\lambda_1=z+w\lambda_1=(1,\lambda_2)=0.$$

$$(2,\lambda_1)\equiv\begin{vmatrix}1 & 1 & k & A\\ \lambda_1 & \lambda_1^2 & 0 & B\\ \lambda_2 & \lambda_2^2 & m(\lambda_2-\lambda_1) & C\\ \lambda_3 & \lambda_3^2 & n(\lambda_3-\lambda_1) & D\end{vmatrix}=0;$$

$$(\sigma,2,\lambda_1):(1,\lambda_1)=(2,\lambda_1)=\frac{S-lT-r[x+y\lambda_1-l(z+w\lambda_1)]}{\lambda-\lambda_1}\Big]_{\lambda\doteq\lambda_1}=0.$$

$$(1,\lambda_2)\equiv x+y\lambda_2-m(z+w\lambda_2)=0;\quad (\sigma,1,\lambda_2):x+y\lambda_2=z+w\lambda_2=(2,\lambda_2)=0.$$

$$(2,\lambda_2)\equiv\begin{vmatrix}1 & 1 & k & A\\ \lambda_1 & \lambda_1^2 & l(\lambda_1-\lambda_2) & B\\ \lambda_2 & \lambda_2^2 & 0 & C\\ \lambda_3 & \lambda_3^2 & n(\lambda_3-\lambda_2) & D\end{vmatrix}=0;$$

$$(\sigma,2,\lambda_2):(1,\lambda_2)=(2,\lambda_2)=\frac{S-mT-s[x+y\lambda_2-m(z+w\lambda_2)]}{\lambda-\lambda_2}\Big]_{\lambda\doteq\lambda_2}=0.$$

$$(1,\lambda_3)\equiv x+y\lambda_3-n(z+w\lambda_3)=0;\quad (\sigma,1,\lambda_3):x+y\lambda_3=z+w\lambda_3=(2,\lambda_3)=0.$$

$$(2,\lambda_3)\equiv\begin{vmatrix}1 & 1 & k & A\\ \lambda_1 & \lambda_1^2 & l(\lambda_1-\lambda_3) & B\\ \lambda_2 & \lambda_2^2 & m(\lambda_2-\lambda_3) & C\\ \lambda_3 & \lambda_3^2 & 0 & D\end{vmatrix}=0;$$

$$(\sigma,2,\lambda_3):(1,\lambda_3)=(2,\lambda_3)=\frac{S-nT-t[x+y\lambda_3-n(z+w\lambda_3)]}{\lambda-\lambda_3}\Big]_{\lambda\doteq\lambda_3}=0.$$

The symbol for the congruence (18) *is* $\mathrm{I}(3,3)_{12,12}$.

19. In the above work care has been taken that no basis point, line, or plane should be allowed to enter. By imposing conditions on the eight arbitrary constants k, l, m, n, p, r, s, t it is possible to introduce one, two, three

or four basis points, thereby obtaining the four pairs of congruences: $I(3,3)^{1,1}_{12,12}$; $I(3,3)^{2,2}_{12,12}$; $I(3,3)^{3,3}_{12,12}$; $I(3,3)^{4,4}_{12,12}$. The introduction of a basis point in a (3, 3) simultaneously brings in a basis plane through the basis point.

The case of three basis points was obtained synthetically by Roccella* and is also mentioned by Snyder.† The case of four basis points and four basis planes is mentioned by Fano‡ and Montesano.§ It is a particular case of Roccella's configuration and also of that of Hirst.‖ C. Corrona¶ seems to have been familiar with it, but the configuration of points and planes is not discussed.

The four basis points and the necessary and sufficient conditions for each are as follows, O_i being the basis point; A, B, C, D as in Art. 18; and I, II, III, IV the following determinants:

$$\mathrm{I}=\begin{vmatrix} 1 & k & k & A \\ \lambda_1 & l & l\lambda_1 & B \\ \lambda_2 & m & m\lambda_2 & C \\ \lambda_3 & n & n\lambda_3 & D \end{vmatrix}; \quad \mathrm{II}=\begin{vmatrix} 1 & k & k & A \\ \lambda_1^2 & l & l\lambda_1 & B \\ \lambda_2^2 & m & m\lambda_2 & C \\ \lambda_3^2 & n & n\lambda_3 & D \end{vmatrix};$$

$$\mathrm{III}=\begin{vmatrix} 1 & 1 & k & A \\ \lambda_1 & \lambda_1^2 & l & B \\ \lambda_2 & \lambda_2^2 & m & C \\ \lambda_3 & \lambda_3^2 & n & D \end{vmatrix}; \quad \mathrm{IV}=\begin{vmatrix} 1 & 1 & k & A \\ \lambda_1 & \lambda_1^2 & l\lambda_1 & B \\ \lambda_2 & \lambda_2^2 & m\lambda_2 & C \\ \lambda_3 & \lambda_3^2 & n\lambda_3 & D \end{vmatrix}.$$

$$O_1 : w=\mathrm{IV}=x+y\lambda_2-mz=0.$$
$$O_2 : x=\mathrm{I}=y-k(z+w)=0.$$
$$O_3 : w=\mathrm{IV}=x+y-kz=0.$$
$$O_4 : x=\mathrm{I}=y\lambda_1-l(z+w\lambda_1)=0.$$

If A_i, B_i, C_i, D_i indicate that the conditions of O_i are imposed, the conditions for O_1 are:

$$-\frac{z}{y}\begin{vmatrix} 1 & k & k & A_1 \\ \lambda_1 & l & l\lambda_1 & B_1 \\ \lambda_2 & m & m\lambda_2 & 0 \\ \lambda_3 & n & n\lambda_3 & D_1 \end{vmatrix}=\begin{vmatrix} 1 & 1 & k & A_1 \\ \lambda_1 & \lambda_1^2 & l & B_1 \\ \lambda_2 & \lambda_2^2 & m & 0 \\ \lambda_3 & \lambda_3^2 & n & D_1 \end{vmatrix}=\frac{z}{x}\begin{vmatrix} 1 & k & k & A_1 \\ \lambda_1^2 & l & l\lambda_1 & B_1 \\ \lambda_2^2 & m & m\lambda_2 & 0 \\ \lambda_3^2 & n & n\lambda_3 & D_1 \end{vmatrix}.$$

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* Roccella: "Sugli enti geometrici dello spazio di rette," Piazza Armerina (1882).

† Snyder: "Surfaces Invariant under Infinite Discontinuous Birational Groups Defined by Line Congruences," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXII (1910), pp. 177–185.

‡ Fano: "Sopra alcune particolari congruenze di rette del 3° ordine," *Atti di Torino*, Vol. XXXVI (1900–1901), pp. 222–236.

§ Montesano: "Su di un complesso di rette di terzo grado," *Mem. di Bologno*, Ser. 5, Vol. III (1893), pp. 549–577.

‖ Hirst: "On Cremonian Congruences," *Proceedings of the London Mathematical Society*, Vol. XIV (1883), pp. 259–301.

¶ Corrona: "Sopra un complesso di rette di quarto grado," *Rend. di Palermo*, Vol. XVI (1902), pp. 359–375.

The conditions for O_2 are:

$$\frac{y}{w}\begin{vmatrix}1 & 1 & k & 0\\ \lambda_1 & \lambda_1^2 & l & B_2\\ \lambda_2 & \lambda_2^2 & m & C_2\\ \lambda_3 & \lambda_3^2 & n & D_2\end{vmatrix}=\begin{vmatrix}1 & k & k & 0\\ \lambda_1^2 & l & l\lambda_1 & B_2\\ \lambda_2^2 & m & m\lambda_2 & C_2\\ \lambda_3^2 & n & n\lambda_3 & D_2\end{vmatrix}=-\frac{y}{z}\begin{vmatrix}1 & 1 & k & 0\\ \lambda_1 & \lambda_1^2 & l\lambda_1 & B_2\\ \lambda_2 & \lambda_2^2 & m\lambda_2 & C_2\\ \lambda_3 & \lambda_3^2 & n\lambda_3 & D_2\end{vmatrix}.$$

The conditions for O_3 are:

$$-\frac{z}{y}\begin{vmatrix}1 & k & k & A_3\\ \lambda_1 & l & l\lambda_1 & 0\\ \lambda_2 & m & m\lambda_2 & C_3\\ \lambda_3 & n & n\lambda_3 & D_3\end{vmatrix}=\begin{vmatrix}1 & 1 & k & A_3\\ \lambda_1 & \lambda_1^2 & l & 0\\ \lambda_2 & \lambda_2^2 & m & C_3\\ \lambda_3 & \lambda_3^2 & n & D_3\end{vmatrix}=\frac{z}{w}\begin{vmatrix}1 & k & k & A_3\\ \lambda_1 & l & l\lambda_1 & 0\\ \lambda_2 & m & m\lambda_2 & C_3\\ \lambda_3 & n & n\lambda_3 & D_3\end{vmatrix}.$$

The conditions for O_4 are:

$$\frac{y}{w}\begin{vmatrix}1 & 1 & k & A_4\\ \lambda_1 & \lambda_1^2 & l & 0\\ \lambda_2 & \lambda_2^2 & m & C_4\\ \lambda_3 & \lambda_3^2 & n & D_4\end{vmatrix}=\begin{vmatrix}1 & k & k & A_4\\ \lambda_1^2 & l & l\lambda_1 & 0\\ \lambda_2^2 & m & m\lambda_2 & C_4\\ \lambda_3^2 & n & n\lambda_3 & D_4\end{vmatrix}=-\frac{y}{z}\begin{vmatrix}1 & 1 & k & A_4\\ \lambda_1 & \lambda_1^2 & l\lambda_1 & 0\\ \lambda_2 & \lambda_2^2 & m\lambda_2 & C_4\\ \lambda_3 & \lambda_3^2 & n\lambda_3 & D_4\end{vmatrix}.$$

Through O_1 pass planes $(1, \lambda_2)=(1, \infty)=(1, \lambda_3)=(2, \lambda_1)=(2, 0)=(2, 1)=0$.
Through O_2: $(1, \lambda_2)=(1, 1)=(1, 0)=(2, \lambda_1)=(2, \lambda_3)=(2, \infty)=0$.
Through O_3: $(1, \infty)=(1, 1)=(1, \lambda_1)=(2, 0)=(2, \lambda_3)=(2, \lambda_2)=0$.
Through O_4: $(1, 0)=(1, \lambda_1)=(1, \lambda_3)=(2, \infty)=(2, 1)=(2, \lambda_2)=0$.

With the four basis points are associated four basis planes indicated by Π_i, where O_i lies in Π_i, $i=1, 2, 3, 4$. The σ vertices lie by sixes in the planes as follows:

In Π_1 lie $(\sigma, 2, \lambda_1)$; $(\sigma, 1, \lambda_2)$; $(\sigma, 2, 1)$; $(\sigma, 1, \lambda_3)$; $(\sigma, 2, 0)$; $(\sigma, 1, \infty)$.
In Π_2 lie $(\sigma, 1, \lambda_2)$; $(\sigma, 2, \lambda_3)$; $(\sigma, 1, 1)$; $(\sigma, 2, \infty)$; $(\sigma, 1, 0)$; $(\sigma, 2, \lambda_1)$.
In Π_3 lie $(\sigma, 2, \lambda_3)$; $(\sigma, 1, 1)$; $(\sigma, 2, 0)$; $(\sigma, 1, \infty)$; $(\sigma, 1, \lambda_1)$; $(\sigma, 2, \lambda_2)$.
In Π_4 lie $(\sigma, 2, \lambda_2)$; $(\sigma, 1, \lambda_1)$; $(\sigma, 1, \lambda_3)$; $(\sigma, 2, 1)$; $(\sigma, 1, 0)$; $(\sigma, 2, \infty)$.

Each basis plane contains six vertices of pencils and one basis point. Dually, each basis point lies in six planes of pencils and in one basis plane. Each vertex of a pencil lies in two basis planes, and in two planes of pencils. Dually, each plane of pencils contains two basis points and two vertices of pencils.

20. This configuration of sixteen points and sixteen planes contains four basis points O_i; four basis planes Π_i; twelve vertices of pencils P_{ik}, Q_{ik}; twelve planes of pencils σ_{ik}, τ_{ik}, where $i=1, \ldots, 4$, $k=1, \ldots, 4$, $i \neq k$ but $ik=ki$. In Π_i lie O_i and six points P_{ik}, Q_{ik}; Π_i and Π_k meet in l_{ik}, which contains P_{ik}, Q_{jl}, $i \neq k \neq j \neq l$; O_i, O_k, P_{ik} determine σ_{ik} and O_j, O_l, Q_{jl} determine τ_{ik};

σ_{ik} and τ_{ik} form a pair of singular planes which compose a quadric cone of the system H.

21. From Art. 9 it is evident that the sum of the orders of cones of all σ and τ lines through the basis points is three. It is easily seen that the σ lines form a plane pencil in each basis plane with vertex at the basis point. Hence the τ lines through the same point must form a quadric cone. Dual statements hold for the basis planes.

The σ lines form a pencil in each basis plane and through each basis point. The τ lines envelop a conic in each basis plane and define a quadric cone through each basis point.

22. If an attempt is made to arrange the basis points in the $\mathrm{I}\,(3,3)^{3,3}_{12,12}$ (Rocella's congruence) in a symmetrical manner, the planes of pencils arrange themselves in two distinct sets, six in each. Of the first set, each contains one and only one basis point, while of the second set each contains two basis points. To arrange them in this way requires an extra condition, leaving a fourth basis point impossible, unless six planes pass through a point, which will be shown later to be the case of eight basis points. The σ lines in the basis planes form pencils through the basis points. The τ lines envelop a conic in each basis plane and form a quadric cone through each basis point. The basis points for this case, together with the necessary and sufficient conditions, are:

O_1: $w=0$, $\mathrm{IV}=0$, $x+y\lambda_2-mz=0$. (Same as before for O_1.)

O_2: $x=0$, $\mathrm{I}=0$, $y-k(z+w)=0$. (Same as before for O_2.)

O_3: $\mathrm{I}=0$, $x+y\lambda_1-l(z+w\lambda_1)=0$, $x+y\lambda_3-n(z+w\lambda_3)=0$.

$$O_4: \begin{vmatrix} 1 & k & k & A_4 \\ \lambda_1^2 & l & l\lambda_1 & B_4 \\ \lambda_2^2 & m & m\lambda_2 & 0 \\ \lambda_3^2 & n & n\lambda_3 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & k & A_4 \\ \lambda_1 & \lambda_1^2 & l & B_4 \\ \lambda_2 & \lambda_2^2 & m & 0 \\ \lambda_3 & \lambda_3^2 & n & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & k & A_4 \\ \lambda_1 & \lambda_1^2 & l\lambda_1 & B_4 \\ \lambda_2 & \lambda_2^2 & m\lambda_2 & 0 \\ \lambda_3 & \lambda_3^2 & n\lambda_3 & 0 \end{vmatrix} = 0.$$

The focal surface* of the various forms of the Roccella congruences is self-dual.

Type II.

23. The general congruence

$$\frac{a+b\lambda+c\lambda^2+d\lambda^3}{x+y\lambda+z\lambda^2} = \frac{a'+b'\lambda}{w} = \sigma, \tag{6}$$

$$\frac{a+b\lambda+c\lambda^2+d\lambda^3}{a'+b'\lambda} = \frac{x+y\lambda+z\lambda^2}{w} = \tau \tag{7}$$

* Cayley: "On a Sibi Reciprocal Surfaces," *Berlin Monatsberichte* (1878), pp. 309–313; *Coll. Papers*, Vol. X, pp. 252–255. Kummer: "Ueber diejenigen Flächen ....," *Berlin Monatsberichte* (1878), pp. 25–37.

has a basis point, $a'=b'=w=0$, and the basis plane $w=0$. To determine the σ configuration at this point, substitute the coordinates of the point in the equation (6); then

$$\frac{\bar{a}+\bar{b}\lambda+\bar{c}\lambda^2+\bar{d}\lambda^3}{\bar{x}+\bar{y}\lambda+\bar{z}\lambda^2}=\bar{\sigma}=\frac{\Phi_3(\lambda)}{F_2(\lambda)},$$

where the $\bar{a}$, etc., indicate constant values when $a'=b'=w=0$. Substituting this value for σ in (6), the equations for all σ lines through the point are:

$$A_0\lambda^5+A_1\lambda^4+\ldots.=0,$$
$$B_0\lambda^3+B_1\lambda^2+\ldots.=0.$$

Eliminating λ, the resulting equation is of degree 8. Similar work for τ gives a plane pencil, which was to be expected, since the class of the congruence is 9.

The σ lines of the congruence (6) *through $a'=b'=w=0$ form a cone of order* 8, *while the τ lines of* (7) *through the same point form a plane pencil in the plane $w=0$.*

Since $w=0$ is a basis plane, containing a line of each quadric of the family H, it contains the vertices of the six quadric cones in the family H. The configuration of σ lines in $w=0$ is evident from equations (6). For $w=0$, $\sigma=\infty$; hence $x+y\lambda+z\lambda^2=0$, and the σ lines in the plane $w=0$ envelop the conic $x+y\lambda+z\lambda^2=0$. The orders of the σ and τ configurations in the basis plane $w=0$ are unchanged by the conditions imposed later.

The symbol for congruences (6), (7) *is* $\mathrm{II}(3,9)^{1,1}_{6,0}$.

25. If, in a manner similar to the work for Type I, conditions are imposed upon the equations (6) and (7) so that the cones degenerate one by one, the class of the congruence also is depressed and the order of the σ cone with vertex $a'=b'=w=0$ decreases by unity for each new condition.

In equations (6) let $a=0$; then

$$\frac{b\lambda+c\lambda^2+d\lambda^3}{x+y\lambda+z\lambda^2}=\frac{a'+b'\lambda}{w}=\sigma. \tag{20}$$

As before for $\lambda=0$, the quadric cone is composed of the planes $x=0$, $a'=0$. The σ vertices are:

$$(1',0)\equiv x=0,\quad (\sigma,1',0): x=w=a'=0;$$
$$(2',0)\equiv a=0,\quad (\sigma,2',0): x=b=a'=0.$$

The congruence has the basis plane $w=0$, the basis point $a'=b'=w=0$ and no additional basis points or basis planes.

The symbol for the congruence (20) *is* $\mathrm{II}(3,8)^{1,1}_{7,2}$.

26. Impose on (20) the new condition $d=0$. The new congruence,

$$\frac{b\lambda+c\lambda^2}{x+y\lambda+z\lambda^2}=\frac{a'+b'\lambda}{w}=\sigma, \tag{21}$$

has two cones, for $\lambda=0, \lambda=\infty$, which are composed of planes. The table of planes and vertices includes those of Art. 22 and

$$(1', \infty)\equiv z=0, \quad (\sigma, 1', \infty) : z=b'=w=0;$$
$$(2', \infty)\equiv b=0, \quad (\sigma, 2', \infty) : z=b'=c=0.$$

There are no new additional basis planes or basis points.

The symbol for the congruence (21) *is* $\mathrm{II}(3,7)^{1,1}_{8,4}$.

27. In order that three cones may be composed of planes, impose the condition that $\lambda-1$ may be a factor;* *i. e.*,

$$b+c=k(x+y+z). \tag{22}$$

$$\frac{[k(x+y+z)-c]\lambda+c\lambda^2}{x+y\lambda+z\lambda^2}=\frac{a'-b'\lambda}{w}=\sigma,$$

whence

$$\frac{(1-\lambda)[ky+kz(1+\lambda)-c\lambda]}{x+y\lambda+z\lambda^2}=\frac{a'+b'\lambda-kw}{w}=\sigma. \tag{23}$$

Then the pencils and planes are those of Art. 21 and the new ones:

$$(1', 1)\equiv x+y+z=0, \quad (\sigma, 1', 1) : a'+b'=w=x+y+z=0;$$
$$(2', 1)\equiv a+b-kw=0, \quad (\sigma, 2', 1) : x+y+z=a'+b'-kw=k(x-z)+c=0.$$

The symbol for this congruence is $\mathrm{II}(3,6)^{1,1}_{9,6}$.

28. In order that a fourth cone may be composed of planes, for $\lambda=\lambda_1$, the condition is

$$(1-\lambda_1)[ky+kz(1+\lambda_1)-c\lambda_1]=l(a'+b'\lambda_1-kw).$$

Then the congruence

$$\frac{(\lambda_1-\lambda)\left[(1-\lambda)(1-\lambda_1)(y+z)k-\dfrac{(a'-kw)(1+\lambda+\lambda_1)l+\lambda\lambda_1 lb'}{\lambda_1(1-\lambda_1)}\right]}{x+y\lambda+z\lambda^2}$$
$$=\frac{a'+b'\lambda-kw}{w}=\sigma \tag{24}$$

has four cones composed of planes. The complete table of planes and vertices of pencils includes those of Art. 24 and the following:

$$(1', \lambda_1)\equiv x+y\lambda_1+z\lambda_1^2-lw=0, \quad (\sigma, 1', \lambda_1) : x+y\lambda_1+z\lambda_1^2=w=a'+b'\lambda_1=0;$$
$$(2', \lambda_1)\equiv a'+b'\lambda_1-kw=0, \quad (\sigma, 2', \lambda_1) : x+y\lambda_1+z\lambda_1^2-lw=a'+b'\lambda_1-kw$$
$$=(1-\lambda_1)(1-\lambda_1)^2(y+z)k\lambda_1-(a'-kw)(1+\lambda_1+\lambda_1)l-\lambda_1\lambda_1 b'l=0.$$

The symbol for this congruence is $\mathrm{II}(3,5)^{1,1}_{10,8}$.

* The choice of $0, \infty, 1$ as three vertices of cones is no restriction on the congruence.

29. The condition to be imposed on (23) that a fifth cone may be composed of planes is:

$$(\lambda_1-\lambda_2)\,[(1-\lambda_2)(1-\lambda_1)^2(y+z)k\lambda_1-(a'-kw)(1+\lambda_1+\lambda_2)l] = m\lambda_1(1-\lambda_1)(a'+b'\lambda_2-kw)+(\lambda_1-\lambda_2)\lambda_1\lambda_2 l b'.$$

Imposing this condition gives the congruence

$$\frac{(\lambda_1-\lambda)\,[(1-\lambda)(1-\lambda_1)^2(y+z)k\lambda_1-(a'-kw)(1+\lambda+\lambda_1)l+\lambda\lambda_1 l b']}{\lambda_1(1-\lambda_1)(x+y\lambda+z\lambda^2)}$$
$$=\frac{k(\lambda_1-\lambda_2)\,[wl(1+\lambda_1+\lambda_2)+(y+z)(1-\lambda_2)(1-\lambda_1)^2\lambda_1-\lambda_1\lambda_2 l b']-m\lambda_1(1-\lambda_1)b'(\lambda-\lambda_2)-kwl(1+\lambda_1+\lambda_2)}{w[l(1+\lambda_1+\lambda_2)+m\lambda_1(1-\lambda_1)]}$$
$$\equiv\frac{A}{B(x+y\lambda+z\lambda^2)}=\frac{C}{Dw}=\sigma. \qquad (25)$$

The complete table of planes and vertices of pencils includes those of Art. 25 and in addition:

$(1',\lambda_2)\equiv x+y\lambda_2+z\lambda_2^2-mw=0$, $(\sigma,1',\lambda_2): x+y\lambda_2+z\lambda_2^2=w=a'+b'\lambda_2=0$;

$(2',\lambda_2)\equiv a'+b'\lambda_2-kw=0$, $(\sigma,2',\lambda_2): x+y\lambda_2+z\lambda_2^2-mw=a'+b'\lambda_2-kw$

$$=\frac{AD-mBC}{\lambda-\lambda_2}\Big]_{\lambda\doteq\lambda_2}=0.$$

The symbol for this congruence is $\mathrm{II}(3,4)^{1,1}_{11,10}$.

30. For the case in which all six cones are composed of pairs of planes, the six necessary and sufficient conditions to be simultaneously imposed are:

$$a=0,\quad d=0,\quad b+c=k(x+y+z),$$
$$b\lambda_1+c\lambda_1^2-l(x+y\lambda_1+z\lambda_1^2)=s(a'+b'\lambda_1-lw),$$
$$b\lambda_2+c\lambda_2^2-m(x+y\lambda_2+z\lambda_2^2)=t(a'+b'\lambda_2-mw),$$
$$b\lambda_3+c\lambda_3^2-n(x+y\lambda_3+z\lambda_3^2)=p(a'+b'\lambda_3-nw).$$

Imposing these conditions on equation (9), there results:

$$\begin{vmatrix} 1 & 1 & 0 & 0 & k(x+y+z) \\ \lambda_1 & \lambda_1^2 & s & s\lambda_1 & l(x+y\lambda_1-sw) \\ \lambda_2 & \lambda_2^2 & t & t\lambda_2 & m(x+y\lambda_2-tw) \\ \lambda_3 & \lambda_3^2 & p & p\lambda_3 & n(x+y\lambda_3-pw) \\ w\lambda & w\lambda^2 & x+y\lambda+z\lambda^2 & (x+y\lambda+z\lambda^2)\lambda & 0 \end{vmatrix}=0.$$

Reducing this determinant to the form of (6) gives:

$$\lambda\begin{vmatrix} 1-\lambda & 0 & 0 & k(x+y+z) \\ \lambda_1(\lambda_1-\lambda) & s & s\lambda_1 & l(x+y\lambda_1+z\lambda_1^2-sw) \\ \lambda_2(\lambda_2-\lambda) & t & t\lambda_2 & m(x+y\lambda_2+z\lambda_2^2-tw) \\ \lambda_3(\lambda_3-\lambda) & p & p\lambda_3 & n(x+y\lambda_3+z\lambda_3^2-pw) \end{vmatrix}$$
$$x+y\lambda+z\lambda^2$$
$$=-\frac{\begin{vmatrix} 1 & 1 & 0 & k(x+y+z) \\ \lambda_1 & \lambda_1^2 & s(\lambda_1-\lambda) & l(x+y\lambda_1+z\lambda_1^2-sw) \\ \lambda_2 & \lambda_2^2 & t(\lambda_2-\lambda) & m(x+y\lambda_2+z\lambda_2^2-tw) \\ \lambda_3 & \lambda_3^2 & p(\lambda_3-\lambda) & n(x+y\lambda_3+z\lambda_3^2-pw) \end{vmatrix}}{w}=\sigma, \qquad (26)$$

or
$$\frac{S}{x+y\lambda+z\lambda^2}=\frac{\tau}{w}=\sigma.$$

Applying Art. 8, and denoting by A, B, C, D the elements of the last column of these determinants, the complete table of planes and vertices of pencils is:

$$(1',0)\ : x=0; \quad (\sigma,1',0)\ : x=w=\begin{vmatrix} 1 & 1 & 0 & A \\ \lambda_1 & \lambda_1^2 & s\lambda_1 & B \\ \lambda_2 & \lambda_2^2 & t\lambda_2 & C \\ \lambda_3 & \lambda_3^2 & p\lambda_3 & D \end{vmatrix}=0.$$

$$(2',0)\equiv\begin{vmatrix} 1 & 1 & 0 & A \\ \lambda_1 & \lambda_1^2 & s\lambda_1 & B \\ \lambda_2 & \lambda_2^2 & t\lambda_2 & C \\ \lambda_3 & \lambda_3^2 & p\lambda_3 & D \end{vmatrix}=0; \quad (\sigma,2',0)\ : (1',0)=\begin{vmatrix} 1 & 0 & 0 & A \\ \lambda_1^2 & s & s\lambda_1 & B \\ \lambda_2^2 & t & t\lambda_2 & C \\ \lambda_3^2 & p & p\lambda_3 & D \end{vmatrix}=0=(2',0).$$

$$(1',\infty)\equiv z=0; \quad (\sigma,1',\infty)\ : (1',\infty)=(2',\infty)=w=0.$$

$$(2',\infty)\equiv\begin{vmatrix} 1 & 1 & 0 & A \\ \lambda_1 & \lambda_1^2 & s & B \\ \lambda_2 & \lambda_2^2 & t & C \\ \lambda_3 & \lambda_3^2 & p & D \end{vmatrix}=0; \quad (\sigma,2',\infty)\ : (1',\infty)=(2',\infty)=\begin{vmatrix} 1 & 0 & 0 & A \\ \lambda_1 & s & s\lambda_1 & B \\ \lambda_2 & t & t\lambda_2 & C \\ \lambda_3 & p & p\lambda_3 & D \end{vmatrix}=0.$$

$$(1',1)\equiv x+y+z=0; \quad (\sigma,1',1)\ : x+y+z=(2',1)=w=0.$$

$$(2',1)\equiv\begin{vmatrix} 1 & 1 & 0 & A \\ \lambda_1 & \lambda_1^2 & s(\lambda_1-1) & B \\ \lambda_2 & \lambda_2^2 & t(\lambda_2-1) & C \\ \lambda_3 & \lambda_3^2 & p(\lambda_3-1) & D \end{vmatrix}=0; \quad (\sigma,2',1)\ : (1',1)=(2',1)=\frac{S+k}{\lambda-1}\Big]_{\lambda=1}=0.$$

$$(1',\lambda_1)\equiv x+y\lambda_1+z\lambda_1^2-sw=0; \quad (\sigma,1',\lambda_1)\ : (1',\lambda_1)=(2',\lambda_1)=w=0.$$

$$(2',\lambda_1)\equiv\begin{vmatrix} 1 & 1 & 0 & A \\ \lambda_1 & \lambda_1^2 & 0 & B \\ \lambda_2 & \lambda_2^2 & t(\lambda_2-\lambda_1) & C \\ \lambda_3 & \lambda_3^2 & p(\lambda_3-\lambda_1) & D \end{vmatrix}=0; \ (\sigma,2',\lambda_1)\ : (1',\lambda_1)=(2',\lambda_1)=\frac{S-sT}{\lambda-\lambda_1}\Big]_{\lambda=\lambda_1}=0.$$

$$(1',\lambda_2)\equiv x+y\lambda_2+z\lambda_2^2-tw=0; \quad (\sigma,1',\lambda_2)\ : (1',\lambda_2)=(2',\lambda_2)=w=0.$$

$$(2',\lambda_2)\equiv\begin{vmatrix} 1 & 1 & 0 & A \\ \lambda_1 & \lambda_1^2 & s(\lambda_1-\lambda_2) & B \\ \lambda_2 & \lambda_2^2 & 0 & C \\ \lambda_3 & \lambda_3^2 & p(\lambda_3-\lambda_2) & D \end{vmatrix}=0; \ (\sigma,2',\lambda_2)\ : (1',\lambda_2)=(2',\lambda_2)=\frac{S-tT}{\lambda-\lambda_2}\Big]_{\lambda=\lambda_2}=0.$$

$$(1',\lambda_3)\equiv x+y\lambda_3+z\lambda_3^2-pw=0; \quad (\sigma,1',\lambda_3)\ : (1',\lambda_3)=(2',\lambda_3)=w=0.$$

$$(2',\lambda_3)\equiv\begin{vmatrix} 1 & 1 & 0 & A \\ \lambda_1 & \lambda_1^2 & s(\lambda_1-\lambda_3) & B \\ \lambda_2 & \lambda_2^2 & t(\lambda_2-\lambda_3) & C \\ \lambda_3 & \lambda_3^2 & 0 & D \end{vmatrix}=0; \ (\sigma,2',\lambda_3)\ : (1',\lambda_3)=(2',\lambda_3)=\frac{S-pT}{\lambda-\lambda_3}\Big]_{\lambda=\lambda_3}=0.$$

The symbol for this congruence is $\Pi(3,3)^{1,1}_{12,12}$.

31. As in the case of Type I, additional basis points and basis planes may be introduced and a configuration of σ and τ lines similar to that of Type I exists. These basis points, together with the necessary and sufficient conditions for their existence, are:

$$O_1\colon\ (1',0)=(1',1)=(2',\infty)=0,$$
$$O_2\colon\ (1',\infty)=(1',\lambda_1)=(2',0)=0,$$
$$O_3\colon\ (1',\lambda_2)=(2',0)=(2',\infty)=0,$$
$$O_4\colon\ a'=b'=w=0.$$

For O_1:

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ \lambda_1 & \lambda_1^2 & s\lambda_1 & B' \\ \lambda_2 & \lambda_2^2 & t\lambda_2 & C' \\ \lambda_3 & \lambda_3^2 & p\lambda_3 & D' \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ \lambda_1 & s & s\lambda_1 & B' \\ \lambda_2 & t & t\lambda_2 & C' \\ \lambda_3 & p & p\lambda_3 & D' \end{vmatrix} = 0.$$

For O_2:

$$\begin{vmatrix} 1 & 0 & 0 & \bar{A} \\ \lambda_1^2 & s & s\lambda_1 & 0 \\ \lambda_2^2 & t & t\lambda_2 & \bar{C} \\ \lambda_3^2 & p & p\lambda_3 & \bar{D} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & \bar{A} \\ \lambda_1 & \lambda_1^2 & s & 0 \\ \lambda_2 & \lambda_2^2 & t & \bar{C} \\ \lambda_3 & \lambda_3^2 & p & \bar{D} \end{vmatrix} = 0.$$

For O_3:

$$\begin{vmatrix} 1 & 1 & 0 & \bar{\bar{A}} \\ \lambda_1 & \lambda_1^2 & s & \bar{\bar{B}} \\ \lambda_2 & \lambda_2^2 & t & 0 \\ \lambda_3 & \lambda_3^2 & p & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & \bar{\bar{A}} \\ \lambda_1^2 & s & s\lambda_1 & \bar{\bar{B}} \\ \lambda_2^2 & t & t\lambda_2 & 0 \\ \lambda_3^2 & p & p\lambda_3 & 0 \end{vmatrix} = 0.$$

32. *It is possible to introduce into any of the preceding congruences one, two, three, four, five or six basis points without reducing the family of quadrics to a net of quadrics.*

Consider the family

$$H(\lambda)\equiv H_0\lambda^3+H_1\lambda^2+H_2\lambda+H_3=0.$$

$H_0=0$, $H_1=0$, $H_2=0$ have eight points in common. By imposing the condition that $H_3=0$ passes through this point, a basis point may be introduced into the family. This operation may be repeated for six distinct points, but if an attempt is made to introduce a seventh basis point, the eighth enters simultaneously, since the eight points commom to the three quadrics are associated points on a quartic curve.

33. It is to be noted that the points so chosen are not entirely arbitrary. If the points chosen are arbitrary points, it is not always possible to introduce the complete number. For example, in the cases of $\mathrm{I}(3,3)^{4,4}_{12,12}$, $\mathrm{II}(3,3)^{4,4}_{12,12}$,

discussed in Arts. 22, 31, it is impossible to introduce even a fifth basis point without the entire eight. An interesting example of this is the case of the congruences (4) and (5) when the vertices of the tetrahedron of reference are chosen as basis points. If these are basis points, the equation (4) takes the form:

$$\frac{(a_1x+a_2y)+(b_1x+b_2y)\lambda+(c_1x+c_2y)\lambda^2}{x+y\lambda} = \frac{(a_3'z+a_4'w)+(b_3'z+b_4'w)\lambda+(c_3'z+c_4'w)\lambda^2}{z+w\lambda},$$

where $a \equiv a_1x+a_2y+a_3z+a_4w$, etc.

Let any other point x_1, y_1, z_1, w_1 be supposed to be a basis point; then any point of $x_1y=y_1x$, $z_1w=w_1z$ will also satisfy the identity. Hence no other point can be introduced as a basis point without introducing an entire line. The configuration of cones of σ and τ lines at the basis point is a K_6 of σ lines* and a K_3 of τ lines at each of the four basis points as a vertex. If, however, care is taken in the choice of points, it is often possible to introduce several rational points as basis points. An example of this is the following special case of

$$\frac{a+b\lambda+c\lambda^2}{x+y\lambda} = \frac{a'+b'\lambda+c'\lambda^2}{z+w\lambda} = \sigma \qquad (4)$$

when (0, 0, 0, 1), (1, 0, 1, 0), (1, 1, 1, 1), (0, 1, 0, 1), (1, 0, 0, 0) are chosen as basis points. These five require the twelve conditions, which leave the congruence still (3, 9):

$$a_4=b_4=c_4=a_1+a_3-a_3'=b_1+b_3-b_3'=c_1+c_3-c_3'$$
$$=a_2-a_2'-a_4'=b_2-b_2'-b_4'=c_2-c_2'-c_4'=a_1'=b_1'=c_1'=0.$$

The resulting congruence is:

$$\frac{a_1x+a_2y+a_3z+(b_1x+b_2y+b_3z)\lambda+(c_1x+c_2y+c_3z)\lambda^2}{a_2'y+(a_1+a_3)z+(a_2-a_2')w+[b_2'y+(b_1+b_3)z+(b_2-b_2')w]\lambda+[c_2y+(c_1+c_3)z+(c_2-c_2')w]\lambda^2} = \frac{x+y\lambda}{z+w\lambda} = \tau.$$

The configurations of lines at the basis points are as follows: at (0, 0, 0, 1) and (1, 0, 0, 0), K_6^σ, and K_3^τ; at the remaining points, K_4^σ, K_5^τ.

For a set of six basis points (0, 0, 0, 1), (1, 0, 1, 0), (1, 1, 1, 1), (0, 1, 0, 1), (1, 0, 0, 0), (0, 1, 1, 0), the following sixteen conditions are necessary and sufficient if imposed on equations (4):

$a_1'=b_1'=c_1'=a_4=b_4=c_4=0$, $a_3=-a_2$, $a_2'=b_2+b_3-a_1+a_2$, $b_2'=b_2+b_3+c_2+c_3$, $c_2'=c_1-c_3$, $a_3'=a_1-a_2$, $b_3'=b_1+b_3$, $c_3'=c_1+c_3$, $a_4'=a_1-b_2-b_3$, $b_4'=-b_3-c_2-c_3$, $c_4'=c_1+c_2+c_3$.

* For definition of symbols, see Snyder: *Transactions, loc. cit.*, p. 380.

The equations for the congruences are then:

$$\frac{N}{D}=\frac{x+y\lambda}{z+w\lambda}=\tau,$$

where

$$N\equiv a_1x+a_2(y-z)+(b_1x+b_2y+b_3z)\lambda+(c_1x+c_2y+c_3z)\lambda^2.$$
$$D\equiv(b_2+b_3-a_1+a_2)y+(a_1-a_2)z+(a_1-b_2-b_3)w$$
$$+[(b_2+b_3+c_2+c_3)y+(b_1+b_3)z-(c_1+c_3+b_3)w]\lambda$$
$$+[-(c_1+c_3)y+(c_1+c_3)z+(c_1+c_2+c_3)w]\lambda^2.$$

The configurations at all of the basis points are K_4^σ, K_5^τ, except at $(0,0,0,1)$ and $(1,0,0,0)$, where they are K_6^σ, K_3^τ. The introduction of any other basis point brings with it the eighth point, and the family of quadrics is contained in a net of quadrics through the basis points.

Congruences on Nets of Quadrics.

32. In order to determine the number of σ lines which coincide with τ lines of the conjugate congruences

$$\frac{a+b\lambda+c\lambda^2}{x+y\lambda}=\frac{a'+b'\lambda+c'\lambda^2}{z+w\lambda}=\sigma, \tag{4}$$

$$\frac{a+b\lambda+c\lambda^2}{a'+b'\lambda+c'\lambda^2}=\frac{x+y\lambda}{z+w\lambda}=\tau, \tag{5}$$

assume that the σ line for λ_1 coincides with the τ line for λ_2. Then

$$a+b\lambda_1+c\lambda_1^2+k(a'+b'\lambda_1+c'\lambda_1^2)-\sigma[x+y\lambda_1+k(z+w\lambda_1)]$$
$$\equiv m[a+b\lambda_2+c\lambda_2^2-\tau(a'+b'\lambda_2+c'\lambda_2^2)];$$

and

$$a+b\lambda_1+c\lambda_1^2+l(a'+b'\lambda_2+c'\lambda_2^2)-\sigma[x+y\lambda_1+l(z+w\lambda_1)]$$
$$\equiv n[x+y\lambda_2-\tau(w+z\lambda_2)].$$

Each of these identities requires three independent conditions; *i.e.*, a total of six conditions on the six constants λ_1, λ_2, σ, τ, k, l.

Hence only a finite number of σ lines can coincide with τ lines in the general (3, 9) *congruences without basis points.*

33. Consider the family of quadrics:

$$H(\lambda)\equiv\lambda^3Q_0+\lambda^2Q_1+\lambda Q_2+Q_3=0.$$

Let the powers of λ be considered as separate parameters, μ_1, μ_2, μ_3, μ_4. If these be considered as point coordinates in space, there is, corresponding to the family of quadrics, a twisted cubic. Taking the discriminant of this cubic, there results a quartic surface which intersects the cubic in twelve points.

From former considerations (Art. 6), these points must occur in coincident pairs; hence the surface and curve have six points of contact. If, then, it were possible for a λ_1 to exist such that all σ lines for that value were equal, respectively, to all τ lines for another value λ_2, *i. e.*, for a quadric surface of σ lines to coincide with a quadric surface of τ lines, or two quadrics of H to be coincident, the corresponding cubic must have a double point. This can happen only if the cubic degenerates.

Hence there are no surfaces of σ lines which coincide with surfaces of τ lines, excepting the six quadric cones of the system when the quadrics Q_i can not be expressed as linear functions of three of them.

34. If the quadrics Q_i of the family

$$H(\lambda)\equiv\lambda^3Q_0+\lambda^2Q_1+\lambda Q_2+Q_3=0$$

can be expressed linearly in terms of three of them, the family must have eight basis points at the eight points of intersection of $Q_0=0$, $Q_1=0$, $Q_2=0$.

The family of quadrics

$$H(\lambda)\equiv\frac{a+b\lambda+c\lambda^2}{x+y\lambda}=\frac{a'+b'\lambda+c'\lambda^2}{z+w\lambda}$$

contains no basis point or basis plane unless restrictions are made on the constants. If this family be rewritten

$$\begin{aligned}H(\lambda)\equiv\lambda^3(cw-c'y)+\lambda^2(cz+bw-b'y-c'x)\\+\lambda(aw+bz-a'y-b'x)+(az-a'x)=0,\end{aligned}$$

and the powers of λ be considered as parameters μ_i, $i=1, \ldots, 4$, the relations which are equivalent to a linear relation among the fundamental quadrics of the family are:

$$\mu_1(cw-c'y)+\mu_2(bw+cz-b'y-c'x)+\mu_3(aw+bz-a'y-b'x)+(az-a'x)\equiv 0.$$

The necessary and sufficient condition for this identity is the consistency of the following ten equations:

$$\begin{aligned}\mu_1a_1'+\mu_2b_1'+\mu_3c_1'&=0,\\ \mu_2a_2'+\mu_3b_2'+\mu_4c_2'&=0,\\ \mu_1a_3+\mu_2b_3+\mu_3c_3&=0,\\ \mu_2a_4+\mu_3b_4+\mu_4c_4&=0,\\ \mu_1a_2'+\mu_2(a_1'+b_2')+\mu_3(b_1'+c_2')+\mu_4c_1'&=0,\\ \mu_1(a_3'-a_1)+\mu_2(b_1-b_3')+\mu_3(c_1-c_3')&=0,\\ \mu_1a_2+\mu_2(b_2-a_3')+\mu_3(c_2-b_3')-\mu_4c_3'&=0,\\ \mu_1a_4'+\mu_2(b_4'-a_1)+\mu_3(c_4'-b_1)-\mu_4c_1&=0,\\ \mu_2(a_2-a_4')+\mu_3(b_2-b_4')+\mu_4(c_2-c_4')&=0,\\ \mu_1a_4+\mu_2(a_3+b_4)+\mu_3(b_3+c_4)+\mu_4c_3&=0.\end{aligned}$$

For this the necessary and sufficient condition is:

$$\left\|\begin{matrix} a_1' & 0 & a_3 & 0 & a_2' & (a_3'-a_1) & a_2 & a_4' & 0 & a_4 \\ b_1' & a_2' & b_3 & a_4 & (a_1'+b_2') & (b_1-b_3') & (b_2-a_3') & (b_4'-a_1) & (a_2-a_4') & (a_3+b_4) \\ c_1' & b_2' & c_3 & b_4 & (b_1'+c_2') & (c_1-c_3') & (c_2-b_3') & (c_4'-b_1) & (b_2-b_4') & (c_4+b_3) \\ 0 & c_2' & 0 & c_4 & (c_1') & (0) & -c_3' & -c_1 & (c_2-c_4') & c_3 \end{matrix}\right\| = 0.$$

The matrix is equal to zero if the following seven determinants are zero:

$$\begin{vmatrix} c_3 & c_1' & -c_3' & m_i \\ b_3+c_4 & b_1'+c_2' & c_2-b_3' & n_i \\ a_3+b_4 & a_1'+b_2' & b_2-a_3' & p_i \\ a_4 & a_2' & a_2 & q_i \end{vmatrix} = 0, \quad i=1, \ldots, 7,$$

the fourth column being replaced in turn by the seven columns of the matrix remaining after picking out the above three.

Hence, by imposing seven independent conditions upon the constants of the congruences (4), (5), *it is possible to introduce eight basis points simultaneously.*

35. Suppose there exist two values of λ (λ_1, λ_2) such that for these values the corresponding quadrics of H are identical. The condition for the possibility of this is the consistency of ten simultaneous equations in the four variables $(\lambda_1^3-k\lambda_2^3)$, $(\lambda_1^2-k\lambda_2^2)$, $(\lambda_1-k\lambda_2)$, $(1-k)$. If these variables are called μ_i, these ten equations are identical with those of the preceding paragraph, the consistency of which is the necessary and sufficient condition for the introduction of a linear relation among the fundamental quadrics.

Every rational pair of conjugate congruences which has its lines arranged on a net of quadrics contains all the lines of one quadric as double lines of the congruence. This quadric will be called the double quadric of the congruence.

It will be shown later that in the congruence $I(3, 9)^8$ every line of the double quadric is either twice a σ line or twice a τ line, *i.e.*, one system of generators is composed of double lines of the σ congruence and the other system is composed of double lines of the τ congruence

36. The quadrics of the family

$$H(\lambda) \equiv \frac{a+b\lambda+c\lambda^2}{x+y\lambda+z\lambda^2} = \frac{a'+b'\lambda}{w} = \sigma$$

contain a basis point $a'=b'=w=0$, and a basis plane, $w=0$. Introducing seven additional basis points in this family introduces simultaneously a double quadric, since in each case the necessary and sufficient condition is:

$$\left\|\begin{matrix} a_1' & 0 & 0 & a_4 & a_2' & a_3' & a_1-a_4' & 0 & a_2 & a_3 \\ b_1' & a_2' & 0 & b_4 & a_1'+b_2' & b_3' & b_1-b_4' & a_3' & b_2-a_4' & b_3 \\ 0 & b_2' & a_3' & c_4 & b_1' & a_1' & c_1 & a_2+b_3' & c_2-b_4' & c_3-a_4' \\ 0 & 0 & b_3' & d_4 & 0 & b_1' & d_1 & b_2' & d_2 & d_3-b_4' \end{matrix}\right\| = 0.$$

The conditions for this are the vanishing of the seven determinants

$$\begin{vmatrix} a_1' & 0 & 0 & m_i \\ b_1' & a_2' & 0 & n_i \\ 0 & b_2' & a_3' & p_i \\ 0 & 0 & b_3' & q_i \end{vmatrix} = 0, \quad i=1, \ldots, 7,$$

the last column being replaced in turn by the seven columns of the matrix remaining after picking out the above three. These seven conditions reduce to six independent ones, since the two determinants

$$\begin{vmatrix} a_1' & 0 & 0 & 0 \\ b_1' & a_2' & 0 & a_3' \\ 0 & b_2' & a_3' & a_2'+b_3' \\ 0 & 0 & b_3' & b_2' \end{vmatrix} = a_1'(a_2'+b_3')(b_2'a_3'-a_2'b_3') = 0;$$

hence $b_2'a_3'=a_2'b_3'$, since $a_2'+b_3' \neq 0$, and $a_1' \neq 0$; and

$$\begin{vmatrix} a_1' & 0 & 0 & a_2' \\ b_1' & a_2' & 0 & a_1'+b_2' \\ 0 & b_2' & a_3' & b_1' \\ 0 & 0 & b_3' & 0 \end{vmatrix} = 0 = b_3'(a_1'+b_2')(a_2'b_1'-a_1'b_2');$$

hence $a_1'b_2'-a_2'b_1'=0$, since $b_3' \neq 0$, and $a_1'+b_2' \neq 0$, are equivalent to the single condition $a_1'b_3'-a_3'b_1'=0$. Hence this insures the vanishing of the determinant:

$$\begin{vmatrix} a_1' & 0 & 0 & a_3' \\ b_1' & a_2' & 0 & b_3' \\ 0 & b_2' & a_3' & a_1' \\ 0 & 0 & b_3' & b_1' \end{vmatrix} = (a_1'a_2'-b_2'b_3')(a_1'b_3'-a_3'b_1').$$

Hence, for the congruences (6) *and* (7), *six independent conditions are necessary and sufficient to introduce seven additional basis points and a double quadric.*

37. In the case of all congruences of Type II, the plane $w=0$ is always a basis plane, and therefore is tangent to each quadric of the family. It will contain one generator of each system on the double quadric. Through this point the two τ lines are $w=a'+b'\lambda_1=0$ and $w=a'+b'\lambda_2=0$. Hence, on the double quadric the basis plane contains a τ line as a generator of each system once. This is equivalent to saying that the τ lines of the quadric $Q(\lambda_1)$ form

one system, and the τ lines of the quadric $Q(\lambda_2)$ form the second system of generators of the double quadric.

Each line of the double quadric for the conjugate congruences $\mathrm{II}(3,9)^{8,1}$ *is once a* τ *line and once a* σ *line.*

The only condition under which a congruence of Type II can contain a double quadric of the $2\sigma, 2\tau$ type, is that one of the six quadric cones should be a double quadric. This can happen only if the congruence is of class less than 9, hence if the cone degenerates to a pair of planes.

38. In any congruence it is possible to determine the relation between the class of the congruence and the order of the singular cone whose vertex is at a basis point.

For any family of quadrics

$$Q_0\lambda^n+Q_1\lambda^{n-1}+Q_2\lambda^{n-2}+\ldots.=0,$$

the tangent plane has the equation

$$V_0\lambda^n+V_1\lambda^{n-1}+V_2\lambda^{n-2}+\ldots.=0,$$

where Q_i is of order 2, V_i is of order 1. The λ eliminant of these two equations is the equation of the singular cone at a basis point. In the general case this will be K_{3n}; but if any cone of the family is composed of planes, there will appear a factor in the equation and the eliminant is K_{3n-i}, where i is the number of cones of the family which are composed of planes. But the class of the congruence is $3n-i$.

The class of a congruence is equal to the order of the singular cone whose vertex is at a basis point; i. e., is equal to the sum of the orders of the σ *and* τ *configurations at any basis point.*

39. Every line joining a pair of basis points is, in general, a triple line of the composite singular cone belonging to a pair of conjugate congruences. If these two points lie in a singular plane, the multiplicity of the line is decreased by one. Hence, if three singular planes pass through a pair of basis points, the line joining them becomes a neutral line. In all the work that follows, the notation $K_n(i; j_r)$ will be used to indicate a cone of order n of σ lines, vertex at basis point i; having the line ij as an r-fold line. The same notation with K' indicates a cone of τ lines.

40. If the coordinates of a basis point be substituted in the equations which define a congruence, these equations become indeterminate. This may happen in any one of three ways: (1) in one of the fractions both numerator and denominator may vanish simultaneously; (2) both numerators may vanish;

or (3) both denominators may vanish. The third of these possibilities has been eliminated in congruences (4), ... , (7) by the choice of coordinate axes.

It is possible, from consideration of the equations and the method of procedure due to Snyder,* to write in full the σ and τ configurations at each basis point and to determine the congruences which may be written in the form of Type I or Type II.

41. For the case of a (3,9) congruence there are four pairs of conjugate congruences distinguished by their configurations. The first of these is that determined by $K_1(1)$, a basis plane through one and only one basis point. When conditions for eight basis points are imposed on $\mathrm{II}(3,9)^{1,1}$, the point $a'=b'=w$ is still a basis point. Then the configuration of τ lines was shown in Art. 23 to be a pencil and the configuration of σ lines a cone of order 8. No other basis point lies in the plane $w=0$. Hence all the other basis points must lie on the K_8. Further, every line joining two basis points is a triple line on K_8. The complete configuration for this congruence is:

$$\begin{cases} K_8(1;\, 2_3 3_3 4_3 5_3 6_3 7_3 8_3) \\ K_1'(1) \end{cases} \qquad \begin{cases} K_4(2;\, 1_3 3_1 4_1 5_1 6_1 7_1 8_1) \\ K_5'(2;\, 3_2 4_2 5_2 6_2 7_2 8_2) \end{cases}$$

Cones similar to those at (2) exist at points 3, ... , 8. The double quadric in this case is composed of lines each of which is once a σ line and once a τ line.

42. For the case of $\mathrm{I}(3,9)^8_6$ the basis points may be classified according as their coordinates satisfy

(A) $a+b\lambda+c\lambda^2-k(a'+b'\lambda+c'\lambda^2)=0, \quad x+y\lambda-k(z+w\lambda)=0;$

(B) $a'+b'\lambda+c'\lambda^2=0, \quad a+b\lambda+c\lambda^2=0.$

If condition (A) is fulfilled,

$$\frac{\bar{a}+\bar{b}\lambda+\bar{c}\lambda^2}{\bar{x}+\bar{y}\lambda}=\frac{F_2(\lambda)}{f_1(\lambda)}=\bar{\sigma},$$

where the dash over the letter indicates its particular value when the coordinates of the basis point have been substituted. The equations for all σ lines through this point are:

$$\frac{a+b\lambda+c\lambda^2}{x+y\lambda}=\frac{a'+b'\lambda+c'\lambda^2}{z+w\lambda}=\frac{F_2(\lambda)}{f_1(\lambda)}.$$

For the configuration of all σ lines, therefore, there exists a cone of order 6. For the τ lines the configuration is the eliminant of the equations

$$a+b\lambda+c\lambda^2=x+y\lambda=0.$$

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* Snyder: *Transactions, loc. cit.*

This is a cubic cone, as was to be expected from the preceding work and Art. 38. This pair of cones fixes an entire configuration as soon as the lines of K_3' are fixed.* The cubic cone must have six lines joining basis points, and it must not pass through all eight basis points, since these eight points lie on a quartic curve and the cubic cone defined by this quartic with vertex on the curve would define a (2,6) congruence. Hence there are left but two possibilities: either the cone has simple lines through six basis points or it has simple lines through four points and a double line through the fifth.

The first case, in which the double quadric has each line either twice σ or twice τ, has the complete configuration:

$$\begin{cases} K_6(1;\, 2_2 3_2 4_2 5_2 6_2 7_2 8_3) \\ K_3'(1;\, 2_1 3_1 4_1 5_1 6_1 7_1) \end{cases} \qquad \begin{cases} K_4(2;\, 1_2 3_1 4_1 5_1 6_1 7_1 8_2) \\ K_5'(2;\, 1_1 3_2 4_2 5_2 6_2 7_2 8_1) \end{cases}$$

$$\begin{cases} K_4(3;\, 1_2 2_1 4_1 5_1 6_1 7_1 8_2) \\ K_5'(3;\, 1_1 2_2 4_2 5_2 6_2 7_2 8_1) \end{cases} \qquad \begin{cases} K_4(4;\, 1_2 2_1 3_1 5_1 6_1 7_1 8_2) \\ K_5'(4;\, 1_1 2_2 3_2 5_2 6_2 7_2 8_1) \end{cases}$$

$$\begin{cases} K_4(5;\, 1_2 2_1 3_1 4_1 6_1 7_1 8_2) \\ K_5'(5;\, 1_1 2_2 3_2 4_2 6_2 7_2 8_1) \end{cases} \qquad \begin{cases} K_4(6;\, 1_2 2_1 3_1 4_1 5_1 7_1 8_2) \\ K_5'(6;\, 1_1 2_2 3_2 4_2 5_2 7_2 8_1) \end{cases}$$

$$\begin{cases} K_4(7;\, 1_2 2_1 3_1 4_1 5_1 6_1 8_2) \\ K_5'(7;\, 1_1 2_2 3_2 4_2 5_2 6_2 8_1) \end{cases} \qquad \begin{cases} K_6(8;\, 1_3 2_2 3_2 4_2 5_2 6_2 7_2) \\ K_3'(8;\, 2_1 3_1 4_1 5_1 6_1 7_1) \end{cases}$$

Since all the cones are rational, each of the cones has a double edge which is not an ik line. This double line must lie on the double quadric. If the basis point is of the type (B), the σ lines form a K_4. The only possibilities for this are $K_4(1;\, 2_2 3_2 4_1 5_1 6_1 7_1 8_1)$ or $K_4(1;\, 2_2 3_2 4_2 5_1 6_1 7_1)$. The second of these requires a K_2 which can not occur, and the first gives the same configuration as $K_3'(1;\, 2_1 3_1 4_1 5_1 6_1 7_1)$. Hence the above configuration is the only possible one for the general congruence $\mathrm{I}(3,9)_6^8$.

43. If, in the congruences (4) and (5), the denominators are taken as linear functions of the variables, the congruences become

$$\frac{a+b\lambda+c\lambda^2}{m+n\lambda} = \frac{a'+b'\lambda+c'\lambda^2}{m'+n'\lambda} = \sigma. \quad [\mathrm{III}(3,9).]$$

If the conditions for eight basis points be imposed on these equations, the possibilities for a configuration will include those of Art. 42 and, in addition to these, a $K_2'(1;\, 3_1 4_1 5_1)$. The complete configurations for these are:

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* Snyder: *Transactions, loc. cit.*

$$\begin{cases} K_7(1;\,2_3 3_2 4_2 5_2 6_3 7_3 8_3) \\ K_2'(1;\,3_1 4_1 5_1) \end{cases} \qquad \begin{cases} K_5(2;\,1_3 3_1 4_1 5_1 6_2 7_2 8_2) \\ K_4'(2;\,3_2 4_2 5_2 6_1 7_1 8_1) \end{cases}$$

$$\begin{cases} K_3(3;\,1_2 2_1 6_1 7_1 8_1) \\ K_6'(3;\,1_1 2_2 4_3 5_3 6_2 7_2 8_2) \end{cases} \qquad \begin{cases} K_3(4;\,1_2 2_1 6_1 7_1 8_1) \\ K_6'(4;\,1_1 2_2 3_3 5_3 6_2 7_2 8_2) \end{cases}$$

$$\begin{cases} K_3(5;\,1_2 2_1 6_1 7_1 8_1) \\ K_6'(5:\,1_1 2_2 3_3 4_3 6_2 7_2 8_2) \end{cases} \qquad \begin{cases} K_5(6;\,1_3 2_2 3_1 4_1 5_1 7_2 8_2) \\ K_4'(6;\,2_1 3_2 4_2 5_2 7_1 8_1) \end{cases}$$

$$\begin{cases} K_5(7;\,1_3 2_2 3_1 4_1 5_1 6_2 8_2) \\ K_4'(7;\,2_1 3_2 4_2 5_2 6_1 8_1) \end{cases} \qquad \begin{cases} K_5(8;\,1_3 2_2 3_1 4_1 5_1 6_2 7_2) \\ K_4'(8;\,2_1 3_2 4_2 5_2 6_1 7_1) \end{cases}$$

The double quadric is a σ, τ in this case. By imposing conditions on the basis points that they shall lie by fours in a pair of planes, etc., it is possible to obtain (3,9) congruences which have the basis points arranged by fours in one, two or three pairs of planes. It is impossible to have a (3,9) congruence with the basis points by fours in more than three pairs of planes.

44. If a (3,8) congruence lies on a net of quadrics, it follows from the previous discussion that the net will include a pair of planes each of which contains a pencil of σ lines and a pencil of τ lines. Hence the basis points will be arranged in two planes. Moreover, they lie four in each plane, since if five or more should lie in one plane, they must all lie on a conic which, being uniquely fixed, would be common to every quadric of the family, which is contrary to our hypothesis. In the case of I(3,8), the basis point satisfies one of the following conditions:

$$\text{(A)}\quad b\lambda+c\lambda^2=a'+b'\lambda+c'\lambda^2=0,$$
$$\text{(B)}\quad a'+b'\lambda+c'\lambda^2=x+y\lambda=0,$$
$$\text{(C)}\quad b\lambda+c\lambda^2=x+y\lambda=0.$$

The only distinction between (A) and (B) is that the σ and τ configurations are interchanged throughout. This gives a configuration K_3, K_5'. For the basis point (C) there exists a K_6, K_2'. In the case of II(3,8), the basis point satisfies

(A) $a'=b'=w=0$, giving the configuration K_7, K_1';
(B) $b\lambda+c\lambda^2+d\lambda^3=x+y\lambda+z\lambda^2=0$, giving K_4, K_4';
(C) $b\lambda+c\lambda^2+d\lambda^3=a'+b'\lambda=0$, giving K_3, K_5'.

The case of III(3,8) does not contribute anything different from the congruences already discussed. The only possible K_2' which gives a (3,8) congruence is $K_2'(1;\,2_1 5_1 8_1)$. The double quadric in this case is of the σ, τ type. The complete configuration is:

$$\begin{cases} K_6(1;\, 2_1 3_2 4_2 5_2 6_3 7_3 8_2) \\ K_2'(1;\, 2_1 5_1 8_1) \end{cases} \qquad \begin{cases} K_2(2;\, 1_1 6_1 7_1) \\ K_6'(2;\, 1_1 3_2 4_2 5_3 6_2 7_2 8_3) \end{cases}$$

$$\begin{cases} K_4(3;\, 1_2 4_1 5_1 6_2 7_2 8_1) \\ K_4'(3;\, 2_2 4_1 5_2 6_1 7_1 8_2) \end{cases} \qquad \begin{cases} K_3(5;\, 1_2 3_1 4_1 6_1 7_1) \\ K_5'(5;\, 1_1 2_3 3_2 4_2 6_1 7_1 8_2) \end{cases}$$

$$\begin{cases} K_5(6;\, 1_3 2_1 3_2 4_2 5_1 7_2 8_1) \\ K_3'(6;\, 2_2 3_1 4_1 5_1 8_1) \end{cases}$$

The cones at (4), (8) and (7) are similar respectively to the ones at (3), (5) and (6).

There are two K_3 which give (3,8) congruences different from the above: $K_3(1;\, 2_1 3_1 5_1 6_1 7_1 8_1)$ and $K_3(1;\, 2_2 5_1 6_1 7_1 8_1)$. For the first of these the double quadric is of the 2σ, 2τ type. The complete configuration is:

$$\begin{cases} K_3(1;\, 2_1 3_1 5_1 6_1 7_1 8_1) \\ K_5'(1;\, 2_1 3_1 4_2 5_2 6_2 7_2 8_2) \end{cases} \qquad \begin{cases} K_5(2;\, 1_1 3_2 4_1 5_2 6_2 7_2 8_2) \\ K_3'(2;\, 1_1 4_1 5_1 6_1 7_1 8_1) \end{cases}$$

$$\begin{cases} K_4(5;\, 1_1 2_2 3_2 4_1 6_1 7_1 8_1) \\ K_4'(5;\, 1_2 2_1 3_1 4_2 6_1 7_1 8_1) \end{cases}$$

The cones for (3) and (4) are similar to those for (2) and (1), respectively, while those for (6), (7) and (8) are similar to (5). The complete configuration for the third case, in which the double quadric is of the σ, τ type, is the same as that determined by the K_1. It is as follows:

$$\begin{cases} K_1'(1) \\ K_7(1;\, 2_2 3_2 4_2 5_3 6_3 7_3 8_3) \end{cases} \qquad \begin{cases} K_4'(5;\, 2_2 3_2 4_2 6_1 7_1 8_1) \\ K_4(5;\, 1_3 2_2 3_2 4_2 6_1 7_1 8_1) \end{cases}$$

$$\begin{cases} K_5'(3;\, 2_2 4_2 5_2 6_2 7_2 8_2) \\ K_3(3;\, 1_2 5_1 6_1 7_1 8_1) \end{cases}$$

The cones for (2) and (4) are similar to that at (3); those for (6), (7), (8) are similar to that at (5).

By imposing conditions on the basis points it is possible to obtain a (3,8) congruence with the basis points arranged by fours on two, three, four or five pairs of planes. It is impossible to have a (3,8) congruence with the basis points arranged on six pairs of planes.

45. In a net of quadrics containing six cones and one double quadric, it is possible that one of the cones may be the double quadric, in case the cone is composed of planes. In this case, there are included in the net only four cones, and one pair of planes each of which is counted twice. In general the σ and τ vertices do not coincide.

If this occurs in the case just considered, it is possible to have a (3,7) congruence in which the basis points lie by fours in only one pair of planes; *i. e.*, in the planes of pencils of lines belonging to the congruence. The complete configuration is as follows:

$$\begin{cases} K_2'(1;\,5_1 6_1 7_1) \\ K_5(1;\,2_1 3_1 4_1 5_2 6_2 7_2 8_3) \end{cases} \qquad \begin{cases} K_4'(2;\,3_1 4_1 5_2 6_2 7_2 8_1) \\ K_3(2;\,1_1 5_1 6_1 7_1 8_2) \end{cases}$$

$$\begin{cases} K_4'(5;\,1_1 2_2 3_2 4_2 6_1 7_1) \\ K_3(5;\,1_2 2_1 3_1 4_1 8_1) \end{cases}$$

The cones for (1) and (8) are similar to (1); those for (3) and (4) are similar to (2); and those for (6) and (7) are similar to (5).

46. If a (3,7) congruence lies on a net of quadrics, it has been shown that the net must in general contain two pairs of planes. Hence the basis points lie by fours in these four planes. As in the preceding cases, the basis points in I(3,7) may be shown to give a K_2, K_5'; those in II(3,7) will give K_6, K_1' and K_4, K_3'. The III(3,7) will add nothing to these. K_1' determines a single pair of congruences with a double quadric of the σ, τ type. The configuration follows:

$$\begin{cases} K_6(1;\,2_1 3_2 4_2 5_3 6_3 7_2 8_2) \\ K_1'(1) \end{cases} \qquad \begin{cases} K_2(2;\,1_1 5_1 6_1) \\ K_5'(2;\,3_2 4_2 5_2 6_2 7_2 8_2) \end{cases}$$

$$\begin{cases} K_3(3;\,1_2 5_1 6_1 7_1 8_1) \\ K_4'(3;\,2_2 4_1 5_1 6_1 7_2 8_2) \end{cases} \qquad \begin{cases} K_4(5;\,1_3 2_1 3_1 4_1 6_1 7_1 8_1) \\ K_3'(5;\,2_2 3_1 4_1 7_1 8_1) \end{cases}$$

The cones for (4), (7) and (8) are similar to (3), and that for (6) is similar to (5).

There are two distinct K_2 which give (3,7) congruences. The first of these, $K_2(1;\,2_1 5_1 6_1)$, is identical with the congruence determined by K_1'. For the second, $K_2(1;\,3_1 5_1 7_1)$, the congruence has the double quadric of the σ, τ type, and the configuration is:

$$\begin{cases} K_2(1;\,3_1 5_1 7_1) \\ K_5'(1;\,2_1 3_1 4_2 5_2 6_3 7_1 8_2) \end{cases} \qquad \begin{cases} K_4(2;\,3_2 4_1 5_2 6_1 7_2 8_1) \\ K_3'(2;\,1_1 4_1 5_1 6_2 8_1) \end{cases}$$

$$\begin{cases} K_5(3;\,1_1 2_2 4_1 5_2 6_1 7_3 8_2) \\ K_2'(3;\,1_1 6_1 8_1) \end{cases} \qquad \begin{cases} K_3(4;\,2_1 3_1 5_1 7_2 8_1) \\ K_4'(4;\,1_2 2_1 5_1 6_2 7_1 8_2) \end{cases}$$

$$\begin{cases} K_2(6;\,2_1 3_1 7_1) \\ K_5'(6;\,1_3 2_2 3_1 4_2 5_1 7_1 8_2) \end{cases}$$

The cones at (5) and (7) are similar to (3), and (8) is similar to (6).

Of the K_3, the only one which gives a new congruence is

$$K_3(1; 3_1 4_1 5_1 6_1 7_1 8_1).$$

The double quadric for this is of the $2\sigma, 2\tau$ type. The configuration is:

$$\begin{cases} K_3(1; 3_1 4_1 5_1 6_1 7_1 8_1) \\ K_4'(1; 2_1 3_1 4_1 5_2 6_2 7_1 8_1) \end{cases} \qquad \begin{cases} K_3(2; 3_1 4_1 5_1 6_1 7_1 8_1) \\ K_4'(2; 1_1 3_1 4_1 5_2 6_2 7_1 8_1) \end{cases}$$

$$\begin{cases} K_3(3; 1_1 2_1 5_1 6_1 7_1 8_1) \\ K_4'(3; 1_1 2_1 4_1 5_1 6_1 7_2 8_2) \end{cases} \qquad \begin{cases} K_3(5; 1_1 2_1 3_1 4_1 7_1 8_1) \\ K_4'(5; 1_2 2_2 3_1 4_1 6_1 7_1 8_1) \end{cases}$$

The cones at (4), (7), (8) are similar to that at (3), and that at (6) is similar to that at (5).

By imposing conditions on the basis points, (3,7) congruences may be built having their basis points by fours in six, eight, ten or twelve planes.

47. Just as in Art. 45 it was possible to obtain a (3,7) with the basis points in only one pair of planes, so, if one of the pairs of singular planes (1278), (3456) of Art. 46 is the double quadric of the net, the configuration $K_2(1; 3_1 5_1 6_1)$ determines a (3,6) congruence, with the pair of planes as the double quadric and the basis points arranged by fours only in two pairs of planes. The configuration follows:

$$\begin{cases} K_2(1; 3_1 5_1 6_1) \\ K_4'(1; 3_1 4_2 5_2 6_2 7_1 8_1) \end{cases} \qquad \begin{cases} K_4(2; 3_2 4_1 5_2 6_2 7_1 8_1) \\ K_2'(2; 4_1 5_1 6_1) \end{cases}$$

$$\begin{cases} K_4(3; 1_1 2_2 5_1 6_1 7_2 8_2) \\ K_2'(3; 1_1 7_1 8_1) \end{cases} \qquad \begin{cases} K_2(4; 2_1 7_1 8_1) \\ K_4'(4; 1_2 2_1 5_1 6_1 7_2 8_2) \end{cases}$$

$$\begin{cases} K_3(5; 1_1 2_2 3_1 7_1 8_1) \\ K_3'(5; 1_2 2_1 4_1 7_1 8_1) \end{cases} \qquad \begin{cases} K_3(7; 2_1 3_2 4_1 5_1 6_1) \\ K_3'(7; 3_1 4_2 5_1 6_1 8_1) \end{cases}$$

The cone at (6) is similar to that at (5), and that at (8) is similar to that at (7).

48. In the case of three pairs of singular planes, *i. e.*, the (3,6) congruences, there arise two distinct classes according as the three pairs of planes have general position or the planes pass by threes through a pair of lines. In either case, from II(3,6), there arise K_5, K_1' and K_2, K_4'. I(3,6) adds nothing new to this, but III(3,6) gives a K_3, K_3'.

For cases of the first class, the double quadrics are all of the σ, τ type. The complete configurations follow:

$$\begin{cases} K_1'(1) \\ K_5(1; 2_1 3_2 4_1 5_3 6_2 7_1 8_2) \end{cases} \qquad \begin{cases} K_4'(2; 3_1 4_2 5_1 6_2 7_2) \\ K_2(2; 1_1 5_1 8_1) \end{cases}$$

$$\begin{cases} K_3'(3; 2_1 4_1 6_1 8_1 7_2) \\ K_3(3; 1_2 5_1 6_1 7_1 8_1) \end{cases} \qquad \begin{cases} K_2'(5; 2_1 4_1 7_1) \\ K_4(5; 1_3 2_1 3_1 4_1 6_1 7_1 8_1) \end{cases}$$

The cone for (4) is similar to that at (2); those for (6), (7) and (8) are similar to that at (3).

$$\begin{cases} K_2'(1;\ 2_1 3_1 5_1) \\ K_4(1;\ 3_1 4_1 5_2 6_2 7_1 8_2) \end{cases} \qquad \begin{cases} K_4'(2;\ 1_1 3_1 4_2 5_1 6_2 7_2) \\ K_2(2;\ 5_1 6_1 8_1) \end{cases}$$

$$\begin{cases} K_3'(3;\ 1_1 2_1 4_1 6_1 7_2) \\ K_3(3;\ 1_1 5_1 6_1 7_1 8_2) \end{cases} \qquad \begin{cases} K_2'(8;\ 4_1 6_1 7_1) \\ K_4(8;\ 1_2 2_1 3_2 4_2 5_1 6_1) \end{cases}$$

The cones for (4), (5) and (6) are similar to that at (3); the cone for (7) is similar to that at (2).

In the second class, there are three distinct congruences. For each of the first two, the double quadric is of the σ,τ type. The configurations are:

$$\begin{cases} K_1'(1) \\ K_5(1;\ 3_2 4_2 5_2 6_2 7_2 8_2) \end{cases} \qquad \begin{cases} K_5'(2;\ 3_2 4_2 5_2 6_2 7_2 8_2) \\ K_1(2) \end{cases}$$

$$\begin{cases} K_3'(3;\ 2_2 5_1 6_1 7_1 8_1) \\ K_3(3;\ 1_2 5_1 6_1 7_1 8_1) \end{cases}$$

The cones at (4), (5), (6), (7), (8) are similar to those at (3).

$$\begin{cases} K_2(1;\ 3_1 4_1 5_1) \\ K_4'(1;\ 3_1 4_1 5_1 6_2 7_2 8_2) \end{cases} \qquad \begin{cases} K_4(2;\ 3_2 4_2 5_2 6_1 7_1 8_1) \\ K_2'(2;\ 6_1 7_1 8_1) \end{cases}$$

$$\begin{cases} K_3(3;\ 1_1 2_2 5_1 7_1 8_1) \\ K_3'(3;\ 1_1 6_2 5_1 7_1 8_1) \end{cases} \qquad \begin{cases} K_2(6;\ 2_1 7_1 8_1) \\ K_4'(6;\ 1_2 2_1 3_2 4_2 7_1 8_1) \end{cases}$$

$$\begin{cases} K_3(7;\ 2_1 3_1 4_1 5_2 6_1) \\ K_3'(7;\ 1_2 2_1 3_1 4_1 6_1) \end{cases}$$

The cones at (4) and (5) are similar to (3), and (8) is similar to (7).

Probably the most interesting of all is the (3, 6) defined by K_3, K_3' with a $2\sigma, 2\tau$ double quadric. In this case the σ lines and τ lines enter in perfect symmetry, and this is one of the two congruences of the list in which all the cones are identical in structure. The complete configuration is:

$$\begin{cases} K_3(1;\ 3_1 4_1 5_1 6_1 7_1 8_1) \\ K_3'(1;\ 3_1 4_1 5_1 6_1 7_1 8_1) \end{cases}$$

and similar configurations for all the other vertices. It is to be noted that the basis point omitted in each cone is the one such that the line joining it to the vertex of the cone is a neutral line, *i. e.*, lies in three singular planes. By imposing conditions, it is possible to put the basis points on eight, ten or twelve planes.

49. In the second class of congruences discussed in Art. 48, it is impossible that any pair of singular planes should be a double quadric, as that would make one line lie in four singular planes, which can not be true with congruences of order 3. In the first class, however, one of the pairs of planes (1467), (2358) may be the double quadric, in which case there arises a (3,5) congruence with the basis points by fours in each of six planes. These are the configurations:

$$\begin{cases} K_2(1;\,3_1 5_1 8_1) \\ K_3'(1;\,2_1 3_1 5_2 6_1) \end{cases} \qquad \begin{cases} K_3(3;\,1_1 4_1 6_1 7_2 8_1) \\ K_2'(3;\,1_1 6_1 7_1) \end{cases}$$

The cones at (2), (5) and (6) are similar to (1), and (4), (7) and (8) are similar to (3).

50. Proceeding as before, there arise from I(3,5) a K_2, K_3' and from II(3,5) an additional K_4, K_1' which gives the same (3,5) congruence, with the basis points lying by fours in eight singular planes. It is impossible to have a $2\sigma, 2\tau$ double quadric for a congruence of class lower than 6. The configurations for this case are:

$$\begin{cases} K_4(1;\,3_2 4_1 5_2 6_1 7_1 8_2) \\ K_1'(1) \end{cases} \qquad \begin{cases} K_1(2) \\ K_4'(2;\,3_1 4_2 5_1 6_2 7_2 8_1) \end{cases}$$

$$\begin{cases} K_3(3;\,1_2 5_1 6_1 7_1 8_1) \\ K_2'(3;\,2_1 6_1 7_1) \end{cases} \qquad \begin{cases} K_2(4;\,1_1 5_1 8_1) \\ K_3'(4;\,2_2 5_1 6_1 7_1 8_1) \end{cases}$$

The cones at (5) and (8) are similar to (3), and (6) and (7) are similar to (4). The double quadric is of the σ, τ type. By imposing conditions on the basis points, (3,5) congruences may be determined which have their basis points arranged on ten or twelve planes.

51. In case the planes (1467), (2358) form the double quadric of the net, $K_2(1;\,3_1 5_1 8_1)$ determines a (3,4) congruence with basis points arranged by fours on eight planes. The complete configuration for this congruence is:

$$\begin{cases} K_2(1;\,3_1 5_1 8_1) \\ K_2'(1;\,3_1 5_1 8_1) \end{cases} \qquad \begin{cases} K_2(2;\,4_1 6_1 7_1) \\ K_2'(2;\,4_1 6_1 7_1) \end{cases}$$

$$\begin{cases} K_2(3;\,1_1 6_1 7_1) \\ K_2'(3;\,1_1 6_1 7_1) \end{cases} \qquad \begin{cases} K_2(4;\,2_1 5_1 8_1) \\ K_2'(4;\,2_1 5_1 8_1) \end{cases}$$

The cones at (5), (6), (7) and (8) are similar to that at (4). This is the second case in which the σ and τ lines enter in perfect symmetry and in which all the cones are identical in structure.

52. There is but one type of (3,4) congruence, and this may be derived from either I(3,5) or II(3,5). These congruences have each a σ,τ double quadric, and the configurations are:

$$\begin{cases} K_3(1;3_1 4_1 5_1 6_1 8_2) \\ K_1'(1) \end{cases} \qquad \begin{cases} K_1(2) \\ K_3'(2;3_1 4_1 5_1 6_1 7_2) \end{cases}$$

$$\begin{cases} K_2(3;1_1 6_1 8_1) \\ K_2'(3;2_1 6_1 7_1) \end{cases}$$

The cone at (7) is similar to (2); (8) is similar to (1); and (4), (5) and (6) are similar to (3).

By imposing the proper condition, a (3,4) congruence can be found which has its basis points arranged on twelve planes.

53. By imposing the conditions for eight basis points, in the case of a (3,3) congruence, there appears one and only one congruence. This may be obtained from I(3,3), II(3,3) or III(3,3). It has eight basis planes and eight basis points in addition to the twelve singular points and the twelve singular planes. All the configurations of τ lines are plane pencils, and all the configurations of σ lines are quadric cones. The typical form of singular cone is:

$$\begin{cases} K_2(1;3_1 5_1 7_4) \\ K_1'(1) \end{cases}$$

It will be noticed that there is no possibility of obtaining a (3,3) with eight basis points on five pairs of planes. This is because the doubling of any one of the five pairs of planes would require at least one of the lines of the congruence to be a fourfold line, which is impossible without a basis line.

54. As a summary of the preceding work, it is seen that a (3,9) congruence may have a double quadric of the $2\sigma, 2\tau$ type or of the σ,τ type, and further may have its points arranged by fours on two, four or six planes.

A (3,8) congruence may have a double quadric of the $2\sigma, 2\tau$ type or the σ,τ type, and have its basis points on i pairs of planes, $i=1,\ldots,4$.

A (3,7) congruence may have a double quadric of the $2\sigma, 2\tau$ type or the σ,τ type, or may have a pair of planes for a double quadric. It may have its basis points on i pairs of planes, $i=1,\ldots,5$.

A (3,6) congruence may have a double quadric of the $2\sigma, 2\tau$ type or the σ,τ type, or may have a pair of planes for a double quadric. It may have its basis points arranged on i pairs of planes, $i=2,\ldots,6$.

A (3,5) congruence may have a σ,τ type of double quadric, or may have a pair of planes for double quadric. It may have its basis points arranged on i pairs of planes, $i=3,\ldots,6$.

A (3,4) congruence may have a σ,τ type of double quadric, or may have a pair of planes for double quadric. It may have its basis points arranged on i pairs of planes, $i=4,\ldots,6$.

A (3,3) congruence has a σ,τ type of double quadric and its basis points by fours in six pairs of planes. It contains eight basis planes also.

ITHACA, N. Y., *June*, 1910.

On the Four-dimensional Angles of the Semiregular Polytopes of S_4.

By P. H. Schoute.

§ 1. *Introduction.*

This paper may be regarded as a continuation of a previous one published in this Journal (Vol. XXXI, p. 303); it is concerned with polytopes of S_4 characterized by the property of admitting one kind of vertex and one length of edge, which polytopes will be called "semiregular." These polytopes, corresponding to the Archimedian semiregular polyhedra of ordinary space, have been deduced from the regular ones by very simple geometrical operations called "expansions" and "contractions" in a masterly memoir of Mr. A. Boole Stott;* they will be indicated here by the symbols introduced in that memoir.

It follows immediately from the property of admitting one kind of vertex and one length of edge that the portion of four-dimensional space occupied by a semiregular polytope at any vertex is the same for all the vertices of that polytope, We wish to determine this angle in parts of 16 right angles or 1440°; to that end we put on duty the space fillings or nets of semiregular polytopes. The nets derived from the regular cells C_8, C_{16}, C_{24} have been tabulated in the memoir quoted, those derived from the simplex C_5 have been added by ourselves.† As there are no nets consisting of polytopes, derived from the cells C_{120}, C_{600}, we discard the semiregular polytopes deduced from them.

There is a remarkable difference between the nets derived from C_8, C_{16}, C_{24}, on one hand, and those derived from C_5 on the other. With exception of the regular nets built up exclusively either of C_8 or of C_{16} or of C_{24}, any net of the first group admits, besides *principal* constituents derived from C_8, C_{16}, C_{24},

* "Geometrical Deduction of Semiregular from Regular Polytopes and Space Fillings," *Verhandelingen* of Amsterdam, Vol. XI (1910), No. 1.

† "Analytical Treatment of the Polytopes regularly Derived from the Regular Polytopes," Section I: The Simplex, *Verhandelingen* of Amsterdam, Vol. XI (1911), No. 3.

Sections II, III, IV (Measure Polytope, Cross Polytope, Half-measure Polytope) will be published this year. The last section (on Extra Polytopes) will appear later on.

either *prisms* or *prismotopes* * or both, whilst there is no simplex net admitting either a prism or a prismotope. Now the four-dimensional angles of prism and prismotope are known; these angles will be tabulated — with indication of the source from which they are taken — at the place where we want them, *i. e.*, in § 4.

Before entering on our task, we have still to remark that for our aim it is not sufficient to know the different kinds of constituents of any net, but that we must know also how many constituents of each kind concur in any vertex of the net. This *multiplicity* of each constituent, which is independent of the choice of the vertex, has been determined in another paper. †

§ 2. *The Simplex Group.*

The simplex group contains 15 polytopes, the simplex itself included; they are represented in the following table by their expansion symbol, the coordinates of their vertices and their characteristic numbers of vertices, edges, faces, limiting polyhedra. Instead of C_5 = five-cell, we use henceforth the more appropriate symbol $S(5)$ = simplex with five vertices.

$S(5)$		(1 0 0 0 0)	(5,	10,	10,	5)	
e_1	"	(2 1 0 0 0)	(20,	40,	30,	10)	
e_2	"	(2 1 1 0 0)	(30,	90,	80,	20)	
e_3	"	*(2 1 1 1 0)	(20,	60,	70,	30)	
$e_1 e_2$	"	(3 2 1 0 0)	(60,	120,	80,	20)	
$e_1 e_3$	"	(3 2 1 1 0)	(60,	150,	120,	30)	
$e_2 e_3$	"	(3 2 2 1 0)	(60,	150,	120,	30)	$= -e_1 e_3 S(5)$
$e_1 e_2 e_3$	"	*(4 3 2 1 0)	(120,	240,	150,	30)	
$c e_1$	"	(1 1 0 0 0)	(10,	30,	30,	10)	
$c e_2$	"	(1 1 1 0 0)	(10,	30,	30,	10)	$= -c e_1 S(5)$
$c e_3$	"	(1 1 1 1 0)	(5,	10,	10,	5)	$= -S(5)$
$c e_1 e_2$	"	*(2 2 1 0 0)	(30,	60,	40,	10)	
$c e_1 e_3$	"	(2 2 1 1 0)	(30,	90,	80,	20)	$= -e_2 S(5)$
$c e_2 e_3$	"	(2 2 2 1 0)	(20,	40,	30,	10)	$= -e_1 S(5)$
$c e_1 e_2 e_3$	"	(3 3 2 1 0)	(60,	120,	80,	20)	$= -e_1 e_2 S(5)$

Here (3 2 1 0 0) means that we get the coordinates of the 60 vertices of the polytope $e_1 e_2 S(5)$ with respect to a regular five-cell by assigning to x_1, x_2, x_3, x_4, x_5 the numerical values 3, 2, 1, 0, 0 taken successively in all the possible

* See the definition of the prismotope $(m; n)$, § 4, and compare, for prismotopes in general: "The Characteristic Numbers of the Prismotope," *Proceedings* of Amsterdam, Vol. XIV (1911), p. 424.

† "On Reciprocal Nets," *Nieuw Archief voor Wiskunde*, Vol. X (1913), p. 273.

orders of succession, whilst the remark $e_2 e_3 S(5) = -e_1 e_3 S(5)$ indicates that $e_1 e_3 S(5)$ and $e_2 e_3 S(5)$ are congruent but of opposite orientation. Finally, the asterisk before the coordinate symbol of a polytope expresses the fact that the latter is central symmetric.

It follows from this table that we have to determine nine four-dimensional angles. But in the first paper on four-dimensional angles* we found for the four-dimensional angle A of $S(5)$ itself the value $-\frac{16}{5} + 4\varepsilon$, where $\varepsilon = 75^\circ\ 31'\ 21''$, *i. e.*, $\cos\varepsilon = \frac{1}{4}$, is the angle between the spaces of any two limiting tetrahedra of the simplex. So the number of unknown angles is reduced to *eight*. On the other hand, there are only *seven* simplex nets; so, even if the relations deduced from these nets are mutually independent — which supposition will prove to be too optimistic — we want one relation more. We start by indicating a way leading to such a relation.

By means of a regular truncation of $S(5)$ at the vertices so as to take away a third part of each edge on either side, we get the polytope $e_1 S(5)$. So we get the solid angle round an edge of the original $S(5)$ by addition of the four-dimensional angles of the $e_1 S(5)$ and that of the smaller $S(5)$ taken away by truncation, *i. e.*, by the generally known rule of the spherical excess† we find $A_1 + A = 2\,(3\varepsilon - 2)$, if A_1 stands for the four-dimensional angle of $e_1 S(5)$. So we find $A_1 = -\frac{4}{5} + 2\varepsilon$.

The seven simplex nets are characterized in the following table by the constituents concurring in any vertex:

I	$10\,S(5)$,	$20\,c\,e_1 S(5)$	
II	$2\,S(5)$,	$8\,e_1 S(5)$,	$6\,c\,e_1 e_2 S(5)$
III	$2\,c\,e_1 S(5)$,	$6\,e_2 S(5)$,	$2\,e_3 S(5)$
IV	$2\,e_1 S(5)$,	$6\,e_1 e_2 S(5)$,	$e_3 S(5)$
V	$2\,e_2 S(5)$,	$3\,e_1 e_3 S(5)$,	$2\,c\,e_1 e_2 S(5)$
VI	$2\,e_1 e_2 S(5)$,	$2\,e_1 e_2 e_3 S(5)$,	$2\,e_1 e_3 S(5)$
VII	$5\,e_1 e_2 e_3 S(5)$		

So, if the four-dimensional angles of $e_i e_k \ldots S(5)$ and $c\,e_i e_k \ldots S(5)$ are indicated by $A_{ik\ldots}$ and $A'_{ik\ldots}$ respectively, we find the equations

$$10\,A + 20\,A'_1 = 2\,A + 8\,A_1 + 6\,A'_{12} = 2\,A'_1 + 6\,A_2 + 2\,A_3 = 2\,A_1 + 6\,A_{12} + A_3$$
$$= 2\,A_2 + 3\,A_{13} + 2\,A'_{12} = 2\,A_{12} + 2\,A_{123} + 2\,A_{13} = 5\,A_{123} = 16.$$

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* Henceforth this study will be quoted as "F. P."

† This angle $3\varepsilon - 2$ occurs already in the small table of "F. P." in the row of C_5; it is the angle $\beta = 46^\circ\ 34'\ 3''$.

These relations allow us to complete the two results already obtained by

$$A_{123}=\frac{16}{5},\quad A_1'=A_2'=\frac{12}{5}-2\varepsilon,\quad A_{12}'=\frac{24}{5}-4\varepsilon,$$

but the four equations

$$\left.\begin{aligned}\frac{24}{5}-4\varepsilon+6A_2+2A_3&=16,\\ -\frac{8}{5}+4\varepsilon+6A_{12}+A_3&=16,\\ \frac{24}{5}-4\varepsilon+2A_2+4A_{13}&=16,\\ \frac{32}{5}+2A_{12}+2A_{13}&=16,\end{aligned}\right\}\qquad(1)$$

with the remaining four unknown angles A_2, A_3, A_{12}, A_{13}, prove to admit one degree of dependence. So we still want another relation from a new source.

§ 3. *The Simplex Group Continued. A New Equation.*

We try to determine one of the four unknown angles figuring in the equations (1) by itself, and fix our choice on A_{12}, as this angle is formed by four edges, whilst five edges concur in the vertex of A_{13} and even six in the vertices of A_2 and A_3.

If the point $3,2,1,0,0$ is chosen as vertex O of angle A_{12}, the four adjacent points are

$$P=2,3,1,0,0,\quad Q=3,1,2,0,0,\quad R_1=3,2,0,1,0,\quad R_2=3,2,0,0,1.$$

So we have to determine the six plane angles POQ, POR_i, QOR_i, R_1OR_2 by means of the mutual distances of the points O, P, Q, R_i. We find

$$OP=OQ=OR_i=R_1R_2=\sqrt{2},\quad PQ=QR_i=\sqrt{6},\quad PR_i=2;$$

and therefore

$$POQ=QOR_i=120^\circ,\quad POR_i=90^\circ,\quad R_1OR_2=60^\circ.$$

We have represented the angle $O(PQR_1R_2)=x$ on the spherical space (Fig. 1) with O as center, and introduce in this diagram three more points:

In the first place, the point Q' diametrically opposite to Q, giving rise to the formation of the new angle $O(PQ'R_1R_2)=y$, forming with $O(PQR_1R_2)$ the solid angle on the edge QQ' of the tangents in Q' to the arcs $Q'P$, $Q'R_1$, $Q'R_2$.

In the second place, the point P' diametrically opposite to P, and in the third place the point Q'' bisecting the arc $Q'P'$, giving rise to the formation of two new angles, *i. e.*, $O(Q'Q''R_1R_2)=A$, as all the six plane angles are 60°, and

$O(Q''P'R_1R_2)=y$. So $2y+A$ represents* the solid angle on the edge PP' of the tangents in P to the arcs PQ', PR_1, PR_2.

So we find

$$x+y=2 \text{ sph. exc. } Q'(PR_1R_2), \quad 2y+A=2 \text{ sph. exc. } P(Q'R_1R_2),$$

and therefore

$$x=\frac{1}{2}A+2 \text{ sph. exc. } Q'(PR_1R_2)-\text{sph. exc. } P(Q'R_1R_2). \tag{2}$$

Calculation of sph. exc. $Q'(PR_1R_2)$. The corresponding spherical triangle admits as *sides* one angle of an equilateral triangle with sides of 60°, and two angles equal to the vertex angle of an isosceles triangle with sides 60°, 60°, 90°; so we get for these sides once $\cos^{-1}\frac{1}{3}$ and twice $\cos^{-1}-\frac{1}{3}$, *i.e.*, δ, $2-\delta$, $2-\delta$, where $\delta=70°\,31'\,44''$ is the angle between any two faces of a tetrahedron.

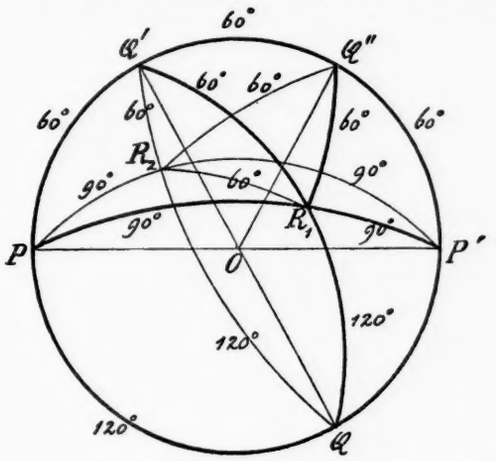


FIG. 1.

From the sides δ, $2-\delta$, $2-\delta$ we have to deduce the *angles*. Here we find once $\cos^{-1}\frac{1}{4}$ and twice $\cos^{-1}-\frac{1}{4}$, *i. e.*, ε, $2-\varepsilon$, $2-\varepsilon$, where $\varepsilon=75°\,31'\,21''$ is the angle introduced in § 2. So the spherical excess of $Q'(PR_1R_2)$ is $2-\varepsilon$.

Calculation of sph. exc. $P(Q'R_1R_2)$. For the *sides* we find one angle of 60° (from triangle PR_1R_2, where PR_1 and PR_2 are right) and two angles equal to the base angle of the isosceles triangle with sides 60°, 60°, 90°; so the sides are once 60° and twice $\cos^{-1}\frac{1}{3}\sqrt{3}$.

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* We owe the ingenious determination of the angle y to Dr. W. A. WYTHOFF, of Amsterdam, whose dissertation "De Biquaternion als Bewerking in de Ruimte van vier Afmetingen" (The Biquaternion as Operation in Four-dimensional Space), Amsterdam, 1898, forms an excellent introduction to the geometry of S_4.

Passing from the sides to the *angles*, we get once the angle ε and twice $\cos^{-1}\frac{1}{6}\sqrt{6} = \frac{1}{2}\cos^{-1}-\frac{2}{3}$. So, if we represent $\cos^{-1}-\frac{2}{3}$ by ϕ, the spherical excess of $P(Q'R_1R_2)$ is $\varepsilon+\phi-2$. As $\cos(\varepsilon+\phi-180°) = \frac{1}{12}(2+5\sqrt{3})$, we use the result $\cos^{-1}\frac{1}{12}(2+5\sqrt{3})$.

Calculation of x. By introducing the value $-\frac{8}{5}+2\varepsilon$ of $\frac{1}{2}A$ and those of the spherical excesses, (2) passes into

$$x = \frac{12}{5} - \cos^{-1}\frac{1}{12}(2+5\sqrt{3}),$$

which leads to the following evaluation:

$$2+5\sqrt{3} = 10{,}6602540, \qquad \log\frac{1}{12}(2+5\sqrt{3}) = 9{,}9485863,$$

$$\frac{1}{12}(2+5\sqrt{3}) = 0{,}8883545, \qquad \cos^{-1}\frac{1}{12}(2+5\sqrt{3}) = 27°\ 19'\ 58''.$$

We denote the latter angle by ε'. Then $x = \frac{12}{5} - \varepsilon'$.

So we find by means of the equations (1)

$$A_2 = \frac{52}{15} + 2\varepsilon - 2\varepsilon', \quad A_3 = \frac{16}{5} - 4\varepsilon + 6\varepsilon', \quad A_{13} = \frac{12}{5} + \varepsilon'$$

by introducing $A_2 = \frac{12}{5} - \varepsilon'$.

These results are tabulated at the end of this paper.

§ 4. *Angles of Four-dimensional Prisms and Prismotopes.*

We find in the measure-polytope nets, prisms on C, O, CO, RCO, tC, tO, tCO and prismotopes (4; 4), (4; 8), (8; 8); in the nets derived from the cell C_{24}, prisms on T, C, O, CO, RCO, tT, tC, tO, tCO and prismotopes (3; 3), (3; 6), (6; 6). As the four-dimensional angle of any of these prisms expressed in right angles is equal to the solid angle of the base of the prism expressed in right angles, these angles can be taken from A. ANDREINI's tables.* As the prismotope $(m; n)$ is generated by placing two regular polygons p_m and p_n in such a way in two planes perfectly normal to each other that the point common to these planes is a common vertex of the polygons, and moving

* "Sulle reti di poliedri regolari e semiregolari e sulle correspondenti reti correlative" (On the Nets of Regular and Semiregular Polyhedra and on the Corresponding Reciprocal Nets), *Memorie* della Società italiana delle scienze (detta dei XL), 3d series, Vol. XIV (1905), p. 75.

one of them so as to bring that vertex of it successively into coincidence with any point in the interior of the other, the four-dimensional angle of that prismotope expressed in right angles is the product of the plane angles of p_m and p_n expressed in right angles. So we find, if P_T stands for "prism on T" and AP_T and $A\,(m;\,n)$ for the four-dimensional angles of P_T and $(m;\,n)$, whilst $\delta = 70^\circ\,31'\,44''$ is the angle introduced in § 3:

$$\begin{array}{l|l|l|l}
AP_T = -2+3\delta & AP_{CO} = 2\delta & AP_{tC} = 1+\delta & A\,(4;4) = 1 \quad A\,(3;3) = \frac{4}{9} \\
AP_C = 1 & AP_{RCO} = 3-\delta & AP_{tO} = 2 & A\,(4;8) = \frac{3}{2} \quad A\,(3;6) = \frac{8}{9} \\
AP_O = 4-4\delta & AP_{tT} = 2-\delta & AP_{tCO} = \frac{5}{2} & A\,(8;8) = \frac{9}{4} \quad A\,(6;6) = \frac{16}{9}
\end{array}$$

§ 5. *The Measure-Polytope Group.*

The 15 polytopes of this group, the measure polytope C_8 itself included, are represented in the following table by their expansion symbol, the coordinates of their vertices and their characteristic numbers;* moreover, a fourth column gives the interpretation of these polytopes as derived from the cross polytope C_{16}.

C_8	[1 1 1 1]	(16, 32, 24, 8)	$c\,e_3\,C_{16}$
e_1 "	[1′ 1′ 1′ 1]	(64, 128, 88, 24)	$c\,e_2\,e_3$ "
e_2 "	[1′ 1′ 1 1]	(96, 288, 248, 56)	$c\,e_1\,e_3$ "
e_3 "	[1′ 1 1 1]	(64, 192, 208, 80)	e_3 "
$e_1\,e_2$ "	[2′ 2′ 1′ 1]	(192, 384, 248, 56)	$c\,e_1\,e_2\,e_3$ "
$e_1\,e_3$ "	[2′ 1′ 1′ 1]	(192, 480, 368, 80)	$e_2\,e_3$ "
$e_2\,e_3$ "	[2′ 1′ 1 1]	(192, 480, 368, 80)	$e_1\,e_3$ "
$e_1\,e_2\,e_3$ "	[3′ 2′ 1′ 1]	(384, 768, 464, 80)	$e_1\,e_2\,e_3$ "
$c\,e_1$ "	[1 1 1 0] $\sqrt{2}$	(32, 96, 88, 24)	$c\,e_2$ "
$c\,e_2$ "	[1 1 0 0] $\sqrt{2}$	(24, 96, 96, 24)	$c\,e_1$ " $= C_{24}$
$c\,e_3$ "	[1 0 0 0] $\sqrt{2}$	(8, 24, 32, 16)	C_{16}
$c\,e_1\,e_2$ "	[2 2 1 0] $\sqrt{2}$	(96, 192, 120, 24)	$c\,e_1\,e_2$ "
$c\,e_1\,e_3$ "	[2 1 1 0] $\sqrt{2}$	(96, 288, 240, 48)	e_2 "
$c\,e_2\,e_3$ "	[2 1 0 0] $\sqrt{2}$	(48, 120, 96, 24)	e_1 "
$c\,e_1\,e_2\,e_3$ "	[3 2 1 0] $\sqrt{2}$	(192, 384, 240, 48)	$e_1\,e_2$ "

Here [2 2 1 0] $\sqrt{2}$ means that we obtain the coordinates of the 96 vertices of $c\,e_1\,e_2\,C_8$ with respect to a system of rectangular axes by assigning to x_1, x_2, x_3, x_4 the numerical values $2\sqrt{2}$, $2\sqrt{2}$, $\sqrt{2}$, 0 taken successively in

* Compare for these data and for the measure-polytope nets: "Analytical Treatment," etc., Section II.

all the possible orders of succession with all the possible combinations of the positive and negative sign; moreover, $1', 2', 3'$ stand for short for $1+\sqrt{2}$, $1+2\sqrt{2}$, $1+3\sqrt{2}$.

As the four-dimensional angles of C_8, C_{16}, C_{24} are known (see the table in "F. P."), we have still to determine 12 angles. To that end we have at our disposal the following sixteen nets, characterized by the constituents concurring in any vertex:

I	$8\,e_1 C_8$	....	....	....	$c\,e_3 C_8 = C_{16}$
II	$e_1 C_8$	$3\,P_{tC}$	$3\,(4;8)$	P_C	$e_3 C_8$
III	$4\,e_2 C_8$	....	....	$2\,P_O$	$c\,e_2 C_8 = C_{24}$
*IV	$e_2 C_8$	$2\,P_{RCO}$	$(4;4)$	$2\,P_{RCO}$	$e_2 C_8$
V	$2\,e_3 C_8$	....	$3\,(4;4)$	$3\,P_{CO}$	$c\,e_1 C_8$
VI	$4\,e_1 e_2 C_8$	....	....	P_O	$c\,e_2 e_3 C_8$
VII	$e_1 e_2 C_8$	$2\,P_{tCO}$	$(4;8)$	P_{RCO}	$e_2 e_3 C_8$
VIII	$2\,e_1 e_3 C_8$	....	$2\,(4;8)$	P_{CO}	$c\,e_1 e_2 C_8$
*IX	$e_1 e_3 C_8$	P_{tC}	$2\,(8;8)$	P_{tC}	$e_1 e_3 C_8$
X	$2\,e_2 e_3 C_8$	....	$(4;4)$	$2\,P_{tO}$	$c\,e_1 e_2 C_8$
XI	$2\,e_1 e_2 e_3 C_8$	....	$(4;8)$	P_{tO}	$c\,e_1 e_2 e_3 C_8$
*XII	$e_1 e_2 e_3 C_8$	P_{tCO}	$(8;8)$	P_{tCO}	$e_1 e_2 e_3 C_8$
XIII	$8\,c\,e_1 C_8$	....	....	....	$2\,c\,e_3 C_8 = 2\,C_{16}$
XIV	$4\,c\,e_1 e_2 C_8$	....	....	....	$2\,c\,e_2 e_3 C_8$
*XV	$2\,c\,e_1 e_3 C_8$	....	$2\,(4;4)$	....	$2\,c\,e_1 e_3 C_8$
*XVI	$2\,c\,e_1 e_2 e_3 C_8$	....	$(4;4)$	....	$2\,c\,e_1 e_2 e_3 C_8$

Here the asterisk before the rank number indicates that the net is symmetric.

Now, if we represent the angles of $e_i e_k \ldots C_8$ and $c\,e_i e_k \ldots C_8$ by $B_{ik\ldots}$ and $B'_{ik\ldots}$ respectively, and indicate the sum of the angles of the constituents concurring in a vertex by an S with the rank of the net as subscript, we get

$$S_1 = 8B_1 + B'_3 \qquad S_9 = 2B_{13} + \frac{13}{2} + 2\delta$$

$$S_2 = B_1 + B_3 + \frac{17}{2} + 3\delta \qquad S_{10} = 2B_{23} + B'_{12} + 5$$

$$S_3 = 4B_2 + B'_2 + 8 - 8\delta \qquad S_{11} = 2B_{123} + B'_{123} + \frac{7}{2}$$

$$S_4 = 2B_2 + 13 - 4\delta \qquad S_{12} = 2B_{123} + \frac{29}{4}$$

$$S_5 = 2B_3 + B'_1 + 3 + 6\delta \qquad S_{13} = 8B'_1 + 2B'_3$$

$$S_6 = 4B_{12} + B'_{23} + 4 - 4\delta \qquad S_{14} = 4B'_{12} + 2B'_{23}$$

$$S_7 = B_{12} + B_{23} + \frac{19}{2} - \delta \qquad S_{15} = 4B_{13} + 2$$

$$S_8 - 2B_{13} + B'_{13} + 3 + 2\delta \qquad S_{16} = 4B'_{123} + 1$$

So we find, as we have already $B'_2 = 2$, $B'_3 = \frac{2}{3}$, from $S_i = 16$:

$$\begin{array}{l|l}
(S_1)\ldots B_1 = \frac{23}{12} & (S_9)\ldots B_{13} = \frac{19}{2} - \delta \\
(S_2)\ldots B_3 = \frac{67}{12} - 3\delta & (\text{either } S_8 \text{ or } S_{15})\ldots B'_{13} = \frac{7}{2} \\
(\text{either } S_3 \text{ or } S_4)\ldots B_2 = \frac{3}{2} + 2\delta & (S_{12})\ldots B_{123} = \frac{35}{6} \\
(\text{“ } S_5 \text{ “ } S_{13})\ldots B'_1 = \frac{11}{6} & (\text{“ } S_{11} \text{ “ } S_{16})\ldots B'_{123} = \frac{15}{4}
\end{array}$$

and the now remaining four equations

$$\left.\begin{array}{rrrr}
4B_{12} + & B'_{23} + & 4 & -4\delta = 16, \\
B_{12} + & B_{23} + & \frac{19}{2} - & \delta = 16, \\
2B_{23} + & B'_{12} + & 5 & = 16, \\
4B'_{12} + & 2B'_{23} & & = 16,
\end{array}\right\} \qquad (3)$$

with the four unknown angles B_{12}, B'_{12}, B_{23}, B'_{23}, prove to admit one degree of dependence. So here also we want another relation from a new source.

By means of a regular truncation of C_{16} at the vertices, so as to take away a third part of each edge on either side, we get the polytope $e_1 C_{16}$. As the four-dimensional angle taken away by truncation at any vertex of this $e_1 C_{16}$ is evidently half that angle of C_{16}, we get the solid angle round an edge of C_{16} by addition of the four-dimensional angle of $e_1 C_{16}$ and half that angle of C_{16}. Now the latter angle (see “F. P.” in the table, the angle β of C_{16}) is $\frac{4}{3}$; as $e_1 C_{16} = c\, e_2 e_3 C_8$, we find therefore $B'_{23} + \frac{1}{2} B'_3 = \frac{8}{3}$, *i. e.*, $B'_{23} = \frac{7}{3}$. Then the equations (3) give $B_{12} = \frac{29}{12} + \delta$, $B_{23} = \frac{49}{12}$, $B'_{12} = \frac{17}{6}$.

These results are tabulated at the end of this paper, both ways (for C_8 and C_{16}).

§ 6. *The Group of the Cell C_{24}.*

The 15 polytopes of this group, the cell C_{24} itself included, are represented in the following table by their expansion symbol and their characteristic number; here we omit the coordinates of their vertices, as some of these polytopes

require two or even three coordinate symbols,* but a third column gives their interpretation as derived either from C_8 or from C_{16} or in a different way from C_{24}.

C_{24}	(24,	96,	96,	24)				
e_1 "	(192,	384,	240,	48)	$= c\,e_1 e_2 e_3 C_8 = e_1 e_2 C_{16}$			
e_2 "	(288,	864,	720,	144)				
e_3 "	(144,	576,	672,	240)				
$e_1 e_2$ "	(576,	1152,	720,	144)				
$e_1 e_3$ "	(576,	1440,	1104,	240)				
$e_2 e_3$ "	(576,	1440,	1104,	240)				$= e_1 e_3 C_{24}$
$e_1 e_2 e_3$ "	(1152,	2304,	1392,	240)				
$c\,e_1$ "	(96,	288,	240,	48)	$=$	$c\,e_1 e_3 C_8 =$	$e_2 C_{16}$	
$c\,e_2$ "	(96,	288,	240,	48)	$=$	$c\,e_1 e_3 C_8 =$	$e_2 C_{16} = c\,e_1 C_{24}$	
$c\,e_3$ "	(24,	96,	96,	24)			$=$	C_4
$c\,e_1 e_2$ "	(288,	576,	336,	48)				
$c\,e_1 e_3$ "	(288,	864,	720,	144)			$=$	$e_2 C_{24}$
$c\,e_2 e_3$ "	(192,	384,	240,	48)			$=$	$e_1 C_{24}$
$c\,e_1 e_2 e_3$ "	(576,	1152,	720,	144)				$= e_1 e_2 C_{24}$

It follows from this table that we have still to determine six angles only, viz., those of $e_2 C_{24}$, $e_3 C_{24}$, $e_1 e_2 C_{24}$, $e_1 e_3 C_{24}$, $e_1 e_2 e_3 C_{24}$, $c\,e_1 e_2 C_{24}$, which angles will be denoted by C_2, C_3, C_{12}, C_{13}, C_{123}, C'_{12}. Now we have at our disposal 29 semiregular nets each of which has for constituents, besides prism and prismotope, one measure-polytope form and one form derived from cell C_{24}, *i. e.*, never more than *one* unknown angle. So we can abridge the work by mentioning only those six nets which contain the angles enumerated above. They are, once more with the constituents concurring in any vertex:

I	$c\,e_1 C_8$	$2\,P_T$	....	$3\,e_2 C_{24}$
II	$c\,e_2 C_8$	$4\,P_O$	$4\,(3;3)$	$2\,e_3 C_{24}$
III	$e_1 C_8$	P_T	....	$3\,e_1 e_2 C_{24}$
IV	$e_2 C_8$	P_O	$2\,(3;6)$	$2\,e_1 e_3 C_{24}$
V	$e_1 e_2 C_8$	P_{tT}	$(6;3)$	$2\,e_1 e_2 e_3 C_{24}$
VI	$2\,e_1 C_8$	....	....	$3\,c\,e_1 e_2 C_{24}$

So we find the six equations

* These symbols will be discussed in "Analytical Treatment, etc.," Section V. However, they figure already in Table VII, belonging to Section III.

$$\left.\begin{aligned}
3C_2 &= 16 - \frac{11}{6} - 2(-2+3\delta),\\
2C_3 &= 16 - 2 - 16(1-\delta) - \frac{16}{9},\\
3C_{12} &= 16 - \frac{23}{12} - (-2+3\delta),\\
2C_{13} &= 16 - \left(\frac{3}{2}+2\delta\right) - 4(1-\delta) - \frac{16}{9},\\
2C_{123} &= 16 - \left(\frac{29}{12}+\delta\right) - (2-\delta) - \frac{8}{9},\\
3C'_{12} &= 16 - \frac{23}{6},
\end{aligned}\right\}$$

giving

$$C_2 = \frac{109}{18} - 2\delta, \quad C_{12} = \frac{193}{36} - \delta, \quad C_{123} = \frac{385}{72},$$
$$C_3 = -\frac{17}{9} + 8\delta, \quad C_{13} = \frac{157}{36} + \delta, \quad C'_{12} = \frac{73}{18}.$$

§ 7. *Results.*

The results are put on record in the following table:

	C_5		C_8		C_{16}		C_{24}	
1	$-\frac{16}{5}+4\varepsilon$	14° 5′ 4″	1	90°	$\frac{2}{3}$	120°	2	180°
e_1	$-\frac{4}{5}+2\varepsilon$	79 2 42	$\frac{23}{12}$	172 30′	$\frac{7}{3}$	210	$\frac{15}{4}$	337 30′
e_2	$\frac{4}{5}+2\varepsilon-2\varepsilon'$	168 22 46	$\frac{3}{2}+2\delta$	141 3 27″	$\frac{7}{2}$	315	$\frac{109}{18}-2\delta$	403 56 33″
e_3	$\frac{16}{5}-4\varepsilon+6\varepsilon'$	149 54 24	$\frac{67}{12}-3\delta$	290 54 49	$\frac{67}{12}-3\delta$	290 54′ 49″	$-\frac{17}{9}+8\delta$	94 13 49
e_1e_2	$\frac{12}{5}-\varepsilon'$	188 40 2	$\frac{29}{12}+\delta$	287 1 44	$\frac{15}{4}$	337 30	$\frac{193}{36}-\delta$	411 58 16
e_1e_3	$\frac{12}{5}+\varepsilon'$	243 19 58	$\frac{19}{4}-\delta$	351 58 16	$\frac{49}{12}$	367 30	$\frac{157}{36}+\delta$	463 1 44
e_2e_3	$\frac{12}{5}+\varepsilon'$	243 19 58	$\frac{49}{12}$	367 30	$\frac{19}{4}-\delta$	351 58 16	$\frac{157}{36}+\delta$	463 1 44
$e_1e_2e_3$	$\frac{16}{5}$	288	$\frac{35}{8}$	393 45	$\frac{35}{8}$	393 45	$\frac{385}{72}$	481 15
ce_1	$\frac{12}{5}-2\varepsilon$	64 57 18	$\frac{11}{6}$	165	2	180	$\frac{7}{2}$	315
ce_2	$\frac{12}{5}-2\varepsilon$	64 57 18	2	180	$\frac{11}{6}$	165	$\frac{7}{2}$	315
ce_3	$-\frac{16}{5}+4\varepsilon$	14 5 24	$\frac{2}{3}$	120	1	90	2	180
ce_1e_2	$\frac{24}{5}-4\varepsilon$	129 54 36	$\frac{17}{6}$	255	$\frac{17}{6}$	255	$\frac{73}{18}$	365
ce_1e_3	$\frac{4}{5}+2\varepsilon-2\varepsilon'$	168 22 46	$\frac{7}{2}$	315	$\frac{3}{2}+2\delta$	141 3 27	$\frac{109}{18}-2\delta$	403 56 33
ce_2e_3	$-\frac{4}{5}+2\varepsilon$	79 2 42	$\frac{7}{3}$	210	$\frac{23}{12}$	172 30	$\frac{15}{4}$	337 30
$ce_1e_2e_3$	$\frac{12}{5}-\varepsilon'$	188 40 2	$\frac{15}{4}$	337 30	$\frac{29}{12}+\delta$	287 1 44	$\frac{193}{36}-\delta$	411 58 16

$\delta = 70° 31′ 43″,624$; $\varepsilon = 75° 31′ 20″,951$; $\varepsilon' = 27° 19′ 58″,037$.

These results are entirely in accordance with the laws of reciprocity,* of C_5 with itself, of C_8 and C_{16} with each other, of C_{24} with itself.

GRONINGEN, *January* 12, 1913.

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* Compare "Reciprocity in Connection with Semiregular Polytopes and Nets," *Proceedings* of Amsterdam, Vol. XIII (1910), p. 384.

As C_{120} and C_{600} are polarly related, the offspring of the one is equal to that of the other; so the law of reciprocity will require the equality of the angles of $e_1 C_{120}$ and $c\, e_1 e_3 C_{600}$, etc.

It may be useful to remember that the Mathematical Society of Amsterdam had proposed the following prize question:

"To determine the four-dimensional angles of the semiregular polytopes of space S_4 by a direct method, and to deduce from it all the possible space fillings by polytopes with one kind of vertex and one length of edge."

This question has been withdrawn, as all the nets are found. Nevertheless it still remains a desideratum: to determine directly the angles of the polytopes belonging to C_{120} and C_{600}.

*A Generalization of Volterra's Derivative of a Function of a Curve.**

By Charles Albert Fischer.

Volterra† has defined the derivative of a function of a curve at a point. He denotes the coordinates of the moving point which generates the curve by $(x, \phi(x))$, and the function and its derivative at the point $x=t$ by the expressions

$$y|[\phi(x)]|, \quad y'|[\phi(x), t]|,$$

respectively, and has proved the theorem that if the derivative is continuous and approached uniformly, then the first variation of the function is expressed by the equation

$$\delta y|[\phi(x)]| = \int_A^B y'|[\phi(x), t]|\delta\phi(t)dt.$$

He does not mention the order of approach to the derivative, but in the statement of his theorem he has assumed the derivative to be unique when the approach is of order zero.‡ In most of the applications it is necessary to assume an approach of higher order if the derivative is to be determined uniquely. Bliss, in an unpublished paper, has proved the same theorem in a slightly different form, introducing the idea of the order of approach to the derivative. He expresses the equations of the curve in parametric form, using the length of arc as the parameter, and assuming the variations to be orthogonal.

In the present paper we assume a set of m differential equations given, of the form

$$z_i' = g_i(x, y, y', z_1, z_2, \ldots, z_m) \qquad (i = 1, 2, \ldots, m),$$

where the functions $g_1, g_2, \ldots, g_m$ satisfy certain continuity conditions, and discuss the Volterra derivative of a function of the curve

$$C: \quad y = y(x) \qquad (x_1 \leqq x \leqq x_2)$$

relative to the set of curves along which the functions $z_1(x), z_2(x), \ldots, z_m(x)$,

* Read before the American Mathematical Society, February 22, 1913.

† Volterra: *Atti della R. Academia dei Lincei*, Ser. 4, Rendiconti III_2, p. 97.

‡ For the definition of the order of approach see § 3.

determined by the differential equations above, have given values at the end-points $x = x_1$ and $x = x_2$.

In § 1 two existence theorems are proved. The first has to do with the continuity of the solution of a set of ordinary differential equations, and the second is a modification of the existence theorem for implicit functions. In § 2 we define the set of curves to be considered and prove the existence of curves of the set having the properties required in the definition of the Volterra derivative. This section also contains a short discussion of the analogy between the theory developed in this paper and some simple theorems in the theory of functions of a finite number of variables. In § 3 will be found the definition of the Volterra derivative of a curve at a point, relative to the set of curves defined in § 2, and a proof of the theorem due to Volterra which has already been mentioned. The statement and proof of the theorem have necessarily been modified because of the special class of curves considered, but the theorem substantially as stated by Volterra falls out as a special case. The theorem as proved in § 3 will be called the generalized Volterra theorem. This section also contains the theorem that if the Volterra derivative of a function of a curve is continuous and approached uniformly with any finite order, then it must vanish identically along a curve which minimizes the value of the function considered. The last section deals with the application of the theory already developed to the Lagrange problem of the calculus of variations.

§ 1. *Auxiliary Theorems.*

We shall frequently consider systems of differential equations of the form

$$z_i' = g_i(x, \omega(x), z_1, z_2, \ldots, z_m) \qquad (i = 1, 2, \ldots, m),$$

where $\omega(x)$ represents a set of arbitrary functions which are always less in numerical value than a constant δ. The well-known existence theorem for the solution of a set of ordinary differential equations implies that if we assume a continuity of a high enough order for the functions g_i and ω there is a unique solution $z_1(x, \omega), z_2(x, \omega), \ldots, z_m(x, \omega)$ with a unique set of initial values and with continuity of any required order. It will also be necessary to know that the functions $z_i(x, \omega)$ approach the functions $z_i(x, 0)$ uniformly as $\omega(x)$ approaches zero with δ. This will be established by means of the following theorem.

Suppose there is given a set of differential equations of the form

$$y_i' = \psi_i(t, y_1, y_2, \ldots, y_m; \eta_1, \eta_2, \ldots, \eta_r) = \psi_i(t, y; \eta)$$
$$(i = 1, 2, \ldots, m; \ t_1 \leqq t \leqq t_2) \qquad (1)$$

containing the arbitrary functions $\eta_1, \eta_2, \ldots, \eta_r$ of t, and which for $\eta = 0$ have a solution

$$y_i = x_i(t) \qquad (i = 1, 2, \ldots, m;\ t_1 \leqq t \leqq t_2). \tag{2}$$

The functions ψ_i are supposed to be continuous and to have continuous first derivatives with respect to the variables y in a neighborhood R of the points $(t, x(t), 0)$ defined by the solutions (2). Let the absolute values of the functions η be restricted so as to satisfy the inequalities

$$|\eta_p(t)| \leqq \delta \qquad (p = 1, 2, \ldots, r;\ t_1 \leqq t \leqq t_2). \tag{3}$$

THEOREM I. *For every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever the functions η satisfy the inequalities* (3), *the solutions $y_i(t)$ of equations* (1) *which have the initial values*

$$y_i(t_1) = x_i(t_1) \qquad (i = 1, 2, \ldots, m),$$

are well defined over the entire interval $t_1 \leqq t \leqq t_2$ and satisfy the inequalities

$$|y_i(t) - x_i(t)| \leqq \varepsilon \qquad (i = 1, 2, \ldots, m;\ t_1 \leqq t \leqq t_2).$$

Let M/m be the largest of the maxima of the absolute values of the first partial derivatives of the functions $\psi_1, \psi_2, \ldots, \psi_m$ with respect to the variables y in the region R, and choose δ, ρ so that the region of points (t, y, η) defined by the inequalities

$$|\eta_p| \leqq \delta;\ x_i(t) - \frac{\rho}{M} e^{M(t_2 - t_1)} \leqq y_i \leqq x_i(t) + \frac{\rho}{M} e^{M(t_2 - t_1)};\ t_1 \leqq t \leqq t_2$$
$$(p = 1, 2, \ldots, r;\ i = 1, 2, \ldots, m)$$

lies entirely within R. The constant δ can furthermore be so restricted that the inequalities

$$|\psi_i(t, x(t); \eta) - \psi_i(t, x(t); 0)| \leqq \rho \qquad (t_1 \leqq t \leqq t_2)$$

are satisfied.

Define the functions $y_i^{(0)}(t), y_i^{(1)}(t), \ldots, y_i^{(n)}(t), \ldots$ by the equations

$$y_i^{(0)}(t) = x_i(t) \qquad (i = 1, 2, \ldots, m),$$

$$y_i^{(n)}(t) = x_i(t) + \int_{t_1}^{t} [\psi_i(t, y^{(n-1)}(t); \eta) - \psi_i(t, x(t); 0)]\, dt$$
$$(n = 1, 2, 3, \ldots).$$

Then, if the functions $y_i^{(n)}(t)$ approach definite limiting functions $y_i(t)$ as n becomes infinite, this set of functions $(y_1(t), y_2(t), \ldots, y_m(t))$ constitute a

solution of equations (1) having the required initial values.* It is evident that the equations

$$|y_i^{(1)}-y_i^{(0)}| = \left|\int_{t_1}^{t} \{\psi_i(t,x;\eta)-\psi_i(t,x;0)\}\,dt\right| \qquad (i=1,2,\ldots,m;\ t_1\leqq t\leqq t_2)$$

are satisfied. They imply the inequalities

$$|y_i^{(1)}-y_i^{(0)}| \leqq \rho\,(t-t_1).$$

Similarly,

$$|y_i^{(2)}-y_i^{(1)}| = \left|\int_{t_1}^{t} \{\psi_i(t,y^{(1)};\eta)-\psi_i(t,y^{(0)};\eta)\}\,dt\right|.$$

Expanding the integrands by Taylor's formula and remembering the definition of the constant M, we find the inequalities

$$|y_i^{(2)}-y_i^{(1)}| \leqq \int_{t_1}^{t} M\rho(t-t_1)\,dt = M\rho\,\frac{(t-t_1)^2}{2!}.$$

By a simple mathematical induction it can be proved that

$$|y_i^{(n)}-y_i^{(n-1)}| \leqq \frac{\rho}{M}\,\frac{M^n(t-t_1)^n}{n!} \qquad (i=1,2,\ldots,m;\ n=1,2,\ldots;\ t_1\leqq t\leqq t_2).$$

Hence the series

$$y_i^{(0)}+(y_i^{(1)}-y_i^{(0)})+\ldots+(y_i^{(n)}-y_i^{(n-1)})+\ldots$$

are uniformly convergent and their limits $y_i(t)$ satisfy the inequalities

$$|y_i(t)-x_i(t)| \leqq \frac{\rho}{M}\,e^{M(t_2-t_1)} \qquad (i=1,2,\ldots,m;\ t_1\leqq t\leqq t_2). \tag{4}$$

If ρ is chosen so that the right member of the inequalities (4) is less than ε, the conditions of Theorem I are satisfied.

The existence theorem for implicit functions states that if there is given a solution

$$x=a;\ y_1=b_1,\ y_2=b_2,\ \ldots,\ y_m=b_m$$

of the algebraic equations

$$\phi_i(x;y_1,y_2,\ldots,y_m)=0 \qquad (i=1,2,\ldots,m),$$

where the functions $\phi_1, \phi_2, \ldots, \phi_m$ satisfy suitable continuity conditions, then for every value of x sufficiently near to a there is a unique set of values for $y_1, y_2, \ldots, y_m$ in the neighborhood of $b_1, b_2, \ldots, b_m$ which satisfy the equations. In the next section we shall have a similar situation, excepting that instead of a single variable x we shall have the infinity of variables represented

* This is the set of approximation functions usually used in Picard's method of approximation. See Bliss, *Annals of Mathematics*, Ser. 2, Vol. VI, p. 49. The proof that the functions defined by these sequences actually solve the differential equations, and the proof of the uniqueness of the solutions so determined, is the same as in the reference just given and is omitted here.

by a function $\eta(x)$ taken over the range $x_1 \leqq x \leqq x_2$. The introduction of the infinity of independent variables necessitates changes in the hypothesis and in the proof of the theorem. The proof given below is similar to a recent proof of the existence theorem for implicit functions due to Bliss.*

Consider a set of functions

$$\phi_i(H; \alpha_1, \alpha_2, \ldots, \alpha_m) = \phi_i(H; \alpha) \qquad (i = 1, 2, \ldots, m)$$

whose values are determined uniquely when the variables $\alpha_1, \alpha_2, \ldots, \alpha_m$ and a curve

$$H: \quad \eta = \eta(x) \qquad (x_1 \leqq x \leqq x_2)$$

in the $x\eta$-plane are given. The functions will be said to be continuous at the values $(\overline{H}; \bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_m)$, provided that for every positive constant e a second positive constant d can be found such that

$$|\phi_i(H; \alpha) - \phi_i(\overline{H}; \bar{\alpha})| \leqq e$$

whenever

$$|\eta(x) - \bar{\eta}(x)| < d, \quad |\alpha_i - \bar{\alpha}_i| \leqq d \qquad (x_1 \leqq x \leqq x_2;\ i = 1, 2, \ldots, m).$$

Let the values $(H; \alpha_1, \alpha_2, \ldots, \alpha_m) = (H_0; 0, 0, \ldots, 0)$, where H_0 is the curve $\eta(x) \equiv 0$, be a solution of the equations

$$\phi_i(H; \alpha) = 0 \qquad (i = 1, 2, \ldots, m), \tag{5}$$

and suppose that, with respect to the variables α, the functions $\phi_1, \phi_2, \ldots, \phi_m$ are continuous and have continuous first partial derivatives in a domain

$$\eta(x) \leqq \varepsilon; \ |\alpha_i| \leqq \alpha^{(0)} \qquad (x_1 \leqq x \leqq x_2;\ i = 1, 2, \ldots, m). \tag{6}$$

At the initial values $(H_0; 0)$ it will be supposed that these functions and their first partial derivatives are continuous with respect to all arguments in the sense described above, and furthermore that the functional determinant

$$\begin{vmatrix} \dfrac{\partial \phi_1(H_0; 0)}{\partial \alpha_1} & \dfrac{\partial \phi_1(H_0; 0)}{\partial \alpha_2} & \cdots & \dfrac{\partial \phi_1(H_0; 0)}{\partial \alpha_m} \\ \dfrac{\partial \phi_2(H_0; 0)}{\partial \alpha_1} & \dfrac{\partial \phi_2(H_0; 0)}{\partial \alpha_2} & \cdots & \dfrac{\partial \phi_2(H_0; 0)}{\partial \alpha_m} \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \dfrac{\partial \phi_m(H_0; 0)}{\partial \alpha_1} & \dfrac{\partial \phi_m(H_0; 0)}{\partial \alpha_2} & \cdots & \dfrac{\partial \phi_m(H_0; 0)}{\partial \alpha_m} \end{vmatrix} \tag{7}$$

is different from zero. Then the following theorem can be proved:

THEOREM II. *If* $(H_0; 0, 0, \ldots, 0)$ *is an initial solution of the kind described above, then two positive constants* ε *and* $\alpha^{(0)}$ *can be determined such*

* *Bulletin of the American Mathematical Society*, Vol. XVIII, No. 4, p. 175.

that no two solutions of equations (5) *in the domain* (6) *have the same argument H. Furthermore, a positive constant* $\delta \leqq \varepsilon$ *can be taken so small that for every curve H satisfying the inequality*

$$|\eta(x)| \leqq \delta \qquad (x_1 \leqq x \leqq x_2) \tag{8}$$

there exists a solution $(H; \alpha_1, \alpha_2, \ldots, \alpha_m)$ *of equations* (5) *in the domain* (6).

Choose the constants ε, $\alpha^{(0)}$ so small that in the domain (6) the inequality

$$\begin{vmatrix} \frac{\partial \phi_1(H;\alpha)}{\partial \alpha_1} & \frac{\partial \phi_1(H;\alpha)}{\partial \alpha_2} & \cdots & \frac{\partial \phi_1(H;\alpha)}{\partial \alpha_m} \\ \frac{\partial \phi_2(H;\alpha)}{\partial \alpha_1} & \frac{\partial \phi_2(H;\alpha)}{\partial \alpha_2} & \cdots & \frac{\partial \phi_2(H;\alpha)}{\partial \alpha_m} \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \frac{\partial \phi_m(H;\alpha)}{\partial \alpha_1} & \frac{\partial \phi_m(H;\alpha)}{\partial \alpha_2} & \cdots & \frac{\partial \phi_m(H;\alpha)}{\partial \alpha_m} \end{vmatrix} \neq 0 \tag{9}$$

is always satisfied. The continuity of the functions $\phi_1, \phi_2, \ldots, \phi_m$ and their partial derivatives and the non-vanishing of the determinant (7) make such a choice of $\varepsilon, \alpha^{(0)}$ possible.

It will first be proved that there can not be two distinct solutions $(H; \alpha_1', \alpha_2', \ldots, \alpha_m')$ and $(H; \alpha_1, \alpha_2, \ldots, \alpha_m)$ of equations (5) in the domain (6). Suppose there are two such solutions. By means of Taylor's formula we can derive the equations

$$\phi_i(H; \alpha') - \phi_i(H; \alpha) = \sum_{j=1}^{m} \frac{\partial \phi_i}{\partial \alpha_j} (\alpha_j' - \alpha_j) = 0 \qquad (i = 1, 2, \ldots, m), \tag{10}$$

where the arguments of the partial derivatives of ϕ_i are

$$H; \alpha_1 + \theta(\alpha_1' - \alpha_1), \ldots, \alpha_m + \theta(\alpha_m' - \alpha_m),$$

in which $0 < \theta < 1$. The determinant of the coefficients of $\alpha_j' - \alpha_j$ in equations (10) cannot vanish on account of the inequality (9). Therefore the only solution of equations (10) is

$$\alpha_j' - \alpha_j = 0 \qquad (j = 1, 2, \ldots, m),$$

and the solutions $(H; \alpha')$, $(H; \alpha)$ of equations (5) are not distinct.

It will next be proved that a constant $\delta \leqq \varepsilon$ can be so restricted that, for every curve H satisfying the inequality (8), the equations (5) have a solution $(H; \alpha_1, \alpha_2, \ldots, \alpha_m)$ lying in the domain (6). Define a function Φ by the equation

$$\Phi(H; \alpha_1, \alpha_2, \ldots, \alpha_m) = \sum_{i=1}^{m} \phi_i^2(H; \alpha_1, \alpha_2, \ldots, \alpha_m).$$

The function $\Phi(H_0; \alpha)$ has a minimum for $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 0$, since it vanishes for those values and is positive everywhere else in the region

$$|\alpha_i| \leqq \alpha^{(0)} \qquad (i = 1, 2, \ldots, m). \tag{11}$$

In particular, there is a positive number e such that the inequality

$$\Phi(H_0; \alpha) - \Phi(H_0; 0) > e$$

is satisfied for all points α on the boundary of the region (11). Then, if δ is sufficiently small, the inequality

$$\Phi(H; \alpha) - \Phi(H; 0) > e$$

is also satisfied for all functions $\eta(x)$ satisfying the condition (8) and all points α on the boundary of the region (11). The minimum of the function $\Phi(H; \alpha)$, considered as a function of $\alpha_1, \alpha_2, \ldots, \alpha_m$ alone, in the closed region (11) is therefore not at a boundary point of the region. The partial derivatives with respect to the variables α must vanish at the particular interior point of the region (11) at which $\Phi(H; \alpha)$ takes its minimum value. At this point we find the equations

$$\begin{aligned}
&\frac{1}{2}\frac{\partial \Phi}{\partial \alpha_1} = \phi_1 \frac{\partial \phi_1}{\partial \alpha_1} + \phi_2 \frac{\partial \phi_2}{\partial \alpha_1} + \ldots + \phi_m \frac{\partial \phi_m}{\partial \alpha_1} = 0, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
&\frac{1}{2}\frac{\partial \Phi}{\partial \alpha_m} = \phi_1 \frac{\partial \phi_1}{\partial \alpha_m} + \phi_2 \frac{\partial \phi_2}{\partial \alpha_m} + \ldots + \phi_m \frac{\partial \phi_m}{\partial \alpha_m} = 0.
\end{aligned}$$

Since the determinant of the coefficients of $\phi_1, \phi_2, \ldots, \phi_m$ is distinct from zero, on account of the inequality (9), we know that equations (5) must be satisfied by the minimizing variables α. Thus the second part of the theorem is proved.

§ 2. *The Set of Curves K.*

Consider a set of differential equations

$$z_i' = g_i(x, y, y', z_1, z_2, \ldots, z_m). \tag{12}$$

Since the subscripts h, i, j always have the range $1, 2, 3, \ldots, m$ and the variable x the range $x_1 \leqq x \leqq x_2$ in this paper, we will hereafter not state the ranges explicitly in every set of equations. When a function $y = y(x)$ is given, equations (12) determine m functions $z_i(x)$ uniquely, if we assume the initial values z_{i1} of the functions $z_i(x)$ at $x = x_1$ to be given in advance. We are to consider the curves

$$C: \quad y = y(x) \tag{13}$$

joining two given points (x_1, y_1) and (x_2, y_2) in the xy-plane. *Such a curve*

is said to belong to the set of curves K if it passes throuyh the points (x_1, y_1) *and* (x_2, y_2), *and if furthermore, when* $y(x)$ *is substituted for* y *and its derivative for* y' *in equations* (12), *the corresponding functions* $z_i(x)$ *not only have the prescribed values* z_{i1} *at* $x = x_1$ *but also given values* z_{i2} *at* $x = x_2$.

We will now take a particular curve C which belongs to the set K and which for the purposes of the following discussion will be assumed to be of class C'' * on the interval $x_1 \leqq x \leqq x_2$.

Associated with C there will be a set of functions

$$z_i = z_i(x) \tag{14}$$

having the given end-values z_{i1}, z_{i2} and satisfying equations (12). The functions $z_i(x)$ must be of class C'' in the discussion which is to follow. This continuity will be assured if in the $(m+3)$-dimensional space of points $(x, y, y', z_1, z_2, \ldots, z_m)$ in the neighborhood of the values defined by equations (13) and (14) the functions g_i are assumed to be of class C''.

There is a set of linear differential equations associated with equations (12), the so-called equations of variation,† which play an important rôle in the following pages. They are the equations

$$\zeta_i' = \frac{\partial g_i}{\partial y}\eta + \frac{\partial g_i}{\partial y'}\eta' + \sum_j \frac{\partial g_i}{\partial z_j}\zeta_j \tag{15}$$

which would be satisfied by the derivatives

$$\eta = \frac{\partial y(x,\alpha)}{\partial \alpha}, \quad \zeta_j = \frac{\partial z_j(x,\alpha)}{\partial \alpha}$$

of a family of systems of functions

$$y = y(x,\alpha), \quad z_i = z_i(x,\alpha)$$

satisfying equations (12), as is evident at once on differentiating those equations. The equations (15) are linear and non-homogeneous in $\zeta_1, \zeta_2, \ldots, \zeta_m$. If the first two terms of their right members are dropped, they become homogeneous, and there is consequently an adjoint system‡

$$\lambda_i' + \sum_j \frac{\partial g_j}{\partial z_i}\lambda_j = 0, \tag{16}$$

which will also be of service.

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* Bolza: "Vorlesungen über Variationsrechnung," p. 14.

† Bolza: *loc. cit.*, p. 560, equations (58). Hadamard: "Leçons sur le Calcul des Variations," p. 226, equations (K).

‡ Hadamard: *loc. cit.*, pp. 33 and 230.

The curve C is said to be normal if there is no solution of equations (16) *which is also a solution of the equation*

$$\sum_j \left\{ \lambda_j \frac{\partial g_j}{\partial y} - \frac{d}{dx}\left(\lambda_j \frac{\partial g_j}{\partial y'} \right) \right\} = 0, \tag{17}$$

where the arguments of the derivatives of g_j are $x, y(x), y'(x), z_1(x), z_2(x), \ldots, z_m(x)$, excepting the solution $(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m) = (0, 0, 0, \ldots, 0)$.

The existence of such a solution of equations (16) and (17) is the first condition which must be satisfied if the curve C is to furnish a minimum for one of the functions, say $z_1(x)$, at $x = x_2$, when the prescribed values are given for all the functions $z_i(x)$ at $x = x_1$, and for all excepting $z_1(x)$ at $x = x_2$. The assumption that C is a normal curve is therefore equivalent to the assumption that C is not an extremal for the minimum problem just described.*

Let the systems $(\lambda_{h1}, \lambda_{h2}, \ldots, \lambda_{hm})$ be the solution of equations (16) which have the initial values †

$$\lambda_{ii}(x_2) = 1, \quad \lambda_{hi}(x_2) = 0 \qquad (i \neq h). \tag{18}$$

Any other solution will then be expressible in the form

$$\lambda_i(x) = \sum_h c_h \lambda_{hi}(x).$$

By means of the solutions $(\lambda_{h1}, \lambda_{h2}, \ldots, \lambda_{hm})$ of the adjoint equations (16), the values $\zeta_i(x_2)$ of the solutions of equations (15) which vanish at $x = x_1$ can readily be expressed in terms of the function $\eta(x)$. Multiply equations (15) by $\lambda_{hi}(x)$ and sum with respect to i. Then applying equations (16), we obtain the equations

$$\frac{d}{dx} \sum_i \lambda_{hi} \zeta_i = \sum_i \lambda_{hi} \left\{ \frac{\partial g_i}{\partial y} \eta + \frac{\partial g_i}{\partial y'} \eta' \right\}. \tag{19}$$

If equations (19) are integrated between the limits x_1 and x_2, and the right members integrated by parts, they become

$$\zeta_h(x_2) = \int_{x_1}^{x_2} N_h(x)\, \eta(x)\, dx, \tag{20}$$

where

$$N_h(x) = \sum_i \left\{ \lambda_{hi} \frac{\partial g_i}{\partial y} - \frac{d}{dx}\left(\lambda_{hi} \frac{\partial g_i}{\partial y'} \right) \right\}, \tag{21}$$

since it is always assumed that the function $\eta(x)$ vanishes at the end-points.

It will now be proved that, if C is a normal curve, a set of points $\xi_1, \xi_2, \ldots, \xi_m$, on the curve C, can always be selected such that the determinant

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* Bolza: *loc. cit.*, p. 564. Hadamard: *loc. cit.*, p. 234, equations (E).

† Hadamard: *loc. cit.*, p. 230–231.

$$\begin{vmatrix} N_1(\xi_1) & N_1(\xi_2) & \dots & N_1(\xi_m) \\ N_2(\xi_1) & N_2(\xi_2) & \dots & N_2(\xi_m) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ N_m(\xi_1) & N_m(\xi_2) & \dots & N_m(\xi_m) \end{vmatrix} \neq 0. \tag{22}$$

It follows directly from the definition of a normal curve that ξ_1 can be taken so that $N_1(\xi_1) \neq 0$. Then the determinant

$$\begin{vmatrix} N_1(\xi_1) & N_1(x) \\ N_2(\xi_2) & N_2(x) \end{vmatrix}$$

can not vanish identically. Otherwise the expressions

$$N_1(\xi_1)\lambda_{2i} - N_2(\xi_1)\lambda_{1i}$$

would be a solution of equations (17) not identically zero, and the curve C would not be normal. Hence ξ_2 can be so chosen that it satisfies the equation

$$\begin{vmatrix} N_1(\xi_1) & N_1(\xi_2) \\ N_2(\xi_1) & N_2(\xi_2) \end{vmatrix} \neq 0.$$

Proceeding in this way with the determinant

$$\begin{vmatrix} N_1(\xi_1) & N_1(\xi_2) & N_1(x) \\ N_2(\xi_1) & N_2(\xi_2) & N_2(x) \\ N_3(\xi_1) & N_3(\xi_2) & N_3(x) \end{vmatrix}$$

and so on, it is easy to show that $\xi_1, \xi_2, \dots, \xi_m$ can be chosen so that the inequality (22) is satisfied.

Consider a constant ε, and any functions $\eta_1(x), \eta_2(x), \dots, \eta_m(x)$ such that $\eta_i(x)$ is of class C'' and vanishes identically everywhere excepting on the interval $\xi_i - \varepsilon < x < \xi_i + \varepsilon$, where it does not change its sign and is not identically zero, and it furthermore satisfies the inequalities

$$|\eta_i(x)| \leqq \varepsilon, \quad |\eta_i'(x)| \leqq \varepsilon, \quad |\eta_i''(x)| \leqq \varepsilon. \tag{23}$$

By applying the mean-value theorem for a definite integral to the determinant

$$\begin{vmatrix} \int_{\xi_1-\epsilon}^{\xi_1+\epsilon} N_1(x)\eta_1(x)\,dx & \int_{\xi_2-\epsilon}^{\xi_2+\epsilon} N_1(x)\eta_2(x)\,dx & \dots & \int_{\xi_m-\epsilon}^{\xi_m+\epsilon} N_1(x)\eta_m(x)\,dx \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \int_{\xi_1-\epsilon}^{\xi_1+\epsilon} N_m(x)\eta_1(x)\,dx & \int_{\xi_2-\epsilon}^{\xi_2+\epsilon} N_m(x)\eta_2(x)\,dx & \dots & \int_{\xi_m-\epsilon}^{\xi_m+\epsilon} N_m(x)\eta_m(x)\,dx \end{vmatrix}$$

it can be reduced to the form

$$\begin{vmatrix} N_1(\xi_1^{(1)}) & N_1(\xi_2^{(1)}) & \dots & N_1(\xi_m^{(1)}) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ N_m(\xi_1^{(m)}) & N_m(\xi_2^{(m)}) & \dots & N_m(\xi_{(m)}^{m}) \end{vmatrix} \prod_{i=1}^{m} \int_{\xi_i-\epsilon}^{\xi_i+\epsilon} \eta_i(x)\,ax,$$

where $\xi_i^{(j)}$ are certain mean values on the respective intervals $\xi_i-\varepsilon<x<\xi_i+\varepsilon$. Therefore, if ε is chosen sufficiently small, the inequality (22) implies the inequality

$$\left|\begin{matrix}\int_{\xi_1-\epsilon}^{\xi_1+\epsilon} N_1(x)\eta_1(x)dx & \int_{\xi_2-\epsilon}^{\xi_2+\epsilon} N_1(x)\eta_2(x)dx & \cdots & \int_{\xi_m-\epsilon}^{\xi_m+\epsilon} N_1(x)\eta_m(x)dx \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \int_{\xi_1-\epsilon}^{\xi_1+\epsilon} N_m(x)\eta_1(x)dx & \int_{\xi_2-\epsilon}^{\xi_2+\epsilon} N_m(x)\eta_2(x)dx & \cdots & \int_{\xi_m-\epsilon}^{\xi_m+\epsilon} N_m(x)\eta_m(x)dx\end{matrix}\right| \neq 0, \quad (24)$$

since $N_h(x)$ is a continuous function of x.

If the points $\xi_1, \xi_2, \ldots, \xi_m$ are considered as fixed and satisfying the conditions imposed above, it is possible to prove the following important theorem:

Suppose $\eta(x)$ is a function of class C'' which vanishes at the end-points $x=x_1$ and $x=x_2$ and satisfies the inequalities

$$|\eta(x)| \leqq \delta, \quad |\eta'(x)| \leqq \delta, \quad |\eta''(x)| \leqq \delta. \quad (25)$$

Suppose, further, that the functions $\eta_1(x), \eta_2(x), \ldots, \eta_m(x)$ are such that $\eta_i(x)$ is of class C'', vanishes for all values of x excepting on the interval $\xi_i-\varepsilon<x<\xi_i+\varepsilon$, where it does not change its sign and is not identically zero, and satisfies the inequalities (23). Then for $\varepsilon>0$ it is possible to choose a positive constant $\delta\leqq\varepsilon$, such that, corresponding to every function $\eta(x)$, there is a set of functions $\eta_1(x), \eta_2(x), \ldots, \eta_m(x)$ which make the curve

$$C_\epsilon: \quad y=y(x)+\eta(x)+\sum_i \eta_i(x) \quad (26)$$

belong to the set K.

Choose an arbitrary set of functions $\bar{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_m$ of class C'' which satisfy the inequalities (23) and vanish everywhere excepting on the specified intervals where they do not change their signs and are not identically zero, and let ε be chosen such that the inequality (24) is satisfied. Consider the curve

$$C_\epsilon: \quad y=y(x)+\eta(x)+\sum_i \alpha_i\bar{\eta}_i(x),$$

where the constants α_i are to be determined. Designate the solution of equations (12), when $y(x)+\eta(x)+\sum_i \alpha_i\bar{\eta}_i(x)$ has been substituted for y and its derivative for y', which has the given set of initial values $z_{11}, z_{21}, \ldots, z_{m1}$ at $x=x_1$, by $z_1(x,\eta;\alpha), z_2(x,\eta;\alpha), \ldots, z_m(x,\eta;\alpha)$, where α represents the set of parameters $\alpha_1, \alpha_2, \ldots, \alpha_m$, and define $\phi_i(H;\alpha)$ by the equations

$$z_i(x_2,\eta;\alpha)-z_{i2}=\phi_i(H;\alpha).$$

The first theorem of § 1 implies that there is a constant $\varepsilon_0>0$, depending on

the magnitude of the region in which the functions g_i are continuous, such that, if we take $\varepsilon \leqq \varepsilon_0$ and $\alpha^{(0)}=1$, the expressions

$$\phi_i(H;\alpha), \quad \frac{\partial \phi_i(H;\alpha)}{\partial \alpha_j}$$

are continuous functions of the arguments α in the neighborhood defined by inequalities (23) and (6). They are also continuous at the values $(H_0; 0, 0, \ldots, 0)$ in the sense described in § 1, with respect to all arguments. Equations (20) imply the equations

$$\frac{\partial \phi_i(H_0;0)}{\partial \alpha_j} = \int_{\xi_j-\epsilon}^{\xi_j+\epsilon} N_i(x)\bar{\eta}_j(x)\,dx,$$

and the inequality (24) implies that the functional determinant

$$\begin{vmatrix} \frac{\partial \phi_1(H_0;0)}{\partial \alpha_1} & \frac{\partial \phi_1(H_0;0)}{\partial \alpha_2} & \cdots & \frac{\partial \phi_1(H_0;0)}{\partial \alpha_m} \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \frac{\partial \phi_m(H_0;0)}{\partial \alpha_1} & \frac{\partial \phi_m(H_0;0)}{\partial \alpha_2} & \cdots & \frac{\partial \phi_m(H_0;0)}{\partial \alpha_m} \end{vmatrix}$$

is different from zero. Thus the hypothesis of the second theorem of § 1 is satisfied, and if δ is taken sufficiently small, there is a unique set of values for the parameters $\alpha_1, \alpha_2, \ldots, \alpha_m$, each less in absolute value than unity, corresponding to every curve H in the $x\eta$-plane which satisfies the condition (25), such that $(H; \alpha_1, \alpha_2, \ldots, \alpha_m)$ is a solution of the equations

$$\phi_i(H;\alpha) = z_i(x_2, \eta; \alpha) - z_{i2} = 0.$$

If we now let $\alpha_i\bar{\eta}_i = \eta_i$, the curve C_ε which is defined by equation (26) belongs to the set K. It is evident that if a constant δ is effective for a particular ε, say ε_0, it is also effective for every ε greater than ε_0.

The proofs in this section are unchanged if the functions considered are assumed to be of class C^r, $r>2$, instead of C'', and the inequalities (23) are replaced by the inequalities

$$|\eta_i(x)| \leqq \varepsilon, \quad |\eta_i'(x)| \leqq \varepsilon, \quad \ldots, \quad |\eta_i^{(r)}(x)| \leqq \varepsilon.$$

There is an interesting analogy between the results which have been obtained in this section, and some of the theorems in the theory of functions of a finite number of variables. The functions $z_i(x)$ which have the given initial values z_{i1} are uniquely determined by equations (12) when the function $y(x)$ is given, and therefore the quantities $z_i(x_2)$ are functions of the value of y at

every point on the range $x_1 \leqq x \leqq x_2$. If, instead of this infinity of variables $y(x)$ $(x_1 \leqq x \leqq x_2)$, we consider a finite set y_p $(p = 1, 2, \ldots, n)$, where the range $p = 1, 2, \ldots, n$ takes the place of the range $x_1 \leqq x \leqq x_2$, the equations $z_i(x_2) = z_{i2}$ have as their analogues a system of equations

$$z_i(y_1, y_2, \ldots, y_n) = z_{i2}. \tag{27}$$

Suppose there is given a set of values for $y_1, y_2, \ldots, y_n$ which satisfy equations (27), corresponding to the initial curve C of the set K. Suppose furthermore that an arbitrary increment η_p be given to each y_p, and that we wish to determine increments ω_{p_i} for m particular variables y_{p_i} such that the values

$$y_p + \eta_p \ (p \neq p_i), \quad y_{p_i} + \eta_{p_i} + \omega_{p_i} \tag{28}$$

shall satisfy equations (27). The condition that this is possible for every choice of η_p sufficiently small is the inequality

$$\begin{vmatrix} \frac{\partial z_1}{\partial y_{p_1}} & \frac{\partial z_1}{\partial y_{p_2}} & \cdots & \frac{\partial z_1}{\partial y_{p_m}} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial z_m}{\partial y_{p_1}} & \frac{\partial z_m}{\partial y_{p_2}} & \cdots & \frac{\partial z_m}{\partial y_{p_m}} \end{vmatrix} \neq 0, \tag{29}$$

which is the condition that equations (27) be solvable for the variables $y_{p_1}, y_{p_2}, \ldots, y_{p_m}$. If the determinant (29) vanishes for every choice of $p_1, p_2, \ldots, p_m$ there is a set of constants $\lambda_1, \lambda_2, \ldots, \lambda_m$, not all zero, which satisfy the equations

$$\sum_i \lambda_i \frac{\partial z_i}{\partial y_p} = 0 \qquad (p = 1, 2, \ldots, n).$$

This constitutes the first necessary condition that this set of values furnish a minimum for one of the functions, say $z_1(y_1, y_2, \ldots, y_m)$, with respect to those values for $y_1, y_2, \ldots, y_m$ which satisfy the equations

$$z_k(y_1, y_2, \ldots, y_n) = z_{k2} \qquad (k = 2, 3, \ldots, m). \tag{30}$$

Such a set of values for $y_1, y_2, \ldots, y_m$ is analogous to one of the curves C which is not normal according to the definition given above.

Suppose then that the set $(y_1, y_2, \ldots, y_n)$ satisfying equations (27) is normal in the sense that not all of the determinants of the matrix

$$\left\| \frac{\partial z_i}{\partial y_p} \right\| \qquad (i = 1, 2, \ldots, m; \ p = 1, 2, \ldots, n)$$

vanish, and suppose for convenience of notation that the determinant

$$\begin{vmatrix} \frac{\partial z_1}{\partial y_1} & \frac{\partial z_1}{\partial y_2} & \cdots & \frac{\partial z_1}{\partial y_m} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial z_m}{\partial y_1} & \frac{\partial z_m}{\partial y_2} & \cdots & \frac{\partial z_m}{\partial y_m} \end{vmatrix}$$

is different from zero. Then we may take the set $(p_1, p_2, \ldots, p_m) = (1, 2, \ldots, m)$, a rearrangement which is not possible for the infinite set of variables $y(x)$, but which does not affect the argument when the number is finite. The theorem that for any set of increments $\eta_1, \eta_2, \ldots, \eta_n$ sufficiently small another set $\omega_1, \omega_2, \ldots, \omega_m$ can be determined, such that the values (28) satisfy equations (27), is analogous to the theorem proved above that the curve whose equation is

$$y = y(x) + \eta(x)$$

can always be made into a curve

$$C_\varepsilon: \quad y = y(x) + \eta(x) + \sum_i \eta_i(x)$$

of the set K by adding a properly chosen set of functions $\eta_i(x)$.

The derivative of another function $L(y_1, y_2, \ldots, y_n)$ with respect to one of the last n-m variables, say y_n, may be defined as the limit

$$\lim_{\eta_n=0} \frac{L(y_1+\omega_1, \ldots, y_m+\omega_m, y_{m+1}, \ldots, y_{n-1}, y_n+\eta_n) - L(y_1, y_2, \ldots, y_n)}{\eta_n}, \quad (31)$$

or with respect to one of the first m variables, say y_1, as

$$\lim_{\eta_1=0} \frac{L(y_1+\eta_1+\omega_1, y_2+\omega_2, \ldots, y_m+\omega_m, y_{m+1}, \ldots, y_n) - L(y_1, y_2, \ldots, y_n)}{\eta_1}. \quad (32)$$

In the former case the derivative has the value

$$\sum_{i=1}^{m} \frac{\partial L}{\partial y_i} \frac{\partial y_i}{\partial y_n} + \frac{\partial L}{\partial y_n},$$

where the factors $\partial y_i / \partial y_n$ are the derivatives found from the solutions of equations (27) for the variables $y_1, y_2, \ldots, y_m$. It must vanish if the function L is to have a minimum for the set of values $y_1, y_2, \ldots, y_n$. In the latter case the derivative is always zero, as can be easily verified. In the succeeding section we will define the derivative of an infinite number of variables $y(x)$ $(x_1 \leqq x \leqq x_2)$ in an analogous way, and prove that it must vanish for a minimum.

It should be noticed that in forming the derivatives (31) and (32) we divide by the arbitrary increment given to the variable with respect to which

we are differentiating, and do not divide by any of the increments which are given to the variables $y_1, y_2, \ldots, y_m$ in order that the restricting equations (27) may be satisfied.

§ 3. *The Derivative of a Function of a Curve.*

The Lagrange problem of the calculus of variations is to find the condition that a curve C shall minimize the value of a definite integral

$$\int_{x_1}^{x_2} f(x, y, y', z_1, z_1', \ldots, z_m, z_m')\, dx$$

taken over the set of curves K defined in the last section. The first necessary condition is the vanishing of an expression which we find to be the derivative of the definite integral considered as a function of the infinity of variables defining the curve C. In this section we will define the derivative for a much more general class of functions and find some of its properties, and in the next section we will discuss its application to the Lagrange problem.

If a number $L(C)$ corresponds to each curve C of a set K, then $L(C)$ is said to be a function of the curve C, as C ranges over K.

If a number $L(C, x)$ corresponds to each point x of each curve C of a set K, then $L(C, x)$ is said to be a function of the curve C at the point x.

If two curves

$$C: \quad y = y(x) \qquad (x_1 \leqq x \leqq x_2),$$
$$\bar{C}: \quad y = \bar{y}(x) \qquad (x_1 \leqq x \leqq x_2)$$

are such that the inequalities

$$|\bar{y}(x) - y(x)| \leqq \varepsilon, \ |\bar{y}'(x) - y'(x)| \leqq \varepsilon, \ \ldots, \ |\bar{y}^{(r)}(x) - y^{(r)}(x)| \leqq \varepsilon \quad (x_1 \leqq x \leqq x_2)$$

are satisfied, then the curve $\bar{C}$ is said to be in the neighborhood $(\varepsilon)_r$ of the curve C.

Suppose C is a curve of class $C^{(r)}$, $r \geqq 2$, belonging to the set K. Take an arbitrary function $\eta(x)$ of class $C^{(r)}$ which vanishes identically everywhere excepting on the interval $\xi - \delta < x < \xi + \delta$, where it does not change its sign and is not identically zero, and which furthermore satisfies the inequalities

$$|\eta(x)| \leqq \delta, \quad |\eta'(x)| \leqq \delta, \quad |\eta''(x)| \leqq \delta.$$

Then choose a set of points $\xi_1, \xi_2, \ldots, \xi_m$ satisfying the conditions of the preceding section. According to the discussion there given, for a given ε the constant δ can be chosen such that a set of functions $\eta_1(x), \eta_2(x), \ldots, \eta_m(x)$, also satisfying the conditions of the preceding section, always exists, for which the curve

$$C_{\xi\epsilon}: \quad y = y(x) + \eta(x) + \sum_i \eta_i(x)$$

belongs to the set K.

If $L(C)$ is a function such that the limit

$$\lim_{\epsilon=0} \frac{L(C_{\xi\epsilon}) - L(C)}{\sigma} = L'(C, \xi; \xi_1, \xi_2, \ldots, \xi_m)$$

exists uniquely, where σ is defined by the equation

$$\sigma = \int_{\xi-\delta}^{\xi+\delta} \eta(x)\,dx,$$

then this limit is called the Volterra derivative of the function $L(C)$ at the point ξ, relative to the set of curves K and the set of points $\xi_1, \xi_2, \ldots, \xi_m$.

Under such circumstances, the Volterra derivative is said to be approached with order r.

If we form this derivative at a point of the set $\xi_1, \xi_2, \ldots, \xi_m$, say ξ_1, one admissible choice of the functions $\eta_i(x)$ is

$$\eta_1(x) = -\eta(x), \quad \eta_2(x) = \eta_3(x) = \ldots = \eta_m(x) = 0.$$

Then the curve $C_{\xi\epsilon}$ coincides with the curve C, and the Volterra derivative must vanish. Therefore, if the derivative exists at the points ξ_i, the equations

$$L'(C, \xi_i; \xi_1, \xi_2, \ldots, \xi_m) = 0$$

are satisfied.

When it will cause no ambiguity, we will let the expression $L'(C, \xi)$ represent the Volterra derivative, omitting the arguments $\xi_1, \xi_2, \ldots, \xi_m$.

We will always let C_ϵ represent a curve of class $C^{(r)}$ belonging to the set K, determined as in § 2. It is evident that every curve C_{ϵ_1} is also a curve C_ϵ, if $\epsilon_1 \leqq \epsilon$. $C_{\xi\epsilon}$ will always represent such a curve which also satisfies the conditions laid down in the definition of the Volterra derivative.

We assume that there is a neighborhood $(d)_r$ of the curve C which has the following property: For every $e>0$ there exists an $\epsilon>0$ such that if $\bar{C}$ and $\bar{C}_{\xi\epsilon}$ are curves of class $C^{(r)}$ belonging to the set K and in the neighborhood $(d)_r$ of C, and if ξ is any point on the interval $x_1 \leqq x \leqq x_2$, then the inequality

$$\left| \frac{L(\bar{C}_{\xi\epsilon}) - L(\bar{C})}{\sigma} - L'(\bar{C}, \xi) \right| \leqq e$$

is always satisfied. Under these circumstances the Volterra derivative is said to be approached uniformly with order r in the neighborhood $(d)_r$ of C.

We assume, further, that for every $e>0$ there is an $\epsilon>0$ such that if $\bar{C}$ is

any curve of class $C^{(r)}$ belonging to the set K in the neighborhood $(\varepsilon)_r$ of the curve C, then the inequality

$$|L'(\bar{C}, x) - L'(C, x)| \leqq e$$

is satisfied. That is, the Volterra derivative is continuous with order r at the curve C, with respect to the set of curves K.

We assume, finally, that the Volterra derivative $L'(C, x)$ is a continuous function of x on the interval $x_1 \leqq x \leqq x_2$, and since this interval is closed it is uniformly continuous.

Suppose $\eta(x, \alpha)$ is an arbitrary function of class $C^{(r)}$ in x which vanishes at the end-points $x = x_1$ and $x = x_2$ identically in α, and satisfies the following conditions. The functions $\eta(x, \alpha), \eta'(x, \alpha), \ldots, \eta^{(r)}(x, \alpha)$ all approach zero uniformly with respect to x when α approaches zero. The function $\eta(x, \alpha)$ has a derivative $\eta_\alpha(x, 0)$ with respect to α at the point $\alpha = 0$ which is continuous and approached uniformly with respect to x. If a small constant value is given to the parameter α, the function $\eta(x, \alpha)$ does not change its sign on the interval $x_1 \leqq x \leqq x_2$.

It follows from these assumptions that when an $\varepsilon > 0$ has been chosen it is always possible to restrict the value of α so that the functions $\eta(x, \alpha), \eta'(x, \alpha), \ldots, \eta^{(r)}(x, \alpha)$ will be less in absolute value than the δ determined in the preceding section, on the whole interval $x_1 \leqq x \leqq x_2$. Then we can determine the functions $\eta_1(x, \alpha), \eta_2(x, \alpha), \ldots, \eta_m(x, \alpha)$ such that the curve

$$C_{\varepsilon\alpha}: \quad y = y(x) + \eta(x, \alpha) + \sum_i \eta_i(x, \alpha)$$

shall belong to the set K, where $\eta_i(x, \alpha)$ satisfies the conditions for $\eta_i(x)$ in the preceding section.

The function $L(C_{\varepsilon\alpha})$ is not uniquely determined by ε and α since the curve $C_{\varepsilon\alpha}$ is not unique. Nevertheless the following theorem, which is referred to in the introduction as the generalized Volterra theorem, can be proved.

If the functions $L(C)$ and the curves C and $C_{\varepsilon\alpha}$ satisfy the assumptions given above, then the expression

$$\lim_{\varepsilon = 0} \frac{L(C_{\varepsilon\alpha}) - L(C)}{\alpha}$$

is unique, and its value is given by the equation

$$\lim_{\varepsilon = 0} \frac{L(C_{\varepsilon\alpha}) - L(C)}{\alpha} = \int_{x_1}^{x_2} L'(C, x) \eta_\alpha(x, 0)\, dx.$$

In particular, suppose $\eta(x, \alpha)$ is an admissible variation; that is, the curves

$$C_\alpha: \quad y = y(x) + \eta(x, \alpha)$$

belong to the set K. Then the functions $\eta_i(x,\alpha)$ are identically zero, and $L(C_\alpha)$ is a uniquely determined function of α, which has a derivative at $\alpha=0$ which is expressed by the equation

$$\left[\frac{dL(C_\alpha)}{d\alpha}\right]_{\alpha=0} = \int_{x_1}^{x_2} L'(C,x)\eta_\alpha(x,0)\,dx.$$

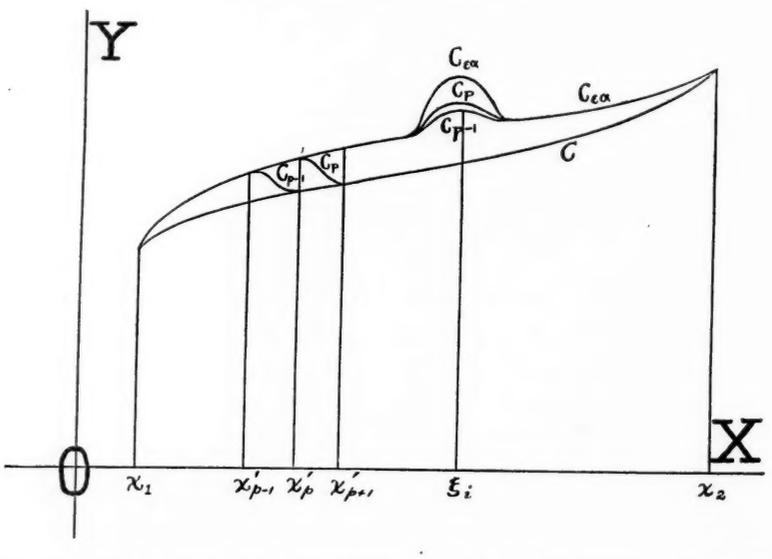
It should be noted that when ε approaches zero the parameter α approaches zero necessarily, and in such a way that every curve $C_{\varepsilon\alpha}$ belongs to the set K.

The theorem can be proved as follows: First choose an ε arbitrarily satisfying the conditions of § 2. Then determine δ, divide the interval $x_1 \leqq x \leqq x_2$ into n segments each of length δ, and call the end-points of the segments $x'_0, x'_1, \ldots, x'_n$, so that $x_1 = x'_0$ and $x_2 = x'_n$. Select a set of functions $\theta_1(x)$, $\theta_2(x), \ldots, \theta_{n-2}(x)$ of class $C^{(r)}$ which satisfy the conditions

$$\begin{aligned} \theta_p(x) &= 1 \qquad (x_1 \leqq x \leqq x'_p), \\ 0 < \theta_p(x) &< 1 \qquad (x'_p < x < x'_{p+1}), \\ \theta_p(x) &= 0 \qquad (x'_{p+1} \leqq x \leqq x_2), \end{aligned}$$

where p has the range $1, 2, \ldots, n-2$, and define $\theta_0(x)$ and $\theta_{n-1}(x)$ by the equations

$$\theta_0(x) \equiv 0, \quad \theta_{n-1}(x) \equiv 1.$$



It follows from the assumed properties of the function $\eta(x,\alpha)$ that for any choice of δ the value of α can be so restricted that the inequalities

$$|\theta_p(x)\eta(x,\alpha)| \leqq \delta, \quad \left|\frac{d}{dx}\{\theta_p(x)\eta(x,\alpha)\}\right| \leqq \delta, \quad \ldots, \quad \left|\frac{d^r}{dx^r}\{\theta_p(x)\eta(x,\alpha)\}\right| \leqq \delta$$

are satisfied on the ranges $x_1 \leqq x \leqq x_2$ and $p = 0, 1, 2, \ldots, n-1$.

Then, according to the principal theorem of § 2 we can choose the functions $\eta_{ip}(x,\alpha)$, where $i=1,2,\ldots,m$ and $p=1,2,\ldots,n-1$, in such a way that the curves

$$C_p:\quad y=y(x)+\theta_p(x)\eta(x,\alpha)+\sum_i \eta_{ip}(x,\alpha)$$

shall belong to the set K. It follows from the proof of that theorem that the function $\eta_{ip}(x,\alpha)$ can be made to differ from the function $\eta_{ip-1}(x,\alpha)$ only by a constant multiplier. Since the curve C_0 is the same as C, and C_{n-1} is the same as $C_{\varepsilon\alpha}$, it is evident that

$$\frac{L(C_{\varepsilon\alpha})-L(C)}{\alpha}=\sum_{p=1}^{n-1}\frac{L(C_p)-L(C_{p-1})}{\alpha}. \tag{33}$$

The curve C_p satisfies all the conditions imposed on the curve $C_{\xi\varepsilon}$ in the definition of the Volterra derivative when that derivative is taken for the curve C_{p-1} at the point x'_p. Since the Volterra derivative is assumed to be approached uniformly along every curve in a neighborhood $(d)_r$ of C, the equations

$$\frac{L(C_p)-L(C_{p-1})}{\alpha}=\frac{\sigma_p}{\alpha}[L'(C_{p-1},x'_p)+\lambda_p]\qquad(p=1,2,\ldots,n-1) \tag{34}$$

are satisfied, where the largest of the quantities $|\lambda_p|$ approaches zero with ε, and σ_p is defined by the equation

$$\sigma_p=\int_{x'_p}^{x'_{p+1}}\eta(x,\alpha)\,dx+\int_{x'_{p-1}}^{x'_p}(1-\theta_{p-1}(x))\eta(x,\alpha)\,dx-\int_{x'_p}^{x'_{p+1}}(1-\theta_p(x))\eta(x,\alpha)\,dx. \tag{35}$$

Since the derivative is continuous with respect to the set of curves K at the curve C, equations (34) may be written

$$\frac{L(C_p)-L(C_{p-1})}{\alpha}=\frac{\sigma_p}{\alpha}[L'(C,x'_p)+\mu_p]\qquad(p=1,2,\ldots,n-1), \tag{36}$$

where the largest of the quantities $|\mu_p|$ approaches zero with ε.

Since the ratio $\eta(x,\alpha)/\alpha$ approaches $\eta_a(x,0)$ uniformly as α approaches zero, we know that the expression

$$\lim_{\varepsilon=0}\frac{1}{\alpha}\int_{x_1}^{x_2}\eta(x,\alpha)\,dx=\int_{x_1}^{x_2}\eta_a(x,0)\,dx, \tag{37}$$

which is finite. The mean-value theorem reduces equations (35) to the equations

$$\begin{aligned}\sigma_p=\eta(x'''_p,\alpha)(x'_{p+1}-x'_p)+(1-\theta_{p-1}(x''_{p-1}))\eta(x''_{p-1},\alpha)(x'_p-x'_{p-1})\\ -(1-\theta_p(x'''_p))\eta(x'''_p,\alpha)(x'_{p+1}-x'_p),\end{aligned} \tag{38}$$

where x''_{p-1} is a point on the interval $x'_{p-1}<x<x'_p$, and x''_p and x'''_p are points

on the interval $x'_p<x<x'_{p+1}$. It follows without difficulty from equations (37) and (38) that the equation

$$\lim_{\varepsilon=0} \sum_{p=1}^{n-1} \mu_p \frac{\sigma_p}{\alpha} = 0 \tag{39}$$

is satisfied. Since the Volterra derivative is a uniformly continuous function of x, the equations

$$\left.\begin{aligned}
&\lim_{\varepsilon=0} \frac{1}{\alpha} \sum_{p=1}^{n-1} L'(C,x'_p)\,[\,(1-\theta_{p-1}(x''_{p-1}))\,\eta(x''_{p-1},\alpha)\,(x'_p-x'_{p-1}) \\
&\qquad\qquad -(1-\theta_p(x''_p))\,\eta(x''_p,\alpha)\,(x'_{p+1}-x'_p)\,] = \\
&\lim_{\varepsilon=0} \Big\{ \sum_{p=2}^{n-1} [L'(C,x_p)-L'(C,x'_{p-1})]\,(1-\theta_{p-1}(x''_{p-1}))\,\frac{\eta(x''_{p-1},\alpha)}{\alpha}\,(x'_p-x'_{p-1}) \\
&\qquad +L'(C,x'_1)\,(1-\theta_0(x''_0))\,\frac{\eta(x''_0,\alpha)}{\alpha}\,(x'_1-x'_0) \\
&\qquad -L'(C,x'_{n-1})\,(1-\theta_{n-1}(x''_{n-1})\,\frac{\eta(x''_{n-1},\alpha)}{\alpha}\,(x'_n-x'_{n-1})\Big\} = 0
\end{aligned}\right\} \tag{40}$$

are easily verified.

If equations (36) and (38) are substituted in (33), the limit taken as ε approaches zero, and then equations (39) and (40) are subtracted, we have the equation

$$\lim_{\varepsilon=0} \frac{L(C_{\varepsilon a})-L(C)}{\alpha} = \lim_{\varepsilon=0} \sum_{p=1}^{n-1} L'(C,x'_p)\,\frac{\eta(x'''_p,\alpha)}{\alpha}\,(x'_{p+1}-x'_p). \tag{41}$$

Since $\eta_a(x,0)$ is approached uniformly and is continuous, we can state the equation

$$\frac{\eta(x'''_p,\alpha)}{\alpha} = \eta_a(x'_p,0)+\nu_p, \tag{42}$$

where the largest of the quantities $|\nu_p|$ approaches zero with ε. If we now substitute equation (42) in (41), it easily reduces to the equation

$$\lim_{\varepsilon=0} \frac{L(C_{\varepsilon a})-L(C)}{\alpha} = \int_{x_1}^{x_2} L'(C,x)\,\eta_a(x,0)\,dx. \tag{43}$$

We will now consider the particular case where $\eta(x,\alpha)$ is a family of admissible variations. Since in the principal theorem of § 2 the multipliers $\alpha_1, \alpha_2, \ldots, \alpha_m$ were determined uniquely, and since in this case $\alpha_1=\alpha_2=\ldots=0$ is a correct determination of them, it is the only determination. Then the functions $\eta_i(x,\alpha)$ must vanish identically. Therefore when α is given, the curve C_a is determined uniquely, and equation (43) becomes

$$\left[\frac{dL(C_a)}{d\alpha}\right]_{a=0} = \int_{x_1}^{x_2} L'(C,x)\,\eta_a(x,0)\,dx.$$

It follows from the generalized Volterra theorem and the fundamental lemma of the calculus of variations* that the identical vanishing of the Volterra derivative is a necessary condition that the curve C furnish a minimum for the value of the function $L(C)$ taken over the set of curves K, provided $L(C)$ is such a function that its Volterra derivative is continuous and approached uniformly with any finite order with respect to the set of curves K, and continuous also with respect to x, on the interval $x_1 \leqq x \leqq x_2$.

§ 4. *Applications to Functions Which Are Defined by Definite Integrals.*

In this section we will let the function $L(C)$ be a definite integral and find its Volterra derivative. It will appear that this derivative is the expression whose vanishing constitutes the first necessary condition for the curve C to minimize the value of the definite integral considered.

Define $L(C)$ by the equation

$$L(C) = \int_{x_1}^{x_2} f(x, y, y', z_1, z_1', \ldots, z_m, z_m')\,dx,$$

where the function f is assumed to be of class C'' in the $(2m+3)$-dimensional space of points $(x, y, y', z_1, \ldots, z_m')$ in the neighborhood of the values defined by the equations (13) and (14). The curve C is assumed to be a normal curve of class C''. Since the functions $f, g_1, g_2, \ldots, g_m$, together with their first and second partial derivatives, are continuous in the neighborhood mentioned above, they are uniformly continuous in every closed region included in that neighborhood.

Choose once for all a set of points $\xi_1, \xi_2, \ldots, \xi_m$ satisfying the conditions of § 2 with respect to the curve C. Then select a function $\eta(x)$ of class C'' which vanishes everywhere excepting on the interval $\xi-\delta < x < \xi+\delta$, where it does not change its sign and is not identically zero, and which satisfies inequalities (25). If ε is given, the value of δ can be restricted so that a set of functions $\eta_1(x), \eta_2(x), \ldots, \eta_m(x)$ satisfying the conditions of § 2 can be found, for which the curve

$$C_{\xi\varepsilon}: \quad y = y(x) + \omega(x)$$

belongs to the set K, where the function $\omega(x)$ is defined by the equation

$$\omega(x) = \eta(x) + \sum_i \eta_i(x).$$

We must compute the Volterra derivative from the equation

$$L'(C, \xi) = \lim_{\varepsilon=0} \frac{L(C_{\xi\varepsilon}) - L(C)}{\sigma},$$

* Bolza: *loc. cit.*, p. 25.

where as usual

$$\sigma = \int_{\xi-\delta}^{\xi+\delta} \eta(x)\,dx.$$

Let $\zeta_i(x)$ represent the difference between the value of the function $z_i(x)$ taken along the curve $C_{\xi\varepsilon}$ and its value taken along the curve C. The expression $L(C_{\xi\varepsilon}) - L(C)$ can be expanded by Taylor's formula,* giving the equation

$$L(C_{\xi\varepsilon}) - L(C) = \int_0^1 \int_{x_1}^{x_2} \left\{ f_y \omega + f_{y'} \omega' + \sum_j (f_{z_j} \zeta_j + f_{z'_j} \zeta'_j) \right\} dx\,du, \qquad (44)$$

where the arguments of the partial derivatives of f are x, $y + u\omega$, $y' + u\omega'$, $z_1 + u\zeta_1$, ..., $z'_m + u\zeta'_m$. Since the integrand is uniformly continuous in both of the arguments x and u, we can change the order of integration whenever it is convenient. It follows immediately from equations (12) that the equations

$$\zeta'_i - [g_i(x, y+\omega, y'+\omega', \ldots, z_m+\zeta_m) - g_i(x, y, \ldots, z_m)] = 0$$

are satisfied for all values of x. Taylor's formula reduces them to the equations

$$\zeta'_i(x) - \int_0^1 \left\{ \frac{\partial g_i}{\partial y} \omega + \frac{\partial g_i}{\partial y'} \omega' + \Sigma \frac{\partial g_i}{\partial z_j} \zeta_j \right\} du, \qquad (45)$$

where the arguments of the partial derivatives of g_i are x, $y + u\omega$, ..., $z_m + u\zeta_m$. If equations (45) are multiplied by the undetermined functions $\lambda_i(x)$ of class C', then summed with respect to i, integrated between the limits x_1 and x_2, and added to equation (44), they give us the equation

$$L(C_{\xi\varepsilon}) - L(C) = \int_{x_1}^{x_2} \int_0^1 \left\{ F_y \omega + F_{y'} \omega' + \sum_j (F_{z_j} \zeta_j + F_{z'_j} \zeta'_j) \right\} du\,dx, \qquad (46)$$

where the function F is defined by the equation

$$F = f + \sum_i \lambda_i (z'_i - g_i), \qquad (47)$$

with the arguments the same as in equations (44) and (45).

Since all the functions in the integrand of equation (46) are uniformly continuous, the terms containing ω' and ζ'_j can be integrated by parts. The functions $\omega(x)$, $\zeta_1(x)$, ..., $\zeta_m(x)$ all vanish at both end-points, and equation (46) becomes

$$L(C_{\xi\varepsilon}) - L(C) = \int_{x_1}^{x_2} \left\{ \omega \int_0^1 (F_y - F'_{y'})\,du + \sum_j \zeta_j \int_0^1 (F_{z_j} - F'_{z'_j})\,du \right\} dx, \qquad (48)$$

where the accent always signifies the total derivative with respect to x. If the order of integration is changed in the first part of the right member of equation

* Jordan: "Cours D'Analyse," Vol. I, p. 247, equation (2).

(48), the function $\omega(x)$ replaced by $\eta(x)+\sum_j \eta_i(x)$, and the mean-value theorem applied, it reduces to the equation

$$L(C_{\xi\epsilon})-L(C)=\sigma\int_0^1 (F_y-F'_{y'})_{x=\xi'}\,du+\sum_j \sigma_j\int_0^1 (F_y-F'_{y'})_{x=\xi'_j}\,du$$
$$+\sum_j\int_{x_1}^{x_2}\zeta_j\int_0^1 (F_{z_j}-F'_{z'_j})\,du\,dx, \qquad (49)$$

where the points ξ' and ξ'_j are on the intervals $\xi-\delta<x<\xi+\delta$ and $\xi_j-\epsilon<x<\xi_j+\epsilon$ respectively, and the functions σ_j are defined by the equations

$$\sigma_j=\int_{\xi_j-\epsilon}^{\xi_j+\epsilon}\eta_j(x)\,dx.$$

We will determine the functions $\lambda_1(x), \lambda_2(x), \ldots, \lambda_m(x)$ by the differential equations

$$\int_0^1 (F_{z_j}-F'_{z'_j})\,du=0,$$

which may be written in the form

$$\lambda'_j+\sum_i \lambda_i\int_0^1 \frac{\partial g_i}{\partial z_j}\,du=\int_0^1 (f_{z_j}-f'_{z'_j})\,du. \qquad (50)$$

The constants of integration will be determined by the equations

$$\int_0^1 \{F_y-F'_{y'}\}_{x=\xi'_j}\,du=0,$$

which may be written in the form

$$\sum_i \left\{\lambda_i(x)\int_0^1 \frac{\partial g_i}{\partial y}\,du-\frac{d}{dx}\left(\lambda_i\int_0^1 \frac{\partial g_i}{\partial y'}\,du\right)\right\}_{x=\xi'_j}=\int_0^1 (f_y-f'_{y'})_{x=\xi'_j}\,du. \qquad (51)$$

This reduces equation (49) to the form

$$L(C_{\xi\epsilon})-L(C)=\sigma\int_0^1 (F_y-F'_{y'})_{x=\xi'}\,du. \qquad (52)$$

The existence theorem for a solution of a set of ordinary differential equations implies that the functions $\lambda_i(x)$ are of class C', if we can show that the constants of integration are finite and determined uniquely. Consider a particular solution $(\mu_1(x), \mu_2(x), \ldots, \mu_m(x))$ of equations (50) having an arbitrarily fixed set of initial values $\mu_{12}, \mu_{22}, \ldots, \mu_{m2}$ at $x=x_2$, and the fundamental system $(\lambda_{h1}, \lambda_{h2}, \ldots, \lambda_{hm})$ of solutions of the homogeneous equations

$$\lambda'_j+\sum_i \lambda_i\int_0^1 \frac{\partial g_i}{\partial z_j}\,du=0, \qquad (53)$$

with the initial values

$$\lambda_{hh}(x_2)=1, \quad \lambda_{hj}(x)=0 \qquad (h\neq j). \qquad (18)$$

The functions μ_i and λ_{hi} depend on ω, but, according to the first theorem of § 1,

as ω approaches zero with ε, they approach as limiting values the corresponding solutions of the systems of equations (56) and (16) which are derived from (50) and (53) by putting $\omega=0$, $\varepsilon=0$. The general solution of equations (50) is expressed by the equation

$$\lambda_j(x) = \mu_j(x) + \sum_h c_h \lambda_{hj}(x). \tag{54}$$

The constants of integration c_h are determined by equations (51) which may be written in the form

$$\sum_h c_h \sum_i \left\{ \lambda_{hi} \int_0^1 \frac{\partial g_i}{\partial y} du - \frac{d}{dx}\left(\lambda_{hi} \int_0^1 \frac{\partial g_i}{\partial y'} du\right)\right\}_{x=\xi'_j} =$$
$$\left\{\int_0^1 (f_y - f'_{y'})\, du - \sum_i \left[\mu_i \int_0^1 \frac{\partial g_i}{\partial y} du - \frac{d}{dx}\left(\mu_i \int_0^1 \frac{\partial g_i}{\partial y'} du\right)\right]\right\}_{x=\xi'_j}.$$

According to the first theorem of § 1 the determinant of the coefficients of the constants $c_1, c_2, \ldots, c_m$ reduces uniformly to the determinant (22) as ε approaches zero. Hence for a sufficiently small value of ε it is distinct from zero, and the constants $c_1, c_1, \ldots, c_m$ are determined uniquely. Evidently as ε approaches zero the constants themselves approach definite limiting values as is indicated by the theorem of § 1 mentioned above. This makes the functions $\lambda_j(x)$ approach uniformly the functions obtained by putting $\varepsilon=0$, $\omega=0$ in the equations determining them, as ε approaches zero.

If we now divide equation (52) by σ and let ε approach zero, we find the Volterra derivative to be

$$L'(C,\xi) = [F_y(x, y, y', z_1, \ldots, z'_m) - F'_{y'}(x, y, y', z_1, \ldots, z'_m)]_{x=\xi}, \tag{55}$$

where the functions $\lambda_i(x)$ are the solutions of the equations

$$F_{z_j} - F'_{z'_j} = -\lambda'_j - \sum_i \frac{\partial g_i}{\partial g_j}\lambda_i + f_{z_j} - f'_{z'_j} = 0 \tag{56}$$

whose constants of integration are determined by the equation

$$[F_y - F'_{y'}]_{x=\xi_j} = \left[f_y - f'_{y'} - \sum_i \left\{\lambda_i \frac{\partial g_i}{\partial y} - \frac{d}{dx}\left(\lambda_i \frac{\partial g_i}{\partial y'}\right)\right\}\right]_{x=\xi_j} = 0. \tag{57}$$

The well-known existence theorem for the solution of a set of ordinary differential equations implies that the functions $\lambda_i(x)$ and their derivatives are continuous. Since all the other functions involved in $L'(C,x)$ are also continuous by hypothesis, it is evident that $L'(C,x)$ is a continuous function of x over the range $x_1 \leqq x \leqq x_2$. It remains to be proved that it is approached uniformly with respect to the set of curves K in the neighborhood of the curve C, and is continuous in the argument C at the initial curve under consideration.

The condition which must be satisfied by the set of points $\xi_1, \xi_2, \ldots, \xi_m$ is expressed by the inequality (22). If we replace the curve C by any other curve

$$\bar{C}: \quad y = \bar{y}(x)$$

belonging to the set K and in a neighborhood $(\varepsilon)_2$ of the curve C, and choose ε sufficiently small, the resulting change in each element of the determinant (22) can be made less than any previously assigned positive constant, as is easily proved by means of the first theorem of § 1. We can therefore take ε so small that this determinant is distinct from zero for every admissible curve in the neighborhood $(\varepsilon)_2$ of C. That is, the set of points $\xi_1, \xi_2, \ldots, \xi_m$ is effective for every curve $\bar{C}$ in that neighborhood.

Since the first and second partial derivatives of f and g_i are uniformly continuous in a closed region including the curve C, the argument of this section remains valid if $\bar{y}(x)$ is substituted for $y(x)$ everywhere. Furthermore it can be shown that the functions $\lambda_i(x)$ approach the limiting values found by putting ε and consequently $\omega(x)$ equal to zero in equations (50) and (51), uniformly with respect to the curves of the set K in the neigborhood $(\varepsilon)_2$ of C. This assures us that the Volterra derivative is approached uniformly in the neighborhood $(\varepsilon)_2$ of the curve C.

To show that the Volterra derivative is continuous with respect to the set of curves K at the curve C we need simply state that if the quantities

$$|\bar{y}(x)-y(x)|, \quad |\bar{y}'(x)-y'(x)|, \quad |\bar{y}''(x)-y''(x)| \qquad (x_1 \leqq x \leqq x_2)$$

are taken sufficiently small, the difference between the values of $\zeta_i(x)$ and $\lambda_i(x)$ for the curve $\bar{C}$ and their values for the curve C can be made arbitrarily small uniformly. Therefore the value of the expression

$$|L'(\bar{C}, x)-L'(C, x)| \qquad (x_1 \leqq x \leqq x_2)$$

can be made less than an arbitrarily small constant by restricting the value of ε properly.

The hypothesis of the generalized Volterra theorem is completely satisfied for $r=2$, and therefore the Volterra derivative whose value we have just found must vanish identically if the curve C minimizes the value of the function $L(C)$.

The expression which we have just found for the Volterra derivative has the same form as the left member of Euler's equation for the Lagrange problem of the calculus of variations.* The differential equations which determine the functions $\lambda_1(x), \lambda_2(x), \ldots, \lambda_m(x)$ in that problem† are our equations (56).

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* Bolza: *loc. cit.*, p. 563, equations (65).

† Bolza: *loc. cit.*, p. 562, equations (62).

The derivative is not determined uniquely by the arguments C, x since it is a function of the arguments $\xi_1, \xi_2, \ldots, \xi_m$ as well. The constants of integration involved in the functions $\lambda_i(x)$ were determined uniquely by the condition that the expression $F_y - F'_{y'}$ should vanish at the points $\xi_1, \xi_2, \ldots, \xi_m$, as expressed in equations (57). It is evident, then, that there is at most one set of constants which can make this expression vanish identically, as it must in case the curve C minimizes the value of the function $L(C)$. In that case the derivative is independent of the arguments $\xi_1, \xi_2, \ldots, \xi_m$. The constants of integration for the solution of equations (56) have been determined by different methods by Lagrange, Meyer, and perhaps others.‡ Since all of these methods must make the expression which I have called the Volterra derivative vanish identically along a minimizing curve, and in particular at the points $\xi_1, \xi_2, \ldots, \xi_m$, it is evident that the solutions of equations (56) which enter into the function F, in case of a minimum, must be the same, whatever method is used to determine the constants of integration.

‡ Bolza: *loc. cit.*, pp. 563 and 566.

*On Differential Invariants.**

By James Byrnie Shaw.

I. *Introduction.*

This paper is concerned with the expression of certain differential operators and resulting differential parameters in vector form. The chief properties of these parameters are thus shown to be due to the operations involved. It generalizes the Hamiltonian ∇ for space of n dimensions, which may be either flat or curved. From one point of view it has to do with the vector algebra in which the units chosen are themselves variable. The vector algebra used is of the most simple type, the only "product" appearing in the paper being the so-called inner product. A general linear vector operator of one-many type appears and is a vital part of the treatment; however, the formulæ involved are simple. The number of dimensions is general and may in parts of the paper even be infinite.

Memoirs most closely related to this paper are due to Ricci,† Ricci and Levi-Civita,‡ and Maschke§. A paper of Ingold's‖ should also be consulted in connection with references to Maschke. The vectors of Ingold's paper will be noticed in the paragraph dealing with the symbolic invariants. The works of Lamé, Christoffel, Beltrami, Bianchi and Darboux should be consulted. Recent memoirs of Bates generalize portions of the theory.

It ought to be evident from a consideration of the results here developed that the method of attack on the problems concerned is natural, free from artifice, and should result in a more penetrating insight into the real nature of these problems.

II. *Vector Algebra of N Dimensions.*¶

1. We shall use the expression called a *vector* in the sense that it represents an ordered set of coordinates, the coordinates being distinguished by the unit attached in each case. Thus we write the vector

$$\rho = x_1 e_1 + \ldots . + x_n e_n ,$$

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* Read in part before the Chicago Section of the American Mathematical Society, December 31, 1911.

† *Bulletin des Sciences Mathématiques*, (2), 16 (1892), pp. 167–189.

‡ *Mathematische Annalen*, 54 (1900), pp. 125–201.

§ "Present Problems of Algebra and Analysis," *Congress of Arts and Sciences*, St. Louis, 1904, Vol. I, pp. 518–530.

‖ *Transactions of the American Mathematical Society*, 11 (1910), pp. 449–474.

¶ Shaw, *Synopsis of Linear Associative Algebra*, pp. 10–15, 32–34.

which enables us to speak of the variables $x_1, \ldots, x_n$, so as to distinguish between them. Two vectors $\rho=\Sigma x_i e_i$ and $\sigma=\Sigma y_i e_i$ are equal if we have the n equations

$$x_i=y_i, \qquad i=1, \ldots, n.$$

2. We define the expression which may be called the inner product of these two, by the equation

$$I\cdot\rho\sigma=x_1y_1+\ldots+x_ny_n.$$

It is obvious that this expression is linear and distributive as to ρ or σ. It is also commutative; that is, $I\cdot\rho\sigma=I\cdot\sigma\rho$. In case $I\cdot\rho\sigma=0$, we shall call ρ and σ *orthogonal.*

3. We will represent by $A\cdot()\,A\cdot\alpha\beta$ the linear vector operator $\alpha I\cdot\beta()-\beta I\cdot\alpha()$ or

$$A\cdot()A\cdot\alpha\beta=\begin{vmatrix}\alpha & \beta\\ I\cdot\alpha() & I\cdot\beta()\end{vmatrix},$$

and by $A\cdot\rho A\cdot\alpha\beta$ the expression

$$\begin{vmatrix}\alpha & \beta\\ I\cdot\alpha\rho & I\cdot\beta\rho\end{vmatrix}.$$

It is obvious that this expression is alternating, changes sign if α and β are interchanged, and vanishes if they are equal or if one is a multiple of the other. Likewise, we use a general operator defined by the form

$$A\cdot()\ldots()A\cdot\beta_1\beta_2\ldots\beta_m=\begin{vmatrix}\beta_1 & \beta_2 & \ldots. & \beta_m\\ I\cdot\beta_1() & I\cdot\beta_2() & \ldots. & I\cdot\beta_m()\\ I\cdot\beta_1() & I\cdot\beta_2() & \ldots. & I\cdot\beta_m()\\ \ldots. & \ldots. & \ldots. & \ldots.\\ I\cdot\beta_1() & I\cdot\beta_2() & \ldots. & I\cdot\beta_m()\end{vmatrix},$$

where the different lines of operating symbols act on a set of $m-1$ vectors given in order. We have thus the vector

$$A\cdot\alpha_1\alpha_2\ldots\alpha_{m-1}\;A\cdot\beta_1\beta_2\ldots\beta_m=\begin{vmatrix}\beta_1 & \ldots\ldots & \beta_m\\ I\cdot\beta_1\alpha_1 & \ldots\ldots & I\cdot\beta_m\alpha_1\\ \ldots\ldots & \ldots\ldots & \ldots\ldots\\ I\cdot\beta_1\alpha_{m-1} & \ldots. & I\cdot\beta_m\alpha_{m-1}\end{vmatrix}.$$

It is evident that this vector expression is alternating in the vectors α as well as in the vectors β. It is distributive as to either set. It vanishes if either set are linearly connected among themselves, or if any α is orthogonal to all the vectors β. It is a vector that is linearly expressed in terms of the vectors β. We shall speak of a vector that is linearly expressible in terms of other

linearly independent vectors as co-regional with them. Thus the expression above is co-regional with the vectors β. Let us represent this expression temporarily by σ. Then if we operate on σ by $I\cdot\alpha_i()$, where i runs from 1 to $m-1$, we see that in every case the determinant has two lines alike, and therefore vanishes. That is to say, σ is orthogonal to each of the vectors α. We have here, then, a vector extension of the quaternion form $V\alpha V\beta\gamma$ which is orthogonal to α and in the plane of β and γ.

4. If we operate on σ by the operator $I\cdot\alpha()$ we have

$$I\cdot\alpha A\cdot\alpha_1\alpha_2\ldots.\alpha_{m-1}A\cdot\beta_1\beta_2\ldots.\beta_m=\begin{vmatrix} I\cdot\alpha\beta_1\ldots.\ldots.I\cdot\alpha\beta_m \\ \ldots.\ldots.\ldots.\ldots.\ldots. \\ I\cdot\alpha_{m-1}\beta_1\ldots.I\cdot\alpha_{m-1}\beta_m \end{vmatrix}.$$

This form, however, involves all the vectors $\alpha, \alpha_1, \alpha_2, \ldots., \alpha_{m-1}$, similarly, and is alternating in all of them, so that we will write it in the form (defining thus the expression)

$$I\cdot A\cdot\alpha\alpha_1\alpha_2\ldots.\alpha_{m-1}A\cdot\beta_1\beta_2\ldots.\beta_m,$$

and in this form the alternation under the A in either case is more evident. This expression vanishes if either set of vectors, the α's or the β's, are linearly connected among themselves, or if any one of either set is orthogonal to all of the other set. If the vectors reduce to the units e, we have

$$I\cdot A\cdot e_{i_1}\ldots.e_{i_m}A\cdot e_{j_1}\ldots.e_{j_m}=0 \text{ or } \pm1$$

according as the subscripts i and j contain members not common to both, or as the two sets are the same subscripts in different order. If the sets agree, the sign is + or — according as the total number of inversions of order in the subscripts is even or odd.

It is obvious that we can expand $A\cdot\alpha_1\alpha_2\ldots.\alpha_{m-1}A\cdot\beta_1\beta_2\ldots.\beta_m$ in the forms

$$\Sigma\beta_1 I\cdot A\cdot\alpha_1\alpha_2\ldots.\alpha_{m-1}A\cdot\beta_2\ldots.\beta_m,$$
$$\Sigma A\cdot\alpha_1 A\cdot\beta_1\beta_2 I\cdot A\cdot\alpha_2\ldots.\alpha_{m-1}A\cdot\beta_3\ldots.\beta_m,$$
$$\Sigma A\cdot\alpha_1\alpha_2 A\cdot\beta_1\beta_2\beta_3 I\cdot A\cdot\alpha_3\ldots.\alpha_{m-1}A\cdot\beta_4\ldots.\beta_m,$$
$$\ldots.\ldots.\ldots.\ldots.\ldots.\ldots.\ldots.\ldots.\ldots.\ldots.,$$
$$\Sigma A\cdot\alpha_1\ldots.\alpha_{m-2}A\cdot\beta_1\ldots.\beta_{m-1}I\cdot\alpha_{m-1}\beta_m,$$

where the Σ in each case signifies that the subscripts on the β's are to be permuted in every possible way with a change of sign for every inversion, according to the usual determinant rule of Laplace's expansion.

5. From the form $A\cdot\alpha_1\alpha_2\ldots.\alpha_{m-1}A\cdot\beta_1\beta_2\ldots.\beta_{m-1}\rho$ we see that we have, by expanding in the first manner above and transposing,

$$(-)^m\rho I\cdot A\cdot\alpha_1\ldots.\alpha_{m-1}A\cdot\beta_1\beta_2\ldots.\beta_{m-1}$$
$$=\Sigma\beta_1 I\cdot A\cdot\alpha_1\ldots.\alpha_{m-1}A\cdot\beta_2\ldots.\beta_{m-1}\rho-A\cdot\alpha_1\ldots.\alpha_{m-1}A\cdot\beta_1\ldots.\beta_{m-1}\rho.$$

In this form the Σ signifies that the subscripts on the β's are to be permuted with change of sign as above. This formula enables us to express ρ uniquely in terms of two components, one co-regional with the β's, the other orthogonal to all the α's. If the regions of the α's and the β's coincide we have ρ resolved in this region and orthogonal to it. If ρ also lies co-regional with the β's the last term vanishes. If ρ is orthogonal to all the α's, all the terms vanish but the last. These forms are very useful.

6. If we operate on each side of the last equation with $I\cdot\sigma()$, where σ is any vector, we have

$$(-)^m I\cdot\sigma\rho A\cdot\alpha_1\ldots.\alpha_{m-1}A\cdot\beta_1\ldots.\beta_{m-1}$$
$$=\Sigma I\cdot\sigma\beta_1 IA\cdot\alpha_1\ldots.\alpha_{m-1}A\cdot\beta_2\ldots.\beta_{m-1}\rho-IA\sigma\alpha_1\ldots.\alpha_{m-1}A\beta_1\ldots.\beta_{m-1}\rho,$$

from which we derive, since ρ is any vector, the important formula

$$(-)^m\sigma I\cdot A\cdot\beta_1\ldots.\beta_{m-1}A\cdot\alpha_1\ldots.\alpha_{m-1}$$
$$=\Sigma A\cdot\beta_2\ldots.\beta_{m-1}A\cdot\alpha_1\ldots.\alpha_{m-1}I\cdot\beta_1\sigma-A\cdot\beta_1\ldots.\beta_{m-1}A\cdot\alpha_1\ldots.\alpha_{m-1}\sigma.$$

This formula resolves σ into a component co-regional with the α's and a component orthogonal to all the β's. If σ is co-regional with the α's the last term vanishes. If it is orthogonal to all the β's all the terms vanish but the last.

It should be noticed that in these two formulæ the vectors α and β must have a common region of order $m-1$, in order that the coefficient of ρ or σ may not vanish.

7. Let us examine now the linear operator $\phi=\alpha_1 I\beta_1+\alpha_2 I\beta_2+\ldots+\alpha_m I\beta_m$. The transverse of this operator we will designate by ϕ', where $\phi'=\beta_1 I\alpha_1+\beta_2 I\alpha_2+\ldots.+\beta_m I\alpha_m$. The *first scalar invariant* of this operator is $m_1=I\alpha_1\beta_1+I\alpha_2\beta_2+\ldots.+I\alpha_m\beta_m$. We will define the related operators given below.

$$\chi_1=\phi'-m_1=A\alpha_1 A\beta_1()+A\alpha_1 A\beta_2()+\ldots.+A\alpha_m A\beta_m().$$

The *second scalar invariant* of ϕ is then the first invariant of $\phi'\chi_1$ divided by 2!,

$$m_2=m_1(\phi'\chi_1)/2!=IA\alpha_1\alpha_2 A\beta_2\beta_1+IA\alpha_1\alpha_3 A\beta_3\beta_1+\ldots.+IA\alpha_2\alpha_3 A\beta_3\beta_2+\ldots.,$$
$$\chi_2=\phi'\chi_1-m_2=A\alpha_1\alpha_2 A\beta_1\beta_2()+A\alpha_1\alpha_3 A\beta_1\beta_3()+\ldots.+A\alpha_2\alpha_3 A\beta_2\beta_3()+\ldots..$$

The *third scalar invariant* follows:

$$m_3=m_1(\phi'\chi_2)/3!=IA\alpha_1\alpha_2\alpha_3 A\beta_1\beta_2\beta_3+\ldots.,$$
$$\chi_3=\phi'\chi_2-m_3=A\alpha_1\alpha_2\alpha_3 A\beta_1\beta_2\beta_3()+\ldots..$$

The other scalar invariants of ϕ and the operators χ are easily written down in a similar form. Evidently the last one would be the invariant m_m, and the last χ would be χ_m. We may therefore look upon the form $A\alpha_1\ldots.\alpha_{m-1}A\beta_1\ldots.\beta_m$ as the form χ_{m-1} for $\phi=\alpha_1 I\beta_1+\ldots.+\alpha_{m-1}I\beta_{m-1}$, which has then operated upon β_m; that is, $\chi_{m-1}\beta_m$. These forms are useful later.

8. We notice next certain symbolic forms which are abbreviations for longer forms.* If $Q(\alpha, \beta)$ is any expression linear and homogeneous in α and β, then we shall designate by $Q(\zeta, \zeta)$ the expression $\Sigma Q(e_i, e_i)$, where $i=1, 2, \ldots, n$, the number of independent units concerned being n and the units e being orthogonal each to each. This expression will also be defined later in a different manner. Different forms of Q lead to a number of useful formulæ. Thus we have with no difficulty

$$\rho=\zeta I\zeta\rho, \quad I\zeta\zeta=n, \quad A\zeta A\rho\zeta=(n-1)\rho, \quad IA\zeta\lambda A\zeta\mu=I\lambda\mu(n-1),$$
$$A\zeta\lambda_1 A\zeta\mu_1\mu_2=-(n-2)A\lambda_1 A\mu_1\mu_2,$$
$$IA\zeta\lambda_1\ldots\lambda_s A\zeta\mu_1\ldots\mu_s=(n-s)IA\lambda_1\ldots\lambda_s A\mu_1\ldots\mu_s,$$
$$A\zeta\lambda_1\ldots\lambda_s A\zeta\mu_1\ldots\mu_{s+1}=(n-s-1)A\lambda_1\ldots\lambda_s A\mu_1\ldots\mu_{s+1}, \quad \text{etc.}$$

By accenting the symbols ζ we may use several pairs.† Thus

$$I\zeta_1\zeta_2 I\zeta_1\zeta_2=n, \quad IA\zeta_1\zeta_2 A\zeta_1\zeta_2=n(n-1), \quad A\zeta_1\zeta_2 A\rho\zeta_1\zeta_2=(n-2)(n-1)\rho,$$
$$IA\zeta_1\zeta_2\lambda_1 A\zeta_1\zeta_2\mu_1=(n-2)(n-1)I\lambda_1\mu_1,$$
$$A\zeta_1\zeta_2\lambda_1 A\zeta_1\zeta_2\mu_1\mu_2=(n-3)(n-2)A\lambda_1 A\mu_1\mu_2, \quad \text{etc.}$$

In general we have the following reduction formulæ for these symbols:

$$IA\zeta_1\zeta_2\ldots\zeta_s A\zeta_1\zeta_2\ldots\zeta_s=n!/(n-s)!,$$
$$IA\zeta_1\ldots\zeta_t\lambda_1\ldots\lambda_s A\zeta_1\ldots\zeta_t\mu_1\ldots\mu_s=(n-s)!/(n-s-t)!\cdot IA\lambda_1\ldots\lambda_s A\mu_1\ldots\mu_s.$$

If $s+t$ exceeds n these expressions vanish identically. If $s+t=n$ we have only the numerator on the right.

$$A\zeta_1\ldots\zeta_t\lambda_1\ldots\lambda_{s-1}A\zeta_1\ldots\zeta_t\mu_1\ldots\mu_s$$
$$=(-)^t(n-s)!/(n-s-t)!\cdot A\lambda_1\ldots\lambda_{s-1}A\mu_1\ldots\mu_s.$$

III. *Region of Order N—1 in Region of Order N.*

1. Let ρ be a vector dependent upon $n-1$ independent parameters, $u_1, u_2, \ldots, u_{n-1}$. That is,

$$\rho=\rho(u_1, \ldots, u_{n-1}).$$

If now we let these parameters vary, and if we assume that the function ρ is differentiable, we have

$$d\rho=\rho_1 du_1+\rho_2 du_2+\ldots+\rho_{n-1}du_{n-1},$$

where ρ_i is the derivative $\partial\rho/\partial u_i$.

2. We shall write as an abbreviation for the orthogonal of $n-1$ vectors

$$(-1)^{i-1}A\alpha_1\alpha_2\ldots\alpha_{i-1}\alpha_{i+1}\ldots\alpha_n Ae_1e_2\ldots e_n/I\cdot A\alpha_1\ldots\alpha_n Ae_1\ldots e_n$$

the vector α_i'. It follows at once that $I\alpha_i\alpha_i'=1$, but $I\alpha_i\alpha_j'=0$ if $i\neq j$.

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*Due to A. McAulay.

†Triple ζ forms, etc., may be used for cubic differential forms, etc.

In case the $n-1$ vectors are $\rho_1, \ldots, \rho_{n-1}$, we shall write ν instead of ρ'_n. The vector ν is thus the normal in the space of n dimensions to the space or region of $n-1$ dimensions. As the parameters are all essential by hypothesis, ν cannot vanish. The vectors $\rho_1, \ldots, \rho_{n-1}$ will be spoken of as tangent to the region. In connection with $\rho_1, \ldots, \rho_{n-1}$ we shall use ρ'_i to represent the orthogonal in the region of $n-1$ dimensions to $n-2$ vectors,

$$\frac{(-1)^{i-1}A\rho_1 \cdot \ldots \cdot \rho_{i-1}\rho_{i+1} \cdot \ldots \cdot \rho_{n-1}A\rho_1 \cdot \ldots \cdot \rho_{n-1}}{I\nu\nu}.$$

We have at once $I\rho_i\rho'_i=1$, $I\rho_i\rho'_j=0$ if $i \neq j$. It is evident that we may expand any vector τ linearly expressible in $\rho_1, \ldots, \rho_{n-1}$ in either of the forms $\tau=\Sigma\rho_i I\rho'_i\tau=\Sigma\rho'_i I\rho_i\tau$.

3. The expressions $I\rho_i\rho_j$, where $i, j=1, 2, \ldots, n-1$, are called the *fundamental quantities of first order*. They may be represented also by a_{ij}. We see that

$$Id\rho d\rho=\Sigma I\rho_i\rho_j du_i du_j, \quad \text{and} \quad I\nu\nu=|a_{ij}|.$$

The expression $Id\rho d\rho$ is called the *first fundamental form*. The partial differentiation of a_{ij} as to u_k gives $I\rho_{ik}\rho_j+I\rho_i\rho_{jk}$. The expression $I\rho_i\rho_{jk}$ is the Christoffel symbol $\left[\begin{matrix} jk \\ i \end{matrix}\right]$ and the Ricci symbol $a_{jk,i}$. Recalling the significance of ρ'_k we see that the Christoffel symbol $\left\{\begin{matrix} ij \\ k \end{matrix}\right\}$ is identical with $I\rho_{ij}\rho'_k$. Thence it is easy to see that we have

$$\left[\begin{matrix} ij \\ k \end{matrix}\right]=\Sigma a_{km}\left\{\begin{matrix} ij \\ m \end{matrix}\right\}, \qquad m=1, \ldots, n-1.$$

Since $I\rho_i\rho'_i=1$, $I\rho_i\rho'_j=0$, $i \neq j$, we have by differentiation

$$I\rho_{ij}\rho'_k+I\rho_i(\rho'_k)_j=0.$$

Therefore

$$I\rho_{ij}\rho'_k=-I\rho_i(\rho'_k)_j=-I\rho_j(\rho'_k)_i, \quad \text{and} \quad (\rho'_k)_j=-\Sigma\rho'_i I\rho'_k\rho_{ij}.$$

Further we have

$$\begin{aligned}(I\rho_{ij}\rho'_k)_l-(I\rho_{il}\rho'_k)_j&=I\rho_{ij}(\rho'_k)_l-I\rho_{il}(\rho'_k)_j \\ &=\Sigma(I\rho_{il}\rho'_h I\rho_{hj}\rho'_k-I\rho_{ij}\rho'_h I\rho_{hl}\rho'_k), \qquad h=1, \ldots, n-1.\end{aligned}$$

4. Again, if we let $Uv=\nu/\sqrt{I\nu\nu}$, since $I\rho_i Uv=0=I\rho_i\nu$, it follows that

$$I\rho_{ij}Uv=-I\rho_i(Uv)_j \quad \text{and} \quad I\rho_i\nu_j=-I\rho_{ij}\nu.$$

Because $IUvUv=1$, therefore $IUv(Uv)_i=0$. Hence $(Uv)_i$ is wholly in the region of $\rho_1, \ldots, \rho_{n-1}$. Indeed we can write at once $(Uv)_i=-\Sigma\rho'_m I\rho_{im}Uv$, *which may also be written in the form of a linear vector function of* ρ_i, namely $N\rho_i$.

The second fundamental form of the region is

$$Id\rho dUv=-\Sigma I\rho_{im}Uv \cdot du_i du_m=Id\rho N d\rho.$$

The coefficients of this form, $-I\rho_{im}Uv$, are called the *fundamental quantities of the second order.* They may be represented by D_{im}.

It is easy to see that if we write v' in place of $A(Uv)_1 \dots (Uv)_{n-1}Ae_1 \dots e_n$, then $Ivv' = |D_{11}, D_{22}, \dots, D_{n-1\,n-1}|$.

Again, if we indicate the partial differentiation of the D's by subscripts,

$$D_{ij,m} = -I\rho_{ijm}Uv - I\rho_{ij}(Uv)_m, \quad \text{and} \quad D_{im,j} = -I\rho_{ijm}Uv - I\rho_{im}(Uv)_j,$$

whence

$$\begin{aligned} D_{ij,m} - D_{im,j} &= I\rho_{im}(Uv)_j - I\rho_{ij}(Uv)_m \\ &= \Sigma(I\rho_{ij}\rho'_k I\rho_{km}Uv - I\rho_{im}\rho'_k I\rho_{jk}Uv) \\ &= \Sigma\left[D_{jk}\begin{Bmatrix} im \\ k \end{Bmatrix} - D_{km}\begin{Bmatrix} ij \\ k \end{Bmatrix}\right]. \end{aligned}$$

These are the extensions of *Codazzi's equations.*

The Christoffel expression $(ijrs)$ is

$$D_{ir}D_{js} - D_{jr}D_{is} = IA\rho_i\rho_j A(Uv)_r(Uv)_s = IA\rho_i\rho_j AN\rho_r N\rho_s.$$

Hence the Christoffel quadrilinear covariant G_4 is

$$G_4 = IAd'\rho d''\rho Ad'''Uvd''''Uv = \Sigma(ijrs)du_i du_j du_r du_s.$$

As it is shown later that N is self-transverse, we have at once

$$(ijrs) = (rsij).$$

5. We can write at once a sextilinear covariant as an extension of such forms,

$$\begin{aligned} G_6 &= IAd^{(1)}\rho d^{(2)}\rho d^{(3)}\rho Ad^{(4)}Uvd^{(5)}Uvd^{(6)}Uv \\ &= \Sigma IA\rho_i\rho_j\rho_k AN\rho_r N\rho_s N\rho_t du_i du_j du_k du_r du_s du_t. \end{aligned}$$

The coefficients of this form furnish new symbols. It is obvious that forms may be constructed of this type up to G_{2n-2}, which will consist of one term, whose coefficient is the Kronecker-Gaussian curvature.

6. The differentiation partially of these forms gives new Christoffel symbols, as

$$(ijrst) = \partial(ijrs)/\partial u_t.$$

In the differentiation of the form when written in the vector notation it is to be remembered that N is to be differentiated, differentiation under these conditions being *covariantive differentiation.*

7. The covariant

$$\frac{I \cdot Ad'\rho d''\rho ANd'\rho Nd''\rho}{I \cdot Ad'\rho d''\rho Ad'\rho d''\rho}$$

is the *Riemann curvature* for the point and the plane of $d'\rho$, $d''\rho$.

Obvious generalizations would be

$$\frac{IAd'\rho d''\rho d'''\rho ANd'\rho Nd''\rho Nd'''\rho}{IAd'\rho d''\rho d'''\rho Ad'\rho d''\rho d'''\rho}, \quad \text{etc.}$$

We arrive thus again at the Kronecker-Gaussian curvature as the final form.

IV. *The Differential Operator* Δ.

1. We define now the differential operator Δ, which turns out to be of great use,

$$\Delta=\rho_1'\partial/\partial u_1+\rho_2'\partial/\partial u_2+\ldots.+\rho_{n-1}'\partial/\partial u_{n-1}.$$

It follows immediately that

$$Id\rho\Delta=du_1\partial/\partial u_1+\ldots.+du_{n-1}\partial/\partial u_{n-1}=d.$$

That is to say, the operator $Id\rho\Delta$ is a total differentiator, and $I\alpha\Delta$, where α is a unit vector in the region of $\rho_1, \ldots., \rho_{n-1}$, gives the rate of change of the operand in the direction of α. So far as the vector properties of Δ are concerned, it may be treated like any other vector. It is the direct extension of the quaternion ∇. Since it depends upon the derivatives of ρ, the question we have to consider first is whether the parameters in terms of which ρ is expressed can be changed without affecting Δ.

Let the parameters u be expressed in terms of the parameters v by the equations

$$u_i=\phi_i(v_1, v_2, \ldots., v_{n-1}), \qquad i=1, \ldots., n-1,$$

where it is assumed that the Jacobian

$$J(u)=\frac{\partial(u_1, u_2, \ldots., u_{n-1})}{\partial(v_1, v_2, \ldots., v_{n-1})}\neq 0.$$

Let the differentiation of any expression as to the parameters v be designated by prefixed subscripts, thus, $\partial\rho/\partial v_2={}_2\rho$, and let the normal of the region defined by ${}_1\rho, {}_2\rho, \ldots., {}_{n-1}\rho$ be designated by $'\nu$. Then we have

$${}_i\rho=\rho_1\,{}_iu_1+\rho_2\,{}_iu_2+\ldots.+\rho_{n-1}\,{}_iu_{n-1}.$$

Hence

$$'\nu=\nu J(u),$$

and therefore

$$U('\nu)=U\nu, \quad \text{and} \quad T('\nu)=T\nu J(u).$$

Further we have (prefixing the $'$ for the v's, instead of affixing it as for the u's)

$$\begin{aligned}{}'_i\rho&=A_1\rho_2\rho\ldots._{i-1}\rho_{i+1}\rho\ldots._{n-1}\rho A_1\rho_2\rho\ldots._{n-1}\rho/I('\nu)('\nu)\\&=\Sigma\rho_j'\frac{\partial(u_1, u_2, \ldots., u_{j-1}, u_{j+1}, \ldots., u_{n-1})}{\partial(v_1, v_2, \ldots., v_{i-1}, v_{i+1}, \ldots., v_{n-1})}/J(u).\end{aligned}$$

Hence, multiplying into

$$\partial/\partial v_j={}_ju_1\partial/\partial u_1+\ldots.+{}_ju_{n-1}\partial/\partial u_{n-1},$$

we have, after summing,

$$'\Delta=\rho_i'\partial/\partial u_1+\ldots.+\rho_{n-1}'\partial/\partial u_{n-1}=\Delta.$$

We have thus established the invariance of the region as well as the invariance of the operator Δ for the region. If the parameters be looked upon as a set of curvilinear coordinates, the formulæ above enable us to pass from one set of coordinates to the other.

2. It follows that such operators as $I()\Delta\cdot Uv$, $I()\Delta\cdot\Delta x$, and the like, are also invariants for a transformation of the parameters, as well as their scalar or vector invariants when considered as linear vector operators. In the latter example, $I()\Delta\cdot\Delta x$, an important theorem needs to be proved. Let Φ represent the operator. Then by expanding we have

$$\Phi=\Sigma_j\Sigma_i(\rho_i' x_i)_j I\rho_j'=\Sigma_j\Sigma_i[(\rho_i')_j x_i+\rho_i' x_{ij}]I\rho_j'.$$

The transverse is

$$\breve{\Phi}=\Sigma_j\Sigma_i[\rho_j' I(\rho_i')_j x_i+\rho_j' I\rho_i' x_{ij}].$$

Thus

$$\Phi-\breve{\Phi}=\Sigma_i\Sigma_j[(\rho_i')_j I\rho_j'-\rho_j' I(\rho_i')_j]x_i.$$

If now we operate upon any vector in the region, say ρ_k, where $k=1, 2, \ldots, n-1$, we find that each coefficient for x_i, where $i=1, 2, \ldots, n-1$, vanishes. Hence, *for the region*, Φ is self-transverse; that is, $I\cdot\sigma\Phi\tau=I\cdot\tau\Phi\sigma$.

3. The expression $Ia\Delta\cdot Q$ gives the rate of variation of Q in the direction a. It is a linear vector function of a, and in the region of $n-1$ dimensions is self-transverse. There are $n-1$ axes and $n-1$ corresponding roots. In the directions of the axes the rate of variation of Q is itself in the direction a, and its measure is the root corresponding. These axes are the directions of extremal variation; they correspond to the axes of a quadric in the region.

Thus, let us consider the operator $N=I()\Delta\cdot Uv$. Then we have $Nd\rho=Id\rho\Delta\cdot Uv$ and $Id'\rho Nd\rho=\Sigma I\rho_i Uv_j d'u_i du_j=\Sigma I\rho_j Uv_i d'u_i du_j=Id\rho Nd'\rho$. Hence in the region N is self-transverse. The axes of N are then the directions of extremal rate of change of the unit normal, are orthogonal to each other, and the roots are the curvatures corresponding. One root of N is 0 corresponding to the axis v. The scalar invariants of N become the mean curvatures, and give us the formulæ

$$\text{First mean curvature} = I\Delta Uv,$$

$$\text{Second mean curvature} = \frac{1}{2!}IA\Delta'\Delta'' AUv'Uv'',$$

$$\text{Third mean curvature} = \frac{1}{3!}IA\Delta'\Delta''\Delta''' AUv'Uv''Uv''',$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\text{Total curvature} = \frac{1}{(n-1)!}IA\Delta'\ldots\Delta^{(n-1)}AUv'\ldots Uv^{(n-1)}$$
$$= (-1)^{n-1}|D_{11}, \ldots, D_{n-1,n-1}|.$$

The accents here merely mark the proper operand for the differentiator, and are to be removed after the operations are performed. They prevent a covariantive differentiation.

4. The well-known formulæ for the relations between the scalar invariants of a linear vector operator and those of its powers give us formulæ connecting these differential parameters dependent on Φ. Thus the second mean curvature above can be written in the form

$$\frac{1}{2}\begin{vmatrix} I\Delta Uv, & I\Delta' Uv'' \\ I\Delta'' Uv', & I\Delta Uv \end{vmatrix} = \frac{1}{2}\left[(I\Delta Uv)^2 - I\Delta' IUv'\Delta'' \cdot Uv'' \right].$$

This is the familiar linear operator formula $m_2\Phi = \frac{1}{2}\left[(m_1\Phi)^2 - m_1(\Phi^2) \right]$.

5. Let us consider the operator $I()\Delta\cdot\Delta x$, which gives us the rate of change of the gradient of x in the direction $()$. We have proved that for the region the operator is self-transverse. The scalar invariants are well-known parameters of differential geometry. Thus, using the conventional notation for differential parameters, Δ_1, Δ_2, Δ_{22}, etc.,

$$m_1(\Phi) = \Delta_2 x, \quad m_2(\Phi) = \Delta_{22} x,$$

and we may add others, in a corresponding notation, as,

$$m_3(\Phi) = \Delta_{222} x, \ \ldots\ldots$$

Syzygies connecting these parameters may be deduced from the usual formulæ connecting the scalar invariants. We have thus for $\Phi_x = \Delta I \Delta x() = \Delta x I \Delta()$ the following corresponding parameters,

$$\Delta_1 x = I\Delta x \Delta x, \quad \Delta_1(x, y) = I\Delta x \Delta y.$$
$$\Delta_1(x, \Delta_1 x) = I\Delta x \Delta I \Delta x \Delta x = 2I\Delta x \Phi_x \Delta x.$$
$$\Delta_2(\Delta_1 x) = I\Delta\Delta I\Delta x\Delta x = 2I\Delta\Phi_x\Delta x.$$

In this formula, as elsewhere, the isolated Δ acts upon all following symbols, unless it is accented.

As an example of a well-known formula take the following. Let $n=3$; then we have $\Phi^2 - \Phi\cdot m_1(\Phi) + m_2(\Phi) = 0$. Therefore if we operate upon Δx and take the product $I\Delta x()$, we have, after a transposition and multiplication by 4,

$$4m_2 I\Delta x\Delta x = 4m_1 I\Delta x\Phi\Delta x - 4I\Delta x\Phi^2\Delta x = 2\Delta_2 x\Delta_1(x, \Delta_1 x) - \Delta_1\Delta_1 x;$$

that is,

$$4\Delta_{22}\cdot\Delta_1 x = 2\Delta_2 x\Delta_1(x, \Delta_1 x) - \Delta_1\Delta_1 x.$$

The derivation of this formula by any other method is scarcely as short. Further, the extensions to space of four or more dimensions is obvious. Thus, for four dimensions we have at once

$$\Delta_{222} x\cdot\Delta_1 x = \frac{1}{2}\Delta_{22} x\cdot\Delta_1(x, \Delta_1 x) - \frac{1}{4}\Delta_2 x\cdot\Delta_1(\Delta_1 x) + I\Delta x\Phi^3\Delta x.$$

The way for further expressions of this type is pointed out here, also an indication that it is better to take as fundamental parameters such forms as

$$I\Delta x\Phi_x^n\Delta x, \quad I\Delta x\Phi_{\Delta_1 x}^n\Delta x,$$

and the like. For example, we have

$$\Phi_{\Delta_1 x}=\Delta I\Delta\Delta_1 x=2\Delta I\Phi_x\Delta x.$$

Thence

$$\Delta_1(x,\ \Delta_1 x)=2I\Delta x\Phi_x\Delta x.$$
$$\Delta_1(\Delta_1 x)=4I\Delta x\Phi_x^2\Delta x.$$
$$\Delta_1(x,\ \Delta_1[x,\ \Delta_1 x])=2I\Delta x\Delta I\Delta x\Phi_x\Delta x=2I\Delta x\Phi_x^2\Delta x+I\Delta x\Phi_{\Delta_1 x}\Delta x.$$
$$\Delta_1(x,\ \Delta_1\Delta_1 x)=4I\Delta x\Phi_{\Delta_1 x}\Phi_x\Delta x.$$
$$\Delta_2\Delta_1 x=m_1(\Phi_{\Delta_1 x}),\ \Delta_{22}(\Delta_1 x)=m_2(\Phi_{\Delta_1 x}),\ \ldots\ldots$$

6. The operator $I(\,)\Delta\cdot U\Delta x=\Phi_x'$ gives the rate of variation of the unit normal of the region of order $n-2$ determined by the levels of the function x, in the region of order $n-1$, and is thus the extension of the geodetic curvature. Its roots and axes determine the extremals of these. The invariants $I\Delta U\Delta x$, etc., give what may be called geodetic mean curvatures. If we write $T\Delta x$ for $\sqrt{I\Delta x\Delta x}$ we have

$$\Phi_x'=\frac{\Phi_x}{T\Delta x}-\frac{1}{2}\Delta x\frac{I\Delta x\Phi_x}{T^3\Delta x}.$$

The first geodetic curvature reduces to

$$I\Delta\Big(\frac{\Delta x}{T\Delta x}\Big)=\frac{I\Delta\Delta x}{T\Delta x}-\frac{1}{2}\frac{I\Delta x\Delta I\Delta x\Delta x}{T^3\Delta x}=\frac{\Delta_2 x}{\sqrt{\Delta_1 x}}+\Delta_1\Big(x,\ \frac{1}{\sqrt{\Delta_1 x}}\Big).$$

This formula is well-known. There will be many other functions related to Φ_x', and others analogous to all these so far considered, but we need not insist upon them here.

V. *Symbolic Invariants.*

1. We return now to the previous double ζ notation. By the use of this we find expressions for all the forms used by Prof. Maschke.* The following examples of Maschke's forms indicate the equivalences. (We abbreviate thus: for the scalar $I\cdot A\alpha_1\alpha_2\ldots\alpha_n Ae_1\ldots e_n$ we use $A\cdot\alpha_1\alpha_2\ldots\alpha_n$. It is clear that if α is linear in $\rho_1, \rho_2, \ldots, \rho_{n-1}$, then $A\cdot\alpha_1\ldots\alpha_{n-1}\nu=IA\alpha_1\ldots\alpha_{n-1}A\rho_1\ldots\rho_{n-1}$.)

$$f_i=I\zeta\rho_i, \quad a_{ik}=I\rho_i\rho_k, \quad f_{kl}=I\zeta\rho_{kl}.$$
$$(f)=A\zeta_1\zeta_2\ldots\zeta_{n-1}U\nu, \quad (f)^2=(n-1)!\,IU\nu U\nu=(n-1)!.$$
$$(uf)=A\Delta u\zeta_1\ldots\zeta_{n-2}U\nu,$$
$$(uf)^2=(n-2)!\,IA\Delta uU\nu A\Delta uU\nu=(n-2)!\,I\Delta u\Delta u=(n-2)!\,\Delta_1 u.$$

* Present Problem of Algebra and Analysis, *Congress of Arts and Sciences*, St. Louis, 1904, Vol. I, pp. 518–530; *Transactions of the American Mathematical Society*, Vols. I, VII.

$$(uf)(vf)=(n-2)!\,\Delta_1(u,v),\quad (uvf)=A\Delta u\Delta v\zeta_1\ldots\zeta_{n-3}Uv,$$
$$(uvf)^2=(n-3)!\,IA\Delta u\Delta vUvA\Delta u\Delta vUv=(n-3)!\,IA\Delta u\Delta vA\Delta u\Delta v.$$
$$\begin{aligned}((uf),f)&=A(\Delta A\Delta u\zeta_1\ldots\zeta_{n-2}Uv)\zeta_1\ldots\zeta_{n-2}Uv\\&=(n-2)!\,IAUv\Delta AUv\Delta u=(n-2)!\,\Delta_2 u.\end{aligned}$$
$$\begin{aligned}(fa)(fv)(ua)&=A\zeta_1\eta_1\ldots\eta_{n-2}UvA\zeta_1\Delta v_1\ldots\Delta v_{n-2}UvA\Delta u\eta_1\ldots\eta_{n-2}Uv\\&=A\Delta u\Delta v_1\ldots\Delta v_{n-2}Uv\cdot(n-2)!.\end{aligned}$$

When we need to discriminate between sets of ζ's we accent them or use η.

2. It is obvious that if we differentiate $(f)^2$ as to u_i we have at once

$$(f)(f)_i=0.$$

Such differentiation may be extended to any of these forms, remembering that the Δ implicit in them is also differentiable (the covariantive differentiation of Maschke), as for example,

$$(fa)_i=A\zeta\eta_1\ldots\eta_{n-2}Uv_i,$$

and

$$(uf)_i=A\Delta u_i\zeta_1\ldots\zeta_{n-2}Uv+A\Delta_i u\zeta_1\ldots\zeta_{n-2}Uv+A\Delta u\zeta_1\ldots\zeta_{n-2}Uv_i.$$

Thus we have such formulæ as

$$f_k(fa)_i(ua)=0,$$
$$(fa)_i(fv)(ua)=0.$$

3. The Maschke symbol $[f]=A\cdot\zeta_1\zeta_2\ldots\zeta_{n-1}z$, where z may be Uv, or $(Uv)_i$, etc. Thus we have

$$\begin{aligned}f_1'(uf)[f]&=(n-2)!\,u_1IUvz,\\f_i(ua)[fa]&=(n-2)!\,I\Delta aq_iIUvz,\\f_k(fa)_i(ua)&=0,\\(fa)(fv)(ua)&=A\Delta u\Delta v_1\ldots\Delta v_{n-2}Uv\cdot(n-2)!,\\(fa)_i(fv)(ua)&=0,\\f_1'f_2''(uvf)[f]&=(n-3)!\,IUvzIA\Delta u\Delta vA\rho_i\rho_j.\end{aligned}$$

4. It is not necessary to point out further the equivalent forms, as they can be easily written down and reduced from the system of Maschke to that of the present paper. The forms given here are at once interpretable in geometric form and for that reason would seem to be more useful in the end. The vectors of Ingold's paper, cited above, are the vectors (other than ζ) appearing above, and his formulæ are not the above equivalents of Maschke's but are their analogues due to suppressing the ζ's and the scalar function.

Applications of these operational forms to differential geometry or mechanics may easily be found in the theorems of the treatises and memoirs cited.

Some Properties of Closed Convex Curves in a Plane.

By Arnold Emch.

1. The main object of this investigation is to prove that at least one square may be inscribed in every closed convex curve in a plane. Instead of "closed convex curve" in the ordinary sense I shall use throughout the equivalent shorter term "oval."

But before the main proposition can be proved it is necessary to give definitions of a convex domain and of an oval in particular, and to establish a number of preliminary theorems.

(1) Minkowsky* has given the following definition for a continuous domain of points enclosed by a convex boundary in a plane:

(1a) The domain contains with any two points also the entire segment between the two points.

(1b) The domain is finite.

(1c) The domain is closed.

The points of the boundary belong to the domain. Through every point of the boundary there is at least one straight line, so that all points of the domain which are not on the line lie on one side of the line only. Minkowsky calls such a line a "supporting line" (*Stützlinie*).

(1d) An *oval* in particular encloses such a domain (including the oval) and can be defined parametrically by two distinct continuous single-valued periodic functions

$$x=\phi(t), \qquad y=\psi(t)$$

of a real parameter t and with the common period ω.† The derivatives $\phi'(t)$ and $\psi'(t)$ are also periodic (same period ω) and are assumed continuous for all definite values of t, which merely requires continuity within the interval $0 \leqq t \leqq \omega$. It is also assumed that $\phi'(t)$ and $\psi'(t)$ do not vanish simultaneously for any values of t. Thus, singular points are excluded. We include furthermore in the definition of an oval that for no parts of the period-interval the

* Theorie der konvexen Körper, *Gesammelte Abhandlungen*, Vol. II, p. 154.

† See Osgood, *Lehrbuch der Funktionentheorie*, Vol. I, pp. 120–123.

functions $\phi(t)$ and $\psi(t)$ remain constant or depend linearly upon t. This excludes any straight portions of the boundary.

From this definition (1d), follows that at every point of the oval there exists a definite tangent (including the cases in which for certain values of t $\lim \left\{\frac{\psi'(t)}{\phi'(t)}\right\} = \pm\infty$). If the point of tangency varies continuously, the direction of the tangent varies continuously.

2. For the proof of some of the theorems that follow, we make use of the following properties of continuous single-valued periodic functions as considered under (1d):

By choosing the origin of t properly we can always assume that for $t=0$, or $t=\omega$, none of the functions $\phi(t)$, $\psi(t)$, $\phi'(t)$, $\psi'(t)$ vanish. At the extremities of the period interval there is $\phi(0)=\phi(\omega)$, $\phi'(0)=\phi'(\omega)$, $\psi(0)=\psi(\omega)$, $\psi'(0)=\psi'(\omega)$. Thus, from Rolle's theorem follows as an application

THEOREM I: *If $\phi(t)$ and $\psi(t)$ have real roots within the period-interval, then their number is in each case even. $\phi'(t)$ and $\psi'(t)$ have always at least two and generally an even number of real roots.*

3. Consider now any direction with the slope τ in the plane of the oval. The question is, are there any tangents to the oval parallel to this direction, and if there are any, what is their number? We evidently have the condition $\frac{dy}{dx}=\tau$, or

$$\psi'(t)-\tau\phi'(t)=0.$$

$\psi'(t)-\tau\phi'(t)$ is the derivative of $\psi(t)-\tau\phi(t)$, and as the latter is a periodic function of the prescribed type, also its derivative is coperiodic and consequently admits according to theorem I an even number of roots. Hence

THEOREM II: *There exist at least two tangents to an oval parallel to any given direction.*

To prove the

THEOREM III: *There are always two and only two tangents to an oval parallel to any given direction.*

Suppose that there were three distinct parallel tangents t_1, t_2, t_3, with t_2 between t_1 and t_3. Then, there would be points belonging to the domain on both sides of t_2, which, according to the property of a supporting line, is impossible.

When, in the equation

$$\psi'(t)-\tau\phi'(t)=0,$$

τ changes continuously, then also the two real roots change continuously. Geometrically, this is equivalent to the fact that *when a given direction changes continuously, then each of the two tangents to the oval changes continuously.*

4. THEOREM IV: *No reentrant quadrangle can be inscribed in an oval.*

As a definition I state that in a reentrant quadrangle $A_1A_2A_3A_4$ one of the vertices, say A_4, lies within the boundary of the triangle formed by the remaining three and in which at each vertex two sides of the quadrangle meet, as in Fig. 1. On the oval the vertices of an inscribed reentrant quadrangle are assumed to follow each other in the order $A_1A_4A_3A_2$. According to (1a) all points of the segments A_1A_2 and A_3A_4 are in the domain, while those outside of these segments on their prolongations are outside of the domain. According to the same condition, A_5, the point where the prolongation of A_3A_4 meets A_1A_2, belongs to the domain. Hence, since A_4 and A_5 belong to the domain, all points of the segment A_4A_5 belong to the domain. This, however, is in contradiction to the assertion that all points outside of the segment A_3A_4 are excluded from the domain. A reentrant quadrangle can therefore never have its vertices on an oval.

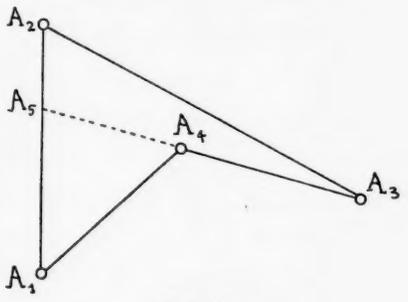


FIG. 1.

THEOREM V: *Two distinct rhombs with corresponding parallel sides or parallel axes can never be inscribed in the same oval.*

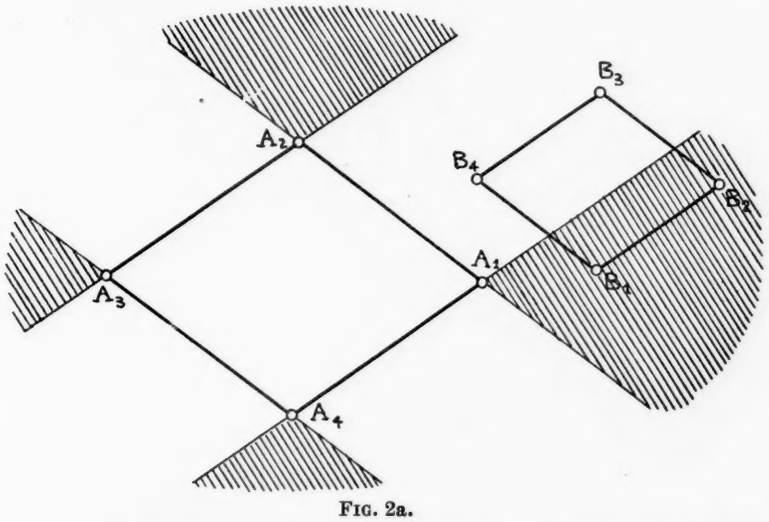


FIG. 2a.

To prove this, assume first that the second rhomb $B_1B_2B_3B_4$ has one vertex, say B_1, in any one of the five shaded regions determined by the first rhomb $A_1A_2A_3A_4$, Fig. 2a. Then there are always three points of the first rhomb which with B_1 form a reentrant quadrangle. The other possibility left for the location of the second rhomb is within the four blank regions of Fig. 2a. In

this second case all vertices of the first rhomb are within the shaded region determined by the second rhombus, Fig. 2b, so that there are always three points B which with any vertex A form a reentrant quadrangle. Both cases include those where points of one rhombus lie on the sides of the other. Hence, no matter what the relative position of the two rhombs may be, there exists always at least one reentrant quadrangle among the eight vertices, and consequently, according to the theorem IV, no oval can pass through them. In a similar manner the proof can be extended without difficulty to figures with parallel axes only.

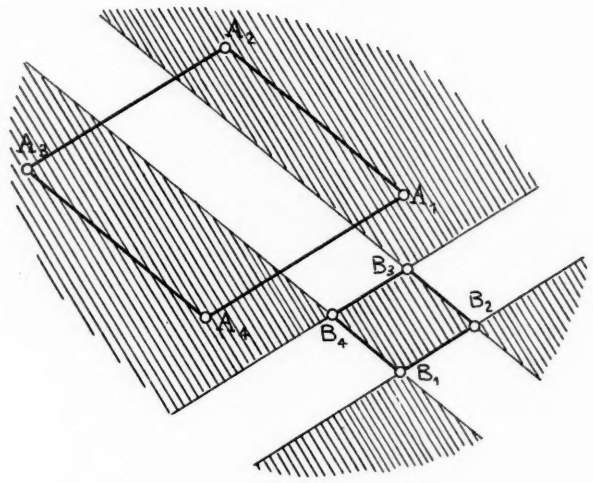


FIG. 2b.

5. In what follows I shall also have to make use of a proposition in function theory which may be stated as follows:

THEOREM VI: *Let a and b be two distinct real numbers, and let $\lambda=\phi_1(\theta)$, $\mu=\psi_1(\theta)$ be two uniform continuous real functions of a parameter θ, subject to the only condition that two distinct values α and β of the parameter θ exist, so that*

$$\phi_1(\alpha)=\psi_1(\beta)=a,$$
$$\phi_1(\beta)=\psi_1(\alpha)=b;$$

then there exists at least one value of θ, say $\theta=\gamma$, for which $\lambda=\mu$, or

$$\phi_1(\gamma)=\psi_1(\gamma).$$

The proof follows immediately from the fact that $\phi_1(\theta)-\psi_1(\theta)$ is continuous and hence takes every value (at least once) between $a-b$ and $b-a$. That is, there is at least one value of θ, $\theta=\gamma$, such that $\phi_1(\gamma)=\psi_1(\gamma)$.

6. It is now possible to prove

THEOREM VII: *It is always possible to inscribe a square in an oval.*

For this purpose assume any point O in the plane of this curve and draw any line l_α through this point, and determine the mid-points of all chords of the oval parallel to l_α and designate the points of tangency of the tangents parallel to l_α by S_α and T_α. The locus of these mid-points is a certain continuous curve C_α, extending from S_α to T_α. Next, draw through O a line $l_\beta \perp l_\alpha$ and repeat the same construction with respect to this direction. The result is a continuous curve C_β extending from S_β to T_β. As the tangents parallel to l_α and l_β form

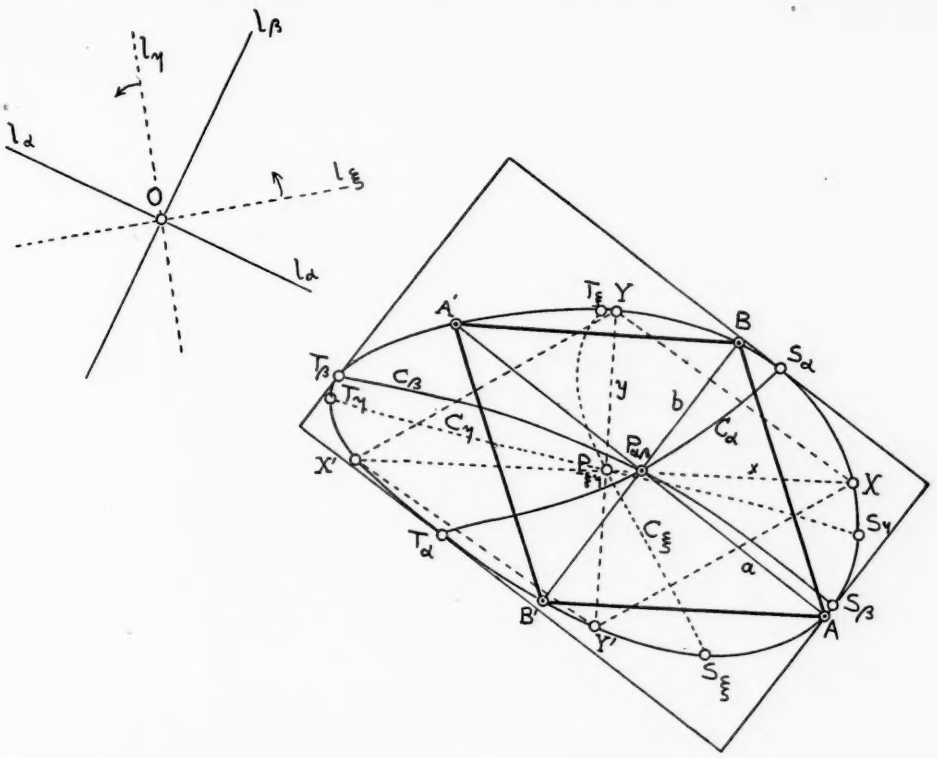


FIG. 3.

a rectangle, it is easily seen that C_α and C_β necessarily intersect within the domain of the oval, Fig. 3. In fact *there is always only one real point of intersection $P_{\alpha\beta}$ between C_α and C_β within the domain.*

To prove this, assume that there are two points of intersection; then there would exist two rhombs with parallel axes inscribed in the oval, which is in contradiction to theorem V. The extremities of the lines through $P_{\alpha\beta}$ parallel to l_α and l_β on the oval form a rhombus $ABA'B'$. Thus, with every pair of orthogonal rays l_α and l_β through O is associated one definite rhombus inscribed in the oval; and the same rhombus is evidently obtained when l_α and l_β are interchanged.

There exists therefore a (1, 1) *correspondence between all pairs of orthogonal lines through O and all rhombs inscribed in the given oval.*

Turning a line l_ξ through O continuously from l_α to l_β, then its orthogonal ray l_η will turn in the same sense from l_β to l_α. The corresponding curves C_ξ and C_η, since their extremities S and T on the oval change continuously, also change continuously. Their point of intersection $P_{\xi\eta}$ describes therefore a continuous curve, and consequently the corresponding rhombus $XYX'Y'$ changes continuously. The axes $\lambda=XX'$ and $\mu=YY'$ of this rhombus may therefore be expressed as uniform and continuous functions of a parameter θ associated with the direction of l_ξ, within the interval between l_α and l_β and including these limits. We may, for instance, choose as θ the positive angle which l_ξ makes with the positive part of the X-axis. Designating the diagonals of the original rhombus by a and b, by α and β the parameters associated with l_α and l_β, by

$$\lambda=\phi(\theta), \qquad \mu=\psi(\theta)$$

the axes of the rhombus as the above uniform and continuous functions of θ within the interval $\alpha\leqq\theta\leqq\beta$, then

$$a=\phi(\alpha), \qquad b=\psi(\alpha).$$

If now the line l_ξ turns from l_α to l_β, $XYX'Y'$ changes from $ABA'B'$ to $BA'B'A$, so that in the second position

$$\lambda=\phi(\beta)=b, \qquad \mu=\psi(\beta)=a.$$

Hence, the situation is exactly as stated in theorem IV. There exists therefore at least one direction, l_γ, for which $\phi(\gamma)=\psi(\gamma)$, or $\lambda=\mu$; *i.e.*, where the rhombus becomes a square.

*Finiteness of the Odd Perfect and Primitive Abundant Numbers with n Distinct Prime Factors.**

By Leonard Eugene Dickson.

1. Denote by $\sigma(a)$ the sum of all the divisors of a positive integer a. Then a is called abundant, perfect, or deficient, according as

$$\sigma(a) > 2a, \ = 2a, \ < 2a,$$

respectively. If $d_1, \ldots, d_k$ are the divisors of a, the divisors of ma include $md_1, \ldots, md_k$ and unity. Hence

$$\sigma(ma) > m\sigma(a) \qquad (m > 1),$$

so that any greater multiple of an abundant or perfect number is abundant. This implies that any smaller divisor of a deficient or perfect number is deficient.

A non-deficient number will be called *primitive* if it is not a multiple of a smaller non-deficient number. The set of all non-deficient numbers is identical with the set of all multiples of the primitive non-deficient numbers. Any perfect number is a primitive non-deficient number.

While there is an infinitude of non-deficient odd numbers having any given number n $(n > 2)$ of distinct prime factors, for example,

$$3^e 5 \cdot 7\, p_4^{e_4} \ldots . p_n^{e_n} \qquad (e \geqq 3, \ p\text{'s distinct primes} > 7),$$

we shall nevertheless prove the following

Theorem. *There is only a finite number of primitive non-deficient odd† numbers having any given number of distinct prime factors.*

Corollary. *There is not an infinitude of odd perfect numbers with any given number of distinct prime factors.*

Sylvester‡ proved that there is no odd perfect number having five or fewer distinct prime factors.

Below are listed all primitive non-deficient odd numbers with four or fewer distinct prime factors. Since none of them are perfect, we obtain anew Sylvester's result, except for the case of five primes.

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* Presented to the American Mathematical Society, December 31, 1912.

† The restriction to odd numbers is essential. For instance, if p is a prime, $2^m p$ is non-deficient if and only if $2^{m+1} > p+1$.

‡ *Collected Math. Papers*, Vol. IV (1912), pp. 588–629 (several articles).

2. LEMMA A. *Any set S of functions of the type*

$$F = x_1^{e_1} x_2^{e_2} \ldots . x_n^{e_n} \qquad (e\text{'s integers} \geqq 0) \tag{1}$$

contains a finite number of functions $F_1, \ldots, F_k$ such that each function F of the set S can be expressed as a product $F_i f$, where f is of the form (1), *but is not necessarily in the set S.*

For $n=1$, we may take $k=1$, $F_1 = x_1^l$, where l is the least e_1. To proceed by induction, let the lemma be true for $n-1$ variables. Select at random a function $x_1^{c_1} \ldots . x_n^{c_n}$ of the set S and call it F_1. Then any function (1) of S is of the desired form $F_1 f$ if $e_1 \geqq c_1, \ldots, e_n \geqq c_n$, simultaneously. If there be further functions in S, those in which $e_i = v$, where i is a fixed integer $\leqq n$ and v a fixed integer $< c_i$, have the desired property. Indeed, after deleting the common factor x_i^v, we have a set S' of functions

$$F' = x_1^{e_1} \ldots . x_{i-1}^{e_{i-1}} x_{i+1}^{e_{i+1}} \ldots . x_n^{e_n},$$

which by hypothesis are expressible as products $F_j' f'$ $(j = 1, \ldots, k_i)$. The number of cases arising by varying i and v is finite.

We may also derive* the lemma from a theorem due to Hilbert,† which may be stated in the following form:

Any set S of homogeneous polynomials in $x_1, \ldots, x_n$ contains a finite number of polynomials $F_1, \ldots, F_k$ such that any polynomial F of the set can be expressed in the form $f_1 F_1 + \ldots . + f_k F_k$, where $f_1, \ldots, f_k$ are homogeneous polynomials in $x_1, \ldots, x_n$, not necessarily in the set S.

Let each F be a monomial form (1). Since the f's are not uniquely determined, the resulting expression $\Sigma f_i F_i$ for F is not necessarily of the form $f_i F_i$. However, there is always at least one determination of the f's which gives this special expression for F. Indeed, we have $f_i \equiv 0$ unless the degree of F in x_j equals or is greater than the degree of F_i in x_j for each $j=1, \ldots, n$. Since not every f_i vanishes identically, there is some value of i for which the monomial F is of the form $g_i F_i$, where g_i is monomial.

3. Lemma A may be stated and proved in an equivalent geometrical form. For example, let $n = 2$ and define the region $R(P)$ to be the quadrant of the plane determined by the two half-lines extending from the point P horizontally to the right and vertically upwards, respectively. Let S be a set of points (A, B) with integral coordinates $\geqq 0$. Let a be the least A, β the least B for the points (a, B), and set $P_1 = (a, \beta)$. Let b be the least B, α the least A for the points (A, b), and set $P_2 = (\alpha, b)$. All the points of S lie in $R(M)$,

* Professor E. H. Moore suggested to me that I undertake this derivation.

† *Mathematische Annalen*, Vol. XXXVI.

where $M = (a, b)$. In particular, if $\beta = b$, then $\alpha = a$ (and conversely), and $P_1 = P_2 = M$, so that the points of S lie in $R(P_1)$. In general, the points of S in neither $R(P_1)$ nor $R(P_2)$ lie *inside* the rectangle determined by P_1, P_2, M, and hence are finite in number.

By 2^n applications of Lemma A we obtain the theorem: Let the signs of the coordinates of a point $X' = (\xi_1', \ldots, \xi_n')$ be such that

$$\varepsilon_1 \xi_1' \geqq 0, \ldots, \varepsilon_n \xi_n' \geqq 0 \qquad (\varepsilon_1^2 = 1, \ldots, \varepsilon_n^2 = 1),$$

and denote by $R_{\varepsilon_1, \ldots, \varepsilon_n}(X')$ the region of n dimensions which is composed of all the points $X = (\xi_1, \ldots, \xi_n)$ for which

$$\varepsilon_i \xi_i \geq \varepsilon_i \xi_i' \qquad (i = 1, \ldots, n).$$

Then any set S of points X with integral coordinates contains a finite number of points $X^{(i)}_{\varepsilon_1, \ldots, \varepsilon_n}$ such that every point of S lies in at least one region $R_{\varepsilon_1, \ldots, \varepsilon_n}(X^{(i)}_{\varepsilon_1, \ldots, \varepsilon_n})$.

The lemma may also be interpreted as a theorem on permutable operations O_i with positive or negative integral exponents. We replace x_i by $O_i^{\varepsilon_i}$ and consider the aggregate of the resulting 2^n sets of operations.

4. From Lemma A we derive at once

LEMMA B. *If $p_1, \ldots, p_n$ are given integers, any set S of integers*

$$p_1^{e_1} p_2^{e_2} \ldots p_n^{e_n} \qquad (e\text{'s integers} \geqq 0) \tag{2}$$

contains a finite number of integers $F_1, \ldots, F_k$ such that every integer of the set S is a multiple of at least one F_i.*

5. Let $0 < k < n$, $1 < p_i' \leqq p_i$ $(i > k)$, and set

$$P = \prod_{i=1}^{n} \frac{\sigma(p_i^{e_i})}{p_i^{e_i}} = \prod_{i=1}^{n} \frac{p_i - 1/p_i^{e_i}}{p_i - 1},$$

$$P_0 = \prod_{i=1}^{n} \frac{p_i}{p_i - 1}, \quad P_k = \prod_{i=1}^{k} \frac{\sigma(p_i^{e_i})}{p_i^{e_i}} \cdot \prod_{i=k+1}^{n} \frac{p_i}{p_i - 1},$$

$$P_0' = \prod_{i=1}^{n} \frac{p_i'}{p_i' - 1}, \quad P_k' = \prod_{i=1}^{k} \frac{\sigma(p_i^{e_i})}{p_i^{e_i}} \cdot \prod_{i=k+1}^{n} \frac{p_i'}{p_i' - 1}.$$

Let $p_1, \ldots, p_n$ be fixed distinct primes. By definition, an integer a, of the form (2), is deficient if $P < 2$, non-deficient if $P \geqq 2$. Since

$$P_m' \geqq P_m > P \qquad (0 \leqq m < n),$$

---

* For a case (like that of all non-deficient numbers built from $p_1, \ldots, p_n$) in which every multiple of any F_i is in the set S, we readily obtain the generators of the complementary set C of all integers (2) not in S. A number (2) is in C if and only if for each $i \leqq k$ there is some value of j such that $e_j < c_{ij}$ where

$$F_i = p_1^{c_{i1}} p_2^{c_{i2}} \ldots p_n^{c_{in}} \qquad (i = 1, \ldots, k).$$

We may separate the numbers $1, \ldots, k$ into sets $I_1, \ldots, I_s$, where $s \leqq k$, and define corresponding distinct positive integers $j_1, \ldots, j_s$, each $\leqq n$, such that

$$e_{j_1} < c_{ij_1} \ (i \text{ in } I_1), \ldots, e_{j_s} < c_{ij_s} \ (i \text{ in } I_s),$$

while the remaining $n - s$ exponents e are subject to no restrictions.

a is deficient if $P_m \leqq 2$, also if $P'_m \leqq 2$. Since P_m is the limit of P for $e_i = \infty$ $(i = m+1, \ldots, n)$ and since $P < 2$ if a is deficient, we conclude that $P_m \leqq 2$ if a is deficient for all values of $e_{m+1}, \ldots, e_n$.

The case $P_m = 2$ occurs only when $a = 2^e$. For, if p is the greatest of the primes $p_{m+1}, \ldots, p_n$, no number in the denominator of P_m is divisible by p. Thus $P_m = 2$ implies that $p = 2$, $m = 0$, $n = 1$, $a = 2^e$.

LEMMA C. *If $P_m \leqq 2$ or $P'_m \leqq 2$, (2) is deficient. If an odd number (2) is deficient for all values of $e_{m+1}, \ldots, e_n$, then $P_m < 2$.*

6. We are now in a position to prove

LEMMA D. *All primitive non-deficient odd numbers having a given number n of distinct prime factors are formed from a finite number of sets of n primes.*

We consider numbers a of the form (2), where $p_1, \ldots, p_n$ are primes in ascending order of magnitude. If r is the positive real n-th root of 2, then $p_1 < r/(r-1)$. For,* if we take $p'_i = p_1$ $(i = 1, \ldots, n)$, we see that a is deficient if

$$P'_0 = \left(\frac{p_1}{p_1 - 1}\right)^n \leqq 2, \quad p_1 \geqq \frac{r}{r-1}.$$

To proceed by induction, assume that $p_1, \ldots, p_\nu$ $(\nu < n)$ is a particular one of a finite number of sets of ν distinct primes. Since a is to be a primitive non-deficient number, its divisor

$$\alpha = p_1^{e_1} p_2^{e_2} \ldots . p_\nu^{e_\nu}$$

must be deficient. Noting that each divisor of a deficient α is deficient (or using the foot-note in § 4), we see that the deficient α's are the numbers in which certain exponents $e_{i_1}, \ldots, e_{i_k}$ are arbitrary, while each remaining exponent takes a limited number of values, and further numbers in which every exponent is limited. We consider one such type of deficient α's, thus treating one of a finite number of analogous cases. After permuting $p_1, \ldots, p_\nu$, we may assume that $\mu (0 \leqq \mu \leqq \nu)$ is an integer such that $e_1, \ldots, e_\mu$ are limited, while e_i $(i = \mu + 1, \ldots, \nu)$ takes all values. By Lemma C, the deficiency of these α's implies that

$$\Pi_\mu = \prod_{i=1}^{\mu} \frac{\sigma(p_i^{e_i})}{p_i^{e_i}} \cdot \prod_{i=\mu+1}^{\nu} \frac{p_i}{p_i - 1} < 2,$$

the second product being absent if $\mu = \nu$. Since there is a limited number of sets $e_1, \ldots, e_\mu$, each Π_μ is less than a constant $M < 2$. The use of P'_μ for

$$p'_i = p_i \ (i = \mu + 1, \ldots, \nu), \qquad p'_i = p_{\nu+1} \ (i = \nu + 1, \ldots, n),$$

---

* A like simple proof by use of P'_0 shows that also p_2 is limited (since $p_1 > 2$) and then that p_3 is limited, but fails for p_4 since

$$\frac{p_1}{p_1 - 1} \frac{p_2}{p_2 - 1} \frac{p_3}{p_3 - 1} > 2 \text{ if } p_1 = 3, \ p_2 = 5, \ p_3 \leqq 13.$$

shows that a is deficient if

$$\Pi_\mu \cdot \left(\frac{p_{\nu+1}}{p_{\nu+1}-1}\right)^{n-\nu} \leqq 2.$$

Hence a is deficient if

$$M\left(\frac{p_{\nu+1}}{p_{\nu+1}-1}\right)^{n-\nu} \leqq 2, \quad p_{\nu+1} \geqq \frac{g}{g-1}, \quad g \equiv \sqrt[n-\nu]{2/M} > 1.$$

Hence in a non-deficient a, $p_{\nu+1}$ is less than the largest of the limits obtained in the various cases, finite in number.

7. Consider the set S of primitive non-deficient numbers having as distinct prime factors $p_1, \ldots, p_n$ a particular one of the finite (Lemma D) number of possible sets of n primes. Since any greater multiple of a non-deficient number is not primitive, the set S is finite by Lemma B. We have therefore proved the theorem of § 1.

8. THEOREM. *There is no odd abundant number with less than three distinct prime factors, and no odd perfect number with less than four distinct prime factors. The primitive odd abundant numbers with three distinct prime factors are:*

$$3^3 5 \cdot 7,\ 3^2 5^2 7,\ 3^2 5 \cdot 7^2,\ 3^3 5^2 11,\ 3^5 5^2 13,\ 3^4 5^3 13,\ 3^4 5^2 13^2,\ 3^3 5^3 13^2. \qquad (3)$$

We have $n > 2$ since $\frac{3}{2} \cdot \frac{5}{4} < 2$. Let $n = 3$. Since $\left(\frac{5}{4}\right)^3 < 2$, we have $p_1 = 3$. We have $p_2 = 5$ and then $p_3 < 17$ since

$$\frac{3}{2} \cdot \frac{7}{6} \cdot \frac{11}{10} = \frac{77}{40} < 2, \quad \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{17}{16} = \frac{255}{128} < 2.$$

For $e_1 = 1$ or 2, we obtain a smaller limit for p_3 by use of $P_1 > 2$, viz., $p_3 < 6$, $p_3 < 11$, respectively. Hence $e_1 > 1$.

First, let $e_1 = 2$, so that $p_3 = 7$. Unless $e_2 = e_3 = 1$ (giving a deficient a), a is a multiple of (3_2) or (3_3).

Next, if $e_1 \geqq 3$ and $p_3 = 7$, a is a multiple of (3_1).

Finally, let $e_1 \geqq 3$, $p_3 = 11$ or 13. If $e_2 = 1$, a is deficient since

$$\frac{\sigma(5)}{5} \cdot \frac{3}{2} \cdot \frac{11}{10} = \frac{99}{50} < 2.$$

If $e_2 > 1$ and $p_3 = 11$, a is a multiple of (3_4). Let therefore $e_1 \geqq 3$, $e_2 > 1$, $p_3 = 13$. The conditions that $3^e 5^f 13$ be non-deficient for $f = 2$ and 3 are $3^e > 217$, $e > 4$; $3^e > 42$, $e > 3$, respectively; while $3^3 5^f 13$ is deficient since

$$\frac{\sigma(3^3)}{3^3} \cdot \frac{\sigma(13)}{13} \cdot \frac{5}{4} = \frac{700}{351} < 2.$$

Hence, for $e_3 = 1$, a is non-deficient if and only if it is a multiple of (3_5) or (3_6).

Next, $3^e 5^2 13^2$ and $3^3 5^f 13^2$ are non-deficient if and only if $3^e > 47$, $e > 3$; $5^f > 76$, $f > 2$, respectively. Hence for $e_3 = 2$, a is non-deficient if and only if it is a multiple of (3_7) or (3_8). For $e_3 \geqq 3$, a is a multiple of one of the latter numbers unless it be $3^3 5^2 13^e$, which is deficient.

9. Passing to the case $n = 4$, we have $p_1 = 3$, $p_2 = 5$ or 7 since

$$\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{12} = \frac{1001}{576} < 2, \quad \frac{3}{2}\left(\frac{11}{10}\right)^3 = \frac{3993}{2000} < 2.$$

THEOREM. *The primitive non-deficient odd numbers with four distinct prime factors, of which the second is 7, are*

$3^2 7^2 11^2 13^3$, $3^2 7^2 11^3 13^2$, $3^2 7^3 11^2 13^2$, $3^3 7 \cdot 11 \cdot 13^2$, $3^3 7 \cdot 11^2 13$, $3^3 7^2 11 \cdot 13$, $3^4 7 \cdot 11 \cdot 13$; (4)

$3^3 7^2 11^2 17$, $3^3 7^3 11 \cdot 17^2$, $3^4 7^2 11 \cdot 17$, $3^5 7 \cdot 11^2 17^3$, $3^5 7 \cdot 11^3 17^2$, $3^6 7 \cdot 11^2 17^2$; (5)

$3^3 7^2 11^3 19^2$, $3^3 7^3 11^3 19$, $3^3 7^3 11^2 19^2$, $3^4 7^2 11 \cdot 19^2$, $3^4 7^2 11^2 19$, $3^4 7^3 11 \cdot 19$, $3^5 7^2 11 \cdot 19$; (6)

$3^4 7^3 11^2 23^2$, $3^4 7^4 11^3 23$, $3^5 7^2 11^2 23^2$, $3^5 7^2 11^4 23$, $3^5 7^3 11^2 23$, $3^6 7^2 11^2 23$; (7)

$3^4 7^2 13^3 17^3$, $3^4 7^3 13^2 17^2$, $3^5 7^2 13^2 17^2$, $3^5 7^3 13^2 17$, $3^6 7^2 13^3 17$, $3^6 7^3 13 \cdot 17^2$, $3^7 7^2 13^2 17$; (8)

$3^6 7^4 13^3 19^3$, $3^7 7^3 13^4 19^4$, $3^7 7^3 13^5 19^3$, $3^7 7^4 13^3 19^2$, $3^8 7^3 13^3 19^3$, $3^8 7^4 13^2 19^3$, $3^{10} 7^3 13^4 19^2$. (9)

Henceforth we shall write a, b, c, d for $e_1, \ldots, e_4$ and p for p_4. Denote (2) by A. We have $p_3 < 17$. Consider first the case $p_3 = 11$. Then A is deficient if

$$\frac{3 \cdot 3^a - 1}{2 \cdot 3^a}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{p}{p-1} < 2, \quad 3^a(240 - 9p) \leqq 77p.$$

Hence, for a non-deficient A, $p < 27$ and $a \geqq 4$ if $p = 23$, $a \geqq 3$ if $p = 19$ or 17, $a \geqq 2$ if $p = 13$. Similarly, $b \geqq 2$ if $p \geqq 19$, $c \geqq 2$ if $p = 23$. First, let $p = 23$. Then A is a multiple of (7_6) unless $a \leqq 5$. For $a = 4$, we have $b > 2$; if $d > 1$, A is a multiple of (7_1); if $d = 1$, then $c > 2$, $b > 3$, and A is a multiple of (7_2). For $a = 5$, A is a multiple of (7_5) unless $b = 2$, in which case it is a multiple of (7_3) if $d > 1$; while, if $d = 1$, then $c \geqq 4$ and A is a multiple of (7_4).

Next, let $p = 19$. Then A is a multiple of (6_7) unless $a = 3$ or 4. First, let $a = 3$. Then $c > 1$. For $b = 2$, we have $c > 2$, $d > 1$, whence A is a multiple of (6_1). For $b \geqq 3$, A is a multiple of (6_3) if $d > 1$; while, if $d = 1$, we have $c \geqq 3$ and A is a multiple of (6_2). Next, let $a = 4$. Then A is a multiple of (6_4), (6_5) or (6_6) unless $b = 2$, $c = d = 1$, when A is deficient.

For $p = 17$, $a \geqq 3$. First, let $b = 1$. Then $c > 1$, $d > 1$, $a \geqq 5$, and A is a multiple of (5_6) unless $a = 5$. If $a = 5$, A is a multiple of (5_4) or (5_5) unless $c = d = 2$, in which case it is deficient. Next, if $b \geqq 2$, $a \geqq 4$, A is a multiple of (5_3). Finally, if $b \geqq 2$, $a = 3$, A is a multiple of (5_1) or (5_2) unless $c = d = 1$, or $c = 1$, $d > 1$, $b = 2$, in each of which two cases A is deficient.

The case $p = 13$ needs no special comment.

It remains to treat the coordinate case $p_3 = 13$. By P_1,

$$3^a(288 - 15p) > 91p,$$

whence $p = 19$, $a \geqq 6$, or $p = 17$, $a \geqq 4$. Similarly, $b \geqq 3$, $c \geqq 2$, $d \geqq 2$ if $p = 19$; $b \geqq 2$ if $p = 17$. For $p = 19$, A is a multiple of (9_1) unless $b = 3$ or $c = 2$ or $d = 2$. First, let $c = 2$. Then A is deficient if $a \leqq 7$ or $b = 3$ or $d = 2$; the remaining A's are multiples of (9_6). Next, let $d = 2$ $(c > 2)$. Then A is deficient if $a = 6$. Thus, if $b \geqq 4$, A is a multiple of (9_4). But if $b = 3$, A is deficient if $c = 3$ or $a \leqq 9$, while the remaining A's are multiples of (9_7). Finally, let $b = 3$ $(c > 2, d > 2)$. Now A is deficient if $a = 6$, and a multiple of (9_5) unless $a = 6$ or 7. For $a = 7$, we must have $c \geqq 4$. Then A is a multiple of (9_2) if $d \geqq 4$, of (9_3) if $d = 3$, unless also $c < 5$, when it is deficient.

Finally, let $p = 17$. Unless A is a multiple of (8_1), $c < 3$ or $d < 3$. First, let $c = 2$. Then A is a multiple of (8_2) or (8_3) unless $d = 1$ or $b = 2$, $a = 4$. In the latter case A is deficient. For $d = 1$, A is a multiple of (8_7) or (8_4) unless $b = 2$, $a < 7$ or $a = 4$, while A is then deficient. Second, let $c = 1$; then $a > 5$, $b > 2$, $d > 1$, and A is a multiple of (8_6). Third, let $d = 2$, $c \geqq 3$. Then A is a multiple (8_2) or (8_3) unless $b = 2$, $a = 4$, in which case A is deficient. Finally, let $d = 1$, $c \geqq 3$. If $a = 4$ or if $a = 5$, $b = 2$, A is deficient. If $a = 5$, $b \geqq 3$, A is a multiple of (8_4). If $a = 6$ or $a \geqq 7$, A is a multiple of (8_5) or (8_7), respectively.

10. In treating the more prolific case $p_2 = 5$, use is made of the

THEOREM. *If the prime p is not a divisor of the deficient number k, then kp^e is non-deficient if and only if*

$$x \equiv \frac{\sigma(k)}{2k - \sigma(k)} \geqq \frac{p^e}{\sigma(p^{e-1})}. \tag{10}$$

Since $2k = \sigma(k)\,(1 + 1/x)$, the condition is

$$\sigma(k)\,\sigma(p^e) \geqq 2kp^e, \quad \sigma(p^e) \geqq p^e(1 + 1/x).$$

But $\sigma(p^e) = p^e + \sigma(p^{e-1})$. Hence the inequality (10) follows.

COROLLARY. *kp (and hence also kp^e) is non-deficient if $p \leqq x$, while kp^e is deficient if $p \geqq x + 1$,*

To obtain the latter part of the corollary, we note that

$$\frac{p^e}{\sigma(p^{e-1})} = p - 1 + \frac{1}{\sigma(p^{e-1})}.$$

To give another proof, note that kp^e is deficient if

$$P_1 = \frac{\sigma(k)}{k} \cdot \frac{p}{p-1} \leqq 2, \quad p \geqq \frac{2k}{2k - \sigma(k)} = x + 1.$$

The corollary gives complete information concerning kp^e unless there be a prime K between x and $x+1$. In the latter case, kK is deficient, while $kK^e (e>1)$ is to be tested by the theorem itself.

11. For $n=4$, there remains the case $p_1=3$, $p_2=5$. Since

$$\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{31}{30}\cdot\frac{37}{36}=\frac{1147}{576}<2,$$

$p_3 \leqq 29$. First, let $p_3=7$. Then A is a multiple of (3_1), (3_2) or (3_3) unless $a=1$ or $a=2$, $b=c=1$. In the latter case, A is abundant if $p \leqq x=104$, deficient if $p>104$. Let next $a=1$. Then A is deficient if $p>36$. For

$$k=3\cdot 5^2 7^2,\quad 3\cdot 5\cdot 7^2,\quad 3\cdot 5^2 7,\quad 3\cdot 5\cdot 7,$$

the integer just $>x$ is 26, 14, 18, 11, respectively. For $k=3\cdot 5\cdot 7$, we test $p=11$ by the theorem and find that (10) holds for $e=2$, since $x=32/3$. Again, A is deficient if $b=1$, $p\geqq 17$; $b=2$, $c=1$, $p\geqq 19$; $c=1$, $p>21$; $b=2$, $c=3$, $p=29$; $b=2$, $p=31$; $b=3$, $c=2$, $d=1$, $p=31$. Hence all the primitives with $p_3=7$ are given in the first part of the following list. As in this case, we have rearranged the primitives, listing together those with the same set of primes. Two or more complete sets of primitives differing only in the value of the fourth prime are combined in the list. Although this rearranged list is somewhat longer than the list as initially determined, it has obvious advantages over the latter.

12. List of the primitive abundant numbers with exactly four distinct prime factors, the second prime being 5.

$3\cdot5\cdot7\cdot11^2$, $3\cdot5\cdot7^2 11$, $3\cdot5^2 7\cdot11$, $3^2 5\cdot7\cdot11$; $3\cdot5\cdot7^2 13$, $3\cdot5^2 7\cdot13$, $3^2 5\cdot7\cdot13$;
$3\cdot5^2 7\cdot17$, $3^2 5\cdot7\cdot17$; $3\cdot5^2 7^2 19$, $3\cdot5^3 7\cdot19$, $3^2 5\cdot7\cdot19$; $3\cdot5^2 7^2 23$, $3^2 5\cdot7\cdot23$;
$3\cdot5^2 7^4 29$, $3\cdot5^3 7^2 29$, $3^2 5\cdot7\cdot29$; $3\cdot5^3 7^2 31^2$, $3\cdot5^3 7^3 31$, $3^2 5\cdot7\cdot31$;
$3^2 5\cdot7p\,(37\leqq p\leqq 103)$; $3^2 5\cdot11\cdot13$; $3^2 5\cdot11\cdot17$;
$3^2 5\cdot11^2 19^2$, $3^2 5\cdot11^3 19$, $3^2 5^2 11\cdot19$, $3^3 5\cdot11\cdot19$;
$3^2 5^2 11p$, $3^3 5\cdot11p\,(p=23,29,31)$;
$3^2 5^2 11p$, $3^3 5\cdot11^2 p$, $3^4 5\cdot11p\,(p=37,41)$;
$3^2 5^2 11\cdot43^2$, $3^2 5^2 11^2 43$, $3^2 5^3 11\cdot43$, $3^3 5\cdot11^2 43^2$, $3^3 5\cdot11^3 43$, $3^4 5\cdot11\cdot43$;
$3^2 5^2 11^2 47$, $3^2 5^3 11\cdot47$, $3^4 5\cdot11^2 47$, $3^5 5\cdot11\cdot47$;
$3^2 5^2 11^2 53$, $3^2 5^3 11\cdot53$, $3^4 5\cdot11^2 53$, $3^6 5\cdot11\cdot53^2$, $3^7 5\cdot11\cdot53$;
$3^2 5^2 11^2 59$, $3^2 5^3 11\cdot59^2$, $3^2 5^3 11^2 59$, $3^2 5^4 11\cdot59$, $3^4 5\cdot11^2 59$;
$3^2 5^2 11^2 61$, $3^2 5^3 11^2 61$, $3^2 5^4 11\cdot61$, $3^4 5\cdot11^2 61$;
$3^2 5^2 11^4 67^2$, $3^2 5^3 11^2 67$, $3^4 5\cdot11^2 67^2$, $3^4 5\cdot11^3 67$, $3^5 5\cdot11^2 67$;
$3^2 5^3 11^2 71$, $3^4 5\cdot11^4 71^2$, $3^5 5\cdot11^2 71$; $3^2 5^3 11^2 p$, $3^5 5.11^2 p\,(p=73,79)$;
$3^2 5^3 11^2 83$, $3^5 5\cdot11^3 83$, $3^6 5\cdot11^2 83$; $3^2 5^3 11^2 89$, $3^6 5\cdot11^2 89^2$, $3^6 5\cdot11^3 89$, $3^7 5\cdot11^2 89$;

$3^2 5^3 11^2 97$, $3^7 5 \cdot 11^3 97^2$, $3^7 5 \cdot 11^4 97$, $3^8 5 \cdot 11^3 97$; $3^2 5^3 11^2 p\,(p = 101, 103)$;
$3^2 5^3 11^2 107^2$, $3^2 5^4 11^2 107$; $3^2 5^3 11^3 p$, $3^2 5^4 11^2 p\,(p = 109, 113)$;
$3^2 5^4 11^3 127$, $3^2 5^5 11^2 127$; $3^2 5^4 11^3 131$; $3^2 5^4 11^4 137^2$, $3^2 5^5 11^3 137$; $3^2 5^5 11^3 139$;
$3^2 5^2 13 p$, $3^3 5 \cdot 13 p\,(p = 17, 19)$; $3^2 5^2 13 \cdot 23$, $3^3 5 \cdot 13 \cdot 23^2$, $3^3 5 \cdot 13^2 23$, $3^4 5 \cdot 13 \cdot 23$;
$3^2 5^2 13^2 29$, $3^2 5^3 13 \cdot 29$, $3^3 5^2 13 \cdot 29$, $3^4 5 \cdot 13^2 29$, $3^5 5 \cdot 13 \cdot 29$;
$3^2 5^2 13^2 31$, $3^2 5^3 13 \cdot 31$, $3^3 5^2 13 \cdot 31$, $3^4 5 \cdot 13^2 31$, $3^5 5 \cdot 13 \cdot 31^2$, $3^6 5 \cdot 13 \cdot 31$;
$3^2 5^3 13^2 37$, $3^3 5^2 13 \cdot 37$, $3^5 5 \cdot 13^2 37^2$, $3^6 5 \cdot 13^2 37$; $3^2 5^3 13^2 41^2$, $3^2 5^3 13^3 41$, $3^3 5^2 13 \cdot 41$;
$3^2 5^4 13^2 43^2$, $3^2 5^4 13^3 43$, $3^2 5^5 13^2 43$, $3^3 5^2 13 \cdot 43$;
$3^3 5^2 13 p\,(47 \leqq p \leqq 89)$; $3^3 5^2 13^2 p$, $3^3 5^3 13 p$, $3^4 5^2 13 p\,(97 \leqq p \leqq 181)$;
$3^3 5^2 13^3 p$, $3^3 5^3 13 p$, $3^4 5^2 13 p\,(191 \leqq p \leqq 199)$; $3^3 5^3 13 p$, $3^4 5^2 13 p\,(p = 211, 223)$;
$3^3 5^4 13 p$, $3^4 5^2 13 p\,(227 \leqq p \leqq 313)$; $3^3 5^5 13 p$, $3^4 5^2 13 p\,(p = 317, 331, 337)$;
$3^3 5^6 13 \cdot 347$, $3^4 5^2 13 \cdot 347$; $3^3 5^6 13 \cdot 349^2$, $3^3 5^7 13 \cdot 349$, $3^4 5^2 13 \cdot 349$;
$3^4 5^2 13 p\,(353 \leqq p \leqq 383)$;
$3^2 5^2 17 \cdot 19^2$, $3^2 5^2 17^2 19$, $3^2 5^3 17 \cdot 19$, $3^3 5^2 17 \cdot 19$, $3^4 5 \cdot 17^2 19$, $3^4 5 \cdot 17 \cdot 19^2$, $3^5 5 \cdot 17 \cdot 19$;
$3^2 5^3 17^2 23^2$, $3^2 5^4 17^2 23$, $3^3 5^2 17 \cdot 23$, $3^3 5^2 17 p\,(p = 29, 31)$;
$3^3 5^2 17^2 37$, $3^3 5^3 17 \cdot 37$, $3^4 5^2 17 \cdot 37$; $3^3 5^2 17^2 41^2$, $3^3 5^3 17 \cdot 41$, $3^4 5^2 17 \cdot 41$;
$3^3 5^3 17 \cdot 43$, $3^4 5^2 17 \cdot 43$; $3^3 5^3 17 \cdot 47^2$, $3^3 5^3 17^2 47$, $3^3 5^4 17 \cdot 47$, $3^4 5^2 17 \cdot 47$;
$3^3 5^3 17^2 53$, $3^4 5^2 17^2 53$, $3^4 5^3 17 \cdot 53$, $3^5 5^2 17 \cdot 53$;
$3^3 5^4 17^2 59^2$, $3^3 5^4 17^3 59$, $3^3 5^5 17^2 59$, $3^4 5^2 17^2 59$, $3^4 5^3 17 \cdot 59$, $3^5 5^2 17 \cdot 59$;
$3^4 5^2 17^3 61$, $3^4 5^2 17^2 61^2$, $3^4 5^3 17 \cdot 61$, $3^5 5^2 17^2 61$, $3^6 5^2 17 \cdot 61$;
$3^4 5^3 17 p$, $3^5 5^2 17^2 p\,(p = 67, 71, 73)$;
$3^4 5^3 17^2 79$, $3^4 5^4 17.79$, $3^5 5^3 17 \cdot 79$, $3^6 5^2 17^2 79^2$, $3^6 5^2 17^3 79$, $3^7 5^2 17^2 79$;
$3^4 5^3 17^2 83$, $3^4 5^4 17 \cdot 83$, $3^5 5^3 17 \cdot 83$, $3^7 5^2 17^3 83^2$, $3^9 5^2 17^3 83$;
$3^4 5^3 17^2 89$, $3^5 5^3 17 \cdot 89$; $3^4 5^3 17^2 97$, $3^5 5^3 17 \cdot 97^2$, $3^5 5^4 17.97$, $3^6 5^3 17 \cdot 97$;
$3^4 5^3 17^2 101$, $3^5 5^4 17 \cdot 101$, $3^6 5^3 17 \cdot 101$;
$3^4 5^3 17^3 103$, $3^4 5^4 17^2 103$, $3^5 5^3 17^2 103$, $3^5 5^4 17 \cdot 103$, $3^6 5^3 17 \cdot 103$;
$3^4 5^4 17^2 107$, $3^5 5^3 17^2 107$, $3^5 5^4 17 \cdot 107$, $3^7 5^3 17 \cdot 107$;
$3^4 5^4 17^2 109$, $3^5 5^3 17^2 109$, $3^5 5^4 17 \cdot 109$, $3^7 5^3 17 \cdot 109^2$, $3^8 5^3 17.109$;
$3^4 5^4 17^2 113$, $3^5 5^3 17^2 113$, $3^5 5^5 17 \cdot 113^2$, $3^5 5^6 17 \cdot 113$, $3^6 5^4 17 \cdot 113$;
$3^5 5^3 17^2 127$, $3^6 5^5 17 \cdot 127^2$, $3^7 5^5 17 \cdot 127$, $3^8 5^4 17 \cdot 127$; $3^5 5^3 17^2 131$, $3^7 5^5 17 \cdot 131$;
$3^5 5^3 17^2 p\,(p = 137, 139)$; $3^4 5^4 17^2 p$, $3^6 5^3 17^2 p\,(p = 149, 151, 157)$;
$3^4 5^4 17^2 163$, $3^6 5^3 17^3 163$, $3^7 5^3 17^2 163$; $3^5 5^4 17^2 167$, $3^6 5^3 17^3 167^2$, $3^7 5^3 17^2 167$;
$3^5 5^4 17^3 173$, $3^5 5^5 17^2 173$, $3^6 5^4 17^2 173$, $3^7 5^3 17^3 173$, $3^8 5^3 17^2 173^2$, $3^9 5^3 17^2 173$;
$3^5 5^5 17^2 179$, $3^6 5^4 17^2 179$, $3^8 5^3 17^3 179^2$, $3^8 5^3 17^4 179$, $3^9 5^3 17^3 179$;
$3^5 5^5 17^3 181$, $3^5 5^6 17^2 181$, $3^6 5^4 17^2 181$, $3^9 5^3 17^4 181^2$, $3^{10} 5^3 17^3 181^2$;
$3^6 5^4 17^2 p\,(p = 191, 193, 197, 199)$; $3^6 5^4 17^3 211$, $3^6 5^5 17^2 211$, $3^7 5^4 17^2 211$;
$3^6 5^5 17^3 223$, $3^7 5^4 17^3 223$, $3^7 5^5 17^2 223$, $3^8 5^4 17^2 223^2$, $3^9 5^4 17^2 223$;
$3^6 5^6 17^3 227^2$, $3^6 5^6 17^4 227$, $3^6 5^7 17^3 227$, $3^7 5^5 17^2 227$, $3^7 5^4 17^3 227^2$, $3^8 5^4 17^3 227$;
$3^6 5^7 17^4 229^2$, $3^7 5^5 17^2 229$, $3^8 5^4 17^3 229$;

$3^7 5^5 17^3 233$, $3^7 5^6 17^2 233^2$, $3^8 5^4 17^3 233^2$, $3^8 5^5 17^2 233$, $3^9 5^4 17^4 233$;
$3^7 5^5 17^3 239$, $3^8 5^6 17^2 239^2$, $3^8 5^7 17^2 239$, $3^9 5^6 17^2 239$, $3^{10} 5^5 17^2 239^2$;
$3^7 5^5 17^3 241^2$, $3^7 5^5 17^4 241$, $3^7 5^6 17^3 241$, $3^8 5^5 17^3 241$, $3^9 5^6 17^2 241^2$, $3^9 5^7 17^2 241$, $3^{10} 5^6 17^2 241$;
$3^8 5^6 17^3 251^2$, $3^8 5^7 17^4 251$, $3^9 5^6 17^3 251$, $3^{10} 5^5 17^4 251^2$, $3^{11} 5^5 17^3 251^2$;
$3^3 5^2 19 p\ (p = 23, 29)$; $3^3 5^2 19^2 31$, $3^3 5^3 19 \cdot 31$, $3^4 5^2 19 \cdot 31$;
$3^3 5^3 19 \cdot 37^2$, $3^3 5^3 19^2 37$, $3^3 5^4 19 \cdot 37$, $3^4 5^2 19 \cdot 37$;
$3^3 5^3 19^2 41^2$, $3^3 5^4 19^2 41$, $3^4 5^2 19^2 41$, $3^4 5^3 19 \cdot 41$, $3^5 5^2 19 \cdot 41$;
$3^3 5^4 19^2 43^2$, $3^3 5^5 19^3 43$, $3^4 5^2 19^2 43$, $3^4 5^3 19 \cdot 43$, $3^5 5^2 19 \cdot 43$;
$3^4 5^3 19 \cdot 47$, $3^5 5^2 19^2 47$, $3^7 5^2 19 \cdot 47^2$;
$3^4 5^3 19 \cdot 53^2$, $3^4 5^3 19^2 53$, $3^5 5^3 19 \cdot 53$, $3^7 5^2 19^2 53^2$, $3^9 5^2 19^2 53$;
$3^4 5^3 19^2 59$, $3^5 5^3 19 \cdot 59$; $3^4 5^3 19^2 61^2$, $3^4 5^3 19^3 61$, $3^5 5^3 19 \cdot 61$;
$3^4 5^4 19^3 67^2$, $3^4 5^5 19^2 67$, $3^5 5^3 19^2 67$, $3^5 5^4 19 \cdot 67^2$, $3^5 5^5 19 \cdot 67$, $3^6 5^4 19 \cdot 67$, $3^7 5^3 19 \cdot 67^2$;
$3^5 5^3 19^2 71$, $3^6 5^5 19 \cdot 71$, $3^7 5^4 19 \cdot 71$; $3^5 5^3 19^2 73$, $3^6 5^6 19 \cdot 73^2$, $3^7 5^5 19 \cdot 73$, $3^9 5^4 19 \cdot 73$;
$3^5 5^4 19^2 79$, $3^6 5^3 19^2 79^2$, $3^6 5^3 19^3 79$, $3^7 5^3 19^2 79$;
$3^5 5^5 19^2 83^2$, $3^6 5^4 19^2 83$, $3^9 5^3 19^3 83^2$; $3^6 5^4 19^3 89^2$, $3^6 5^5 19^2 89$, $3^7 5^4 19^2 89$;
$3^3 5^3 23^2 29^2$, $3^3 5^4 23 \cdot 29^2$, $3^3 5^4 23^2 29$, $3^4 5^2 23 \cdot 29^2$, $3^4 5^2 23^2 29$, $3^4 5^3 23 \cdot 29$;
$3^3 5^5 23^2 31^2$, $3^4 5^2 23^2 31^2$, $3^4 5^3 23 \cdot 31$, $3^5 5^2 23 \cdot 31$; $3^4 5^3 23^2 37$, $3^4 5^4 23 \cdot 37$, $3^5 5^3 23 \cdot 37$;
$3^4 5^4 23^2 41^2$, $3^5 5^3 23^2 41$, $3^5 5^4 23 \cdot 41$, $3^6 5^3 23 \cdot 41$;
$3^5 5^3 23^2 43^2$, $3^5 5^4 23^2 43$, $3^5 5^5 23 \cdot 43^2$, $3^6 5^3 23^2 43$, $3^6 5^4 23 \cdot 43$; $3^5 5^6 23^3 47^2$, $3^6 5^4 23^2 47$;
$3^5 5^3 29^2 31^2$, $3^5 5^4 29 \cdot 31^2$, $3^5 5^4 29^2 31$, $3^6 5^3 29 \cdot 31^2$, $3^6 5^3 29^2 31$, $3^6 5^4 29 \cdot 31$.

13. THEOREM. *The only primitive odd abundant numbers* < 15000 *are:*

$3^3 5 \cdot 7 = 945$, $3^2 5^2 7 = 1575$, $3^2 5 \cdot 7^2 = 2205$, $3^2 5 \cdot 7 \cdot 11 = 3465$,
$3^2 5 \cdot 7 \cdot 13 = 4095$, $3^2 5 \cdot 7 \cdot 17 = 5355$, $3 \cdot 5^2 7 \cdot 11 = 5775$, $3^2 5 \cdot 7 \cdot 19 = 5985$,
$3^2 5 \cdot 11 \cdot 13 = 6435$, $3 \cdot 5^2 7 \cdot 13 = 6825$, $3^2 5 \cdot 7 \cdot 23 = 7245$, $3^3 5^2 11 = 7425$,
$3 \cdot 5 \cdot 7^2 11 = 8085$, $3^2 5 \cdot 11 \cdot 17 = 8415$, $3 \cdot 5^2 7 \cdot 17 = 8925$, $3^2 5 \cdot 7 \cdot 29 = 9135$,
$3 \cdot 5 \cdot 7^2 13 = 9555$, $3^2 5 \cdot 7 \cdot 31 = 9765$, $3^2 5 \cdot 7 \cdot 37 = 11655$, $3 \cdot 5 \cdot 7 \cdot 11^2 = 12705$,
$3^2 5 \cdot 7 \cdot 41 = 12915$, $3^2 5 \cdot 7 \cdot 43 = 13545$, $3^2 5 \cdot 7 \cdot 47 = 14805$.

The number n of distinct prime factors is < 5, since

$$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 15015.$$

For $n < 4$, the primitives are given by (3). Since $3^4 5^2 13 = 26325$, the last four exceed 78000. Finally, let $n = 4$. The primitives (§ 9) with $p_2 = 7$ evidently exceed $3^2 7^2 11 \cdot 13 = 63063$. Next, let $p_2 = 5$. The primitives (§ 11) with $p_3 = 7$, which are < 15000, are readily listed. The primitives with $p_3 \geqq 11$ have the factor 3^2 since $3 \cdot 5^b p_3^c p^d$ is deficient. Since $3^2 5 \cdot 11 \cdot 13 = 6435$, whose product by 3 exceeds our limit, it remains to examine only $3^2 5\, p_3 p_4$. The latter is deficient if $p_3 > 11$, and if $p_3 = 11$, $p_4 > 17$.

Even Abundant Numbers.

By L. E. Dickson.

1. Theorem. *There is only a finite number of primitive non-deficient numbers having a given number n of distinct odd prime factors and a given number m of factors 2.*

Let $p_1, \ldots, p_n$ be primes in ascending order. Then

$$a = 2^m p_1^{e_1} p_2^{e_2} \ldots p_n^{e_n}$$

is deficient if

$$G \cdot \left(\frac{p_i}{p_1 - 1}\right)^n < 2, \quad G \equiv \frac{2^{m+1}-1}{2^m},$$

and hence if $p_1 > r/(r-1)$, where r is the positive real n-th root of $2/G$, whence $r > 1$. Thus the least odd prime factor p_1 of a non-deficient number is limited. To prove by induction a like result for each p_i, let $p_1, \ldots, p_\nu$ be given primes, and $\nu < n$. The divisor

$$\alpha = 2^m p_1^{e_1} \ldots p_\nu^{e_\nu}$$

of a primitive a must be deficient. By Lemma B of the former paper, each deficient α is a product of some one of a finite number of α's by a number having no prime factor other than $p_1, \ldots, p_\nu$. We may now complete the proof as in the former paper, inserting in Π_μ the additional factor G.

For example, the primitive non-deficient $2^m p^e$ are $2^m p$, where $2^m - 1 < p \leqq 2^{m+1} - 1$.

2. In the determination of all even non-deficient numbers not exceeding a given number L, use may be made of the following theorems.

Theorem. *Let $2^l \leqq L < 2^{l+1}$ and $\lambda = l/2$ or $(l+1)/2$, according as l is even or odd. If n is any integer for which $l > n \geqq \lambda$ and k is any odd integer for which $2^n k \leqq L$, then $2^n k$ is non-deficient.*

We have $2^\lambda k \leqq 2^n k \leqq L < 2^{l+1}$. Now $l+1 = 2\lambda + 1$ or 2λ, according as l is even or odd. Thus $k < 2^{n+1}$ or 2^n, in the respective cases. In each case, $k \leqq N$, where $N = 2^{n+1} - 1$. Thus k has a prime factor $p \leqq N$. Since $2^n p$ is non-deficient, the same is true of its multiple $2^n k$.

3. Theorem. *Let ν be the least integer for which*

$$2^\nu (2^{\nu+1} - 1)^2 \geqq L,$$

and n any integer $\geqq \nu$. Let c be any odd composite number for which $2^n c < L$. Then $2^n c$ is abundant.

Set $N = 2^{n+1} - 1$. Then $2^n N^2 \geqq L$ and $c < N^2$. Hence the composite number c has a prime factor $p < N$. Since $2^n p$ is abundant, its multiple $2^n c$ is abundant.

This theorem, in connection with the fact that $2^n p$ (p an odd prime) is abundant if and only if $p < N$, enables one to write down immediately all abundant numbers $2^n k < L$, where $n \geqq \nu$ (cf. § 8).

4. THEOREM. *Set* $s = 2^{\nu+1} - 1$, $t = 2^\nu - 1$. *The non-deficient even numbers* $2^{\nu-1} k$ *less than** $2^\nu s^2$ *are those in which* k *has a prime factor* $\leqq t$ *and those in which* k *is a product of two primes* $t + P$ *and* $t + Q$, *where* $0 < P < t$, $P < Q \leqq (t^2 + t)/P$. *If* $P = t - 1$, *then* $Q = t + 1$.

Let every prime factor of k exceed t. Since

$$(2^\nu + 1)^3 \geqq 2 s^2$$

for every $\nu > 0$, k can not have three prime factors. If p is an odd prime, $2^n p^e$ is deficient if

$$\frac{2^{n+1} - 1}{2^n} \frac{p}{p - 1} < 2, \quad p > 2^{n+1}.$$

Hence $2^{\nu-1} p^e$ is deficient if $p > t$. Finally, if p, q are distinct primes $> t$ and $q > p$, then $2^{\nu-1} p q$ is non-deficient if and only if

$$p q \leqq t (p + q + 1).$$

Set $p = t + P$, $q = t + Q$. Then the condition becomes $PQ \leqq t^2 + t$. But $Q \geqq P + 2$. Hence $P < t$. If $P = t - 1$, then $Q < t + 3$, so that $Q = t + 1$. For, if $Q \geqq t + 3$,

$$PQ \geqq t^2 + t + t - 3, \quad PQ > t^2 + t.$$

Indeed, $\nu \geqq 2$, $t \geqq 3$; while if $t = 3$ and if $Q = t + 3$, then $q = 9$.

5. THEOREM. *The non-deficient numbers* $2^{\nu-2} k$ *less than* $2^\nu s^2$ *and having†* $\nu > 3$ *are those in which* k *has a prime factor* $\leqq r$, *where* $r = 2^{\nu-1} - 1$, *those in which* k *is the product of two primes* $r + P$ *and* $r + Q$, *where* $0 < P < r$, $P < Q \leqq (r^2 + r)/P$, *and the following abundant numbers:*

$$2^4 37 \cdot 41^2, \ 2^4 37^2 q (q = 41, 43, 47), \ 2^3 \cdot 17 \cdot 19 l (23 \leqq l \leqq 47),$$
$$2^3 17 \cdot 23 l (l = 29, 31, 37), \ 2^3 17 \cdot 29 \cdot 31, \ 2^3 19 \cdot 23 l (l = 29, 31),$$
$$2^3 17^2 q (q < 55), \ 2^3 19^2 q (q < 45), \ 2^3 23^2 q (q < 31), \ 2^3 29^2 17,$$
$$2^2 11^2 q (q < 26), \ 2^2 13^2 q (q < 19), \ 2^2 17^2 q (q < 14),$$
$$2^2 11 \cdot 13 l (l = 17, 19, 23), \ 2^2 11 \cdot 17 \cdot 19,$$

where q *and* l *are primes,* $q > 7$ *in the fourth line,* $q > 15$ *in the third line.*

Let every prime factor of k exceed r. Since

$$(2^{\nu-1} + 1)^4 > 2^{4\nu-4} \geqq 4 \cdot 2^{2\nu+2} > 4 s^2$$

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* Including all $< L$, since $L \leqq 2^\nu s^2$ by § 3.

† Those with $\nu = 3$ are given in § 7.

for $\nu > 3$, k can not have four prime factors. Since

$$(2^{\nu-1}+1)^3 > 2^{3\nu-3} \geqq 4 \cdot 2^{2\nu+2} > 4s^2$$

for $\nu \geqq 7$, k has at most two prime factors unless $\nu = 6, 5$ or 4. As in § 4, $2^{\nu-2}p^e$ is deficient if $p > r$, since then $p > 2^{\nu-1}$. The case in which k is the product of two distinct prime factors may be treated as in § 4. We shall next consider $2^{\nu-2}p^2q$, where $\nu = 6, 5$ or 4, and p, q are distinct primes $> r$. First, let $\nu = 6$, whence $r = 31$, $s = 127$. The least p is 37; by $37^2q < 4s^2$, $q < 48$; $2^4 37^2 q$ is abundant if $q < 229$. For $p = 41$, $4s^2/p^2 < 39$, whence $q = 37$. Next, let $\nu = 5$, whence $r = 15$, $s = 63$. For $p = 17, 19, 23, 29, 31$, $p^2q < 4s^2$ for $q < 55, 45, 31, 19, 17$, respectively, the final p being therefore excluded; while 2^3p^2q is abundant for $q < 243, 94, 50, 34, 31$, respectively. Finally, let $\nu = 4$, whence $r = 7$, $s = 31$. For $p = 11, 13, 17, 19$, $p^2q < 4s^2$ for $q < 32, 23, 14, 11$; while 2^2p^2q is abundant for $q < 26, 19, 14, 13$.

It remains only to consider $2^{\nu-2}pql$, where p, q, l are distinct primes $> r$, arranged in ascending order. First, let $\nu = 6$. If $p \geqq 41$, $pql > 41^3 > 4s^2$. Hence $p = 37$. But $37 \cdot 41 \cdot 43 > 4s^2$. For $\nu = 5$ or 4, the numbers $< 4s^2$ are listed in the theorem, all being abundant.

6. While we might treat similarly the cases ν—3, etc., the results already obtained, together with those in § 7 for non-deficient numbers $2k$, enable us to tabulate in § 8 the even non-deficient numbers less than $2^\nu s^2$ for $\nu = 4$, namely, $< 2^4 31^2 = 15376$. Indeed, under this limit L, there is no primitive non-deficient number $2k$, where k is an odd number with more than three distinct prime factors (a case not treated in § 7). First, 3 is not a factor. If 5 is a factor, 7 is not, and $2k \geqq 2 \cdot 5 \cdot 11 \cdot 13 \cdot 17 = 24310 > L$. If 5 is not a factor, $2k \geqq 2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$, which exceeds the preceding.

7. THEOREM. *The primitive non-deficient numbers $2k$, where k is an odd number with at most three distinct prime factors, are:*

$2 \cdot 3$, $2 \cdot 5 \cdot 7$, $2 \cdot 5^2 11$, $2 \cdot 5^2 13$, $2 \cdot 5 \cdot 11\, p\,(13 \leqq p \leqq 53)$,
$2 \cdot 5 \cdot 11^2\, p\,(59 \leqq p \leqq 89)$, $2 \cdot 5 \cdot 11^3 97$, $2 \cdot 5 \cdot 13\, p\,(17 \leqq p \leqq 31)$,
$2 \cdot 5 \cdot 13^2 37$, $2 \cdot 5 \cdot 17 \cdot 19$, $2 \cdot 5^2 17\, p\,(23 \leqq p \leqq 61)$, $2 \cdot 5^2 17^2\, p\,(67 \leqq p \leqq 79)$,
$2 \cdot 5^2 17^3 83$, $2 \cdot 5^3 17\, p\,(67 \leqq p \leqq 109)$, $2 \cdot 5^3 17^2 p\,(113 \leqq p \leqq 173)$,
$2 \cdot 5^3 17^3 179$, $2 \cdot 5^3 17^3 181^2$, $2 \cdot 5^3 17^4 181$, $2 \cdot 5^4 17\, p\,(p = 113, 127)$,
$2 \cdot 5^4 17^2\, p\,(179 \leqq p \leqq 223)$, $2 \cdot 5^4 17^3\, p\,(p = 227, 229, 233)$, $2 \cdot 5^5 17 \cdot 131$,
$2 \cdot 5^5 17^2\, p\,(p = 227, 229, 233)$, $2 \cdot 5^5 17^2 239^2$, $2 \cdot 5^5 17^3\, p\,(p = 239, 241)$,
$2 \cdot 5^5 17^3 251^2$, $2 \cdot 5^6 17^2\, p\,(p = 239, 241)$, $2 \cdot 5^6 17^3 251$,
$2 \cdot 5^2 19\, p\,(23 \leqq p \leqq 43)$, $2 \cdot 5^2 19 \cdot 47^2$, $2 \cdot 5^2 19^2\, p\,(p = 47, 53)$,
$2 \cdot 5^3 19\, p\,(47 \leqq p \leqq 61)$, $2 \cdot 5^3 19 \cdot 67^2$, $2 \cdot 5^3 19^2\, p\,(67 \leqq p \leqq 79)$,

$2\cdot5^3 19^3 83^2$, $2\cdot5^4 19\,p\,(p=67,71,73)$, $2\cdot5^4 19^2\,p\,(p=83,89)$,
$2\cdot5^2 23\,p\,(p=29,31)$, $2\cdot5^3 23\,p\,(p=37,41)$, $2\cdot5^3 23^2 43$, $2\cdot5^4 23^2 47$,
$2\cdot5^3 29\cdot31^2$, $2\cdot5^3 29^2 31$, $2\cdot5^4 29\cdot31$,
$2\cdot7\cdot11\cdot13$, $2\cdot7^2 13^2 17$, $2\cdot7^3 13\cdot17^2$, $2.7^3 13^3 19^3$, $2\cdot7^3 13^4 19^2$, $2\cdot7^4 13^2 19^3$,
$2\cdot7^4 13^3 19^2$.

The proof is similar to that used in the former paper.

8. We are now in a position to give the even non-deficient numbers* $< L = 15000$. This L is just under the maximum limit given by $\nu = 4$ (§ 6). Moreover, this L was the convenient limit used in listing the primitive odd abundant numbers in the former paper.

The non-deficient numbers $2^n k < L$, with $n \geqq 4$, are (§§ 2, 3):

$2^{12}3$, $2^{11}k\,(k\leqq 7)$, $2^{10}k\,(k\leqq 13)$, $2^9 k\,(k\leqq 29)$, $2^8 k\,(k\leqq 57)$, $2^7 k\,(k\leqq 117)$,
$2^6 c\,(c\leqq 233)$, $2^6 p\,(p\leqq 127)$, $2^5 c\,(c\leqq 467)$, $2^5 p\,(p<63)$, $2^4 c\,(c\leqq 937)$, $2^4 p\,(p\leqq 31)$,

where k, c, p are odd and >1, c is composite and p prime.

The non-deficient $2^3 k < L$ are (§ 4) those with k having a prime factor $\leqq 13$ ($k\leqq 1875$) and $k = 29\cdot31$, $23\,q\,(29\leqq q\leqq 43)$, $19\,q\,(23\leqq q\leqq 73)$, $17\,q\,(19\leqq q\leqq 109)$. In the last case only the limit for abundance ($q\leqq 131$) exceeded the limit required by L.

The non-deficient $2^2 k < L$ are (§ 5) those with $k < 3750$ having a factor 3, 5 or 7; those with $k = 11\,q\,(q = 13, 17, 19)$; and

$2^2 11^2 q\,(q = 13, 17, 19, 23)$, $2^2 13^2 q\,(q = 11, 17)$, $2^2 17^2 q\,(q = 11, 13)$,
$2^2 11\cdot13\,l\,(l = 17, 19, 23)$, $2^2 11\cdot17\cdot19$.

The primitive non-deficient $2k < L$ are (§ 7) those in the first two lines of the theorem below.

The primitives are now found at once. For instance, $2^2 k$ with k having a factor 3, 5 or 7 is a multiple of one of the primitives $2\cdot3$, $2^2 5$, $2^2 7$. In the list of non-deficients $2^n k$, $n\geqq 4$, k or c is always less than the square of the maximum prime p giving a non-deficient $2^n p$. Hence we obtain the

THEOREM. *The 98 primitive even non-deficient numbers < 15000 are:*

$2\cdot3$, $2\cdot5\cdot7$, $2\cdot5^2 11$, $2\cdot5^2 13$, $2\cdot5\cdot11\,p\,(13\leqq p\leqq 53)$,
$2\cdot5\cdot13\,p\,(17\leqq p\leqq 31)$, $2\cdot5\cdot17\cdot19$, $2\cdot7\cdot11\cdot13$,
$2^2 5$, $2^2 7$, $2^2 11\,q\,(q = 13, 17, 19)$, $2^2 11^2 23$, $2^2 13\cdot17^2$, $2^2 13^2 17$,
$2^3 11$, $2^3 13$, $2^3 17\,q\,(19\leqq q\leqq 109)$, $2^3 19\,q\,(23\leqq q\leqq 73)$, $2^3 23\,q\,(29\leqq q\leqq 43)$, $2^3 29\cdot31$,
$2^4\,p\,(17\leqq p\leqq 31)$, $2^5\,p\,(37\leqq p\leqq 61)$, $2^6\,p\,(67\leqq p\leqq 127)$.

* All <6232 are tabulated in *Quar. Jour. Math.*, 1913, pp. 274–7.

The Identical Relations between the Elements of any Oblique Triple System of Surfaces.

By Henry D. Thompson.

The Lamé identical relations between the elements of an orthogonal triple system of surfaces, are particular cases of six identical relations holding between the elements of a general oblique triple system of surfaces.

Consider a triple system of surfaces, with the parameters ρ_1, ρ_2, ρ_3; and let $M(\rho_1, \rho_2, \rho_3)$ be any point in space, and M_1, M_2, M_3, M_{123} be $(\rho_1+d\rho_1, \rho_2, \rho_3)$, $(\rho_1, \rho_2+d\rho_2, \rho_3)$, $(\rho_1, \rho_2, \rho_3+d\rho_3)$, $(\rho_1+d\rho_1, \rho_2+d\rho_2, \rho_3+d\rho_3)$, respectively, etc.; let x_{ij} represent the derivative with respect to ρ_i and ρ_j, and the summation Σx represent the sum of like terms in x, y, z. And let

$$H_i^2=\Sigma x_i^2; \qquad k_{ij}=\Sigma x_i x_j. \tag{A}$$

Let $\partial H_i/\partial\rho_j$ be represented by H_{ij}, $\partial k_{ij}/\partial\rho_l$ by k_{ijl}, etc.; so that the subscripts after the first for the capital letters H, and after the second for the lower case letters k, represent direct differentiation. Represent $k_{123}+k_{231}+k_{312}$ by $2s$; and $\partial s/\partial\rho_i$ by s_i. With this notation the differentiation of the equations (A) gives, after certain combinations in some cases, the following eighteen equations (B)

$$\left.\begin{aligned}\Sigma x_1x_{11}&=H_1H_{11}\\ \Sigma x_2x_{11}&=k_{121}-H_1H_{12}\\ \Sigma x_3x_{11}&=k_{131}-H_1H_{13}\end{aligned}\right\}(B_{11}) \qquad \left.\begin{aligned}\Sigma x_1x_{22}&=k_{122}-H_2H_{21}\\ \Sigma x_2x_{22}&=H_2H_{22}\\ \Sigma x_3x_{22}&=k_{232}-H_2H_{23}\end{aligned}\right\}(B_{22}) \qquad \left.\begin{aligned}\Sigma x_1x_{33}&=k_{133}-H_3H_{31}\\ \Sigma x_2x_{33}&=k_{233}-H_3H_{32}\\ \Sigma x_3x_{33}&=H_3H_{33}\end{aligned}\right\}(B_{33})$$

$$\left.\begin{aligned}\Sigma x_1x_{23}&=s-k_{231}\\ \Sigma x_2x_{23}&=H_2H_{23}\\ \Sigma x_3x_{23}&=H_3H_{32}\end{aligned}\right\}(B_{23}) \qquad \left.\begin{aligned}\Sigma x_1x_{13}&=H_1H_{13}\\ \Sigma x_2x_{13}&=s-k_{132}\\ \Sigma x_3x_{13}&=H_3H_{31}\end{aligned}\right\}(B_{13}) \qquad \left.\begin{aligned}\Sigma x_1x_{12}&=H_1H_{12}\\ \Sigma x_2x_{12}&=H_2H_{21}\\ \Sigma x_3x_{12}&=s-k_{123}\end{aligned}\right\}(B_{12})$$

The first derivatives of the equations (B), added and subtracted in such a way that the terms containing a third derivative of x, y, z are eliminated, give exactly six relations, namely:

$$H_2H_{233}+H_3H_{322}+H_{23}^2+H_{32}^2-k_{2323}=\Sigma x_{23}^2-\Sigma x_{22}x_{33} \tag{G_1}$$
$$H_3H_{311}+H_1H_{133}+H_{31}^2+H_{13}^2-k_{3131}=\Sigma x_{31}^2-\Sigma x_{33}x_{11} \tag{G_2}$$
$$H_1H_{122}+H_2H_{211}+H_{12}^2+H_{21}^2-k_{1212}=\Sigma x_{12}^2-\Sigma x_{11}x_{22} \tag{G_3}$$
$$H_1H_{123}+H_{12}H_{13}+k_{2311}-s_1=\Sigma x_{12}x_{13}-\Sigma x_{11}x_{23} \tag{C_1}$$
$$H_2H_{213}+H_{21}H_{23}+k_{3122}-s_2=\Sigma x_{23}x_{21}-\Sigma x_{22}x_{31} \tag{C_2}$$
$$H_3H_{312}+H_{31}H_{32}+k_{1233}-s_3=\Sigma x_{31}x_{32}-\Sigma x_{33}x_{12} \tag{C_3}$$

The terms on the right of the equations (G) and (C) can be obtained in terms of the first derivatives of the expressions H and k as follows: Let (x_1, y_1, z_1), (x_{23}, y_{23}, z_{23}), etc. be considered as proportional to the direction cosines of lines named l_1, l_{23}, etc., and represent Σx_{23}^2 by L_{23}^2, etc.; then the cosines of the angles between l_1, l_2, l_3 and l_{23}, l_{31}, l_{12}; l_{11}, l_{22}, l_{33} are given by the equations (B) each divided by the proper H and L; namely, for example, cos (l_2, l_{23}) is H_2H_{23}/H_2L_{23}, etc. The identical relation between two systems of four directions in space (cf. Pascal-Timerding, *Repertorium der höheren Geom.*, 2d ed., 1910, p. 74), applied to the directions l_1, l_2, l_3, l_{22} as the elements of the first system and l_1, l_2, l_3, l_{33} as the elements of the second system, gives

$$0=\begin{vmatrix} 1 & \frac{k_{12}}{H_1H_2} & \frac{k_{13}}{H_1H_3} & \frac{k_{122}-H_2H_{21}}{H_1L_{22}} \\ \frac{k_{12}}{H_1H_2} & 1 & \frac{k_{23}}{H_2H_3} & \frac{H_2H_{22}}{H_2L_{22}} \\ \frac{k_{13}}{H_1H_3} & \frac{k_{23}}{H_2H_3} & 1 & \frac{k_{232}-H_2H_{23}}{H_3L_{22}} \\ \frac{k_{133}-H_3H_{31}}{H_1L_{33}} & \frac{k_{233}-H_3H_{32}}{H_2L_{33}} & \frac{H_3H_{33}}{H_3L_{33}} & \frac{\Sigma x_{22}\cdot x_{33}}{L_{22}L_{33}} \end{vmatrix},$$

or

$$0=\begin{vmatrix} H_1^2 & k_{12} & k_{13} & k_{122}-H_2H_{21} \\ k_{12} & H_2^2 & k_{23} & H_2H_{22} \\ k_{13} & k_{23} & H_3^2 & k_{232}-H_2H_{23} \\ k_{133}-H_3H_{31} & k_{233}-H_3H_{32} & H_3H_{33} & \Sigma x_{22}x_{33} \end{vmatrix}.$$

Use V as a symbol for

$$\begin{vmatrix} H_1^2 & k_{12} & k_{13} \\ k_{12} & H_2^2 & k_{23} \\ k_{13} & k_{23} & H_3^2 \end{vmatrix};$$

then the identical relation above is

$$\Sigma x_{22}x_{33}=\frac{-1}{V}\begin{vmatrix} H_1^2 & k_{12} & k_{13} & k_{122}-H_2H_{21} \\ k_{12} & H_2^2 & k_{23} & H_2H_{22} \\ k_{13} & k_{23} & H_3^2 & k_{232}-H_2H_{23} \\ k_{133}-H_3H_{31} & k_{233}-H_3H_{32} & H_3H_{33} & 0 \end{vmatrix}. \qquad (D_1)$$

In precisely the same way, it can be proved that

$$\Sigma x_{23}^2=\frac{-1}{V}\begin{vmatrix} H_1^2 & k_{12} & k_{13} & s-k_{231} \\ k_{12} & H_2^2 & k_{23} & H_2H_{23} \\ k_{13} & k_{23} & H_3^2 & H_3H_{32} \\ s-k_{231} & H_2H_{23} & H_3H_{32} & 0 \end{vmatrix}, \qquad (D_1')$$

$$\Sigma x_{12}x_{13}=\frac{-1}{V}\begin{vmatrix} H_1^2 & k_{12} & k_{13} & H_1H_{12} \\ k_{12} & H_2^2 & k_{23} & H_2H_{21} \\ k_{13} & k_{23} & H_3^2 & s-k_{123} \\ H_1H_{13} & s-k_{132} & H_3H_{31} & 0 \end{vmatrix},$$

$$\Sigma x_{11}x_{23}=\frac{-1}{V}\begin{vmatrix} H_1^2 & k_{12} & k_{13} & H_1H_{11} \\ k_{12} & H_2^2 & k_{23} & k_{121}-H_1H_{12} \\ k_{13} & k_{23} & H_3^2 & k_{131}-H_1H_{13} \\ s-k_{231} & H_2H_{23} & H_3H_{32} & 0 \end{vmatrix}.$$

The eight other terms on the right of the equations (G) and (C) can be obtained from these by cyclic permutation of the subscripts.

The equations (D_1), (D_1'), for example, can be easily changed to the form

$$\Sigma x_{22}x_{33}=\frac{1}{V\begin{vmatrix} H_2^2 & k_{23} \\ k_{23} & H_3^2 \end{vmatrix}}\begin{vmatrix} k_{122}-H_2H_{21} & k_{12} & k_{13} \\ H_2H_{22} & H_2^2 & k_{23} \\ k_{232}-H_2H_{23} & k_{23} & H_3^2 \end{vmatrix}\cdot\begin{vmatrix} k_{133}-H_3H_{31} & k_{12} & k_{13} \\ k_{233}-H_3H_{32} & H_2^2 & k_{23} \\ H_3H_{33} & k_{23} & H_3^2 \end{vmatrix}$$

$$-\frac{1}{\begin{vmatrix} H_2^2 & k_{23} \\ k_{23} & H_3^2 \end{vmatrix}}\begin{vmatrix} 0 & k_{233}-H_3H_{32} & H_3H_{33} \\ H_2H_{22} & H_2^2 & k_{23} \\ k_{232}-H_2H_{23} & k_{23} & H_3^2 \end{vmatrix}, \quad (E_1)$$

$$\Sigma x_{23}^2=\frac{1}{V\begin{vmatrix} H_2^2 & k_{23} \\ k_{23} & H_3^2 \end{vmatrix}}\begin{vmatrix} s-k_{231} & k_{12} & k_{13} \\ H_2H_{23} & H_2^2 & k_{23} \\ H_3H_{32} & k_{23} & H_3^2 \end{vmatrix}^2-\frac{1}{\begin{vmatrix} H_2^2 & k_{23} \\ k_{23} & H_3^2 \end{vmatrix}}\begin{vmatrix} 0 & H_2H_{23} & H_3H_{32} \\ H_2H_{23} & H_2^2 & k_{23} \\ H_3H_{32} & k_{23} & H_3^2 \end{vmatrix}. \quad (E_1')$$

Since by bordering, from (D_1), can be obtained

$$\Sigma x_{22}x_{33}=\frac{-1}{V\begin{vmatrix} H_2^2 & k_{23} \\ k_{23} & H_3^2 \end{vmatrix}}\begin{vmatrix} H_1^2 & k_{12} & k_{13} & k_{122}-H_2H_{21} & 0 & 0 \\ k_{12} & H_2^2 & k_{23} & H_2H_{22} & 0 & 0 \\ k_{13} & k_{23} & H_3^2 & k_{232}-H_2H_{23} & 0 & 0 \\ k_{133}-H_3H_{31} & k_{233}-H_3H_{32} & H_3H_{33} & 0 & k_{233}-H_3H_{32} & H_3H_{33} \\ k_{12} & H_2^2 & k_{23} & H_2H_{22} & H_2^2 & k_{23} \\ k_{13} & k_{23} & H_3^2 & k_{232}-H_2H_{23} & k_{23} & H_3^2 \end{vmatrix},$$

and this expanded gives (E_1). And so also for all the other terms on the right of the equations (G) and (C).

The equations (G) and (C) are the identical relations between the second derivatives of the elements of a general oblique system of surfaces. On the right of these expressions the first derivatives only appear, in the form given

by the equations (D) or (E). And when $k_{12}=k_{13}=k_{23}=0$, these identical relations reduce to the relations given by Lamé, *Coord. Curv.*, Paris, 1859, (8) p. 76, (9) p. 78. For each of the three surfaces of the triple system, one of the equations (G) is the Gauss relation given in *Disq. gen. c. superficies curvas*, Got., Oct. 8, 1827, p. 119, Art. 11. This is seen at once; since, for example, taking a surface $\rho_1=$const., by a method analogous to that used above in deriving the equation (D_1) the Gauss D, D', D'' are

$$D=\frac{1}{V^{\frac{1}{2}}}\begin{vmatrix} k_{122}-H_2H_{21} & k_{12} & k_{13} \\ H_2H_{22} & H_2^2 & k_{23} \\ k_{232}-H_2H_{23} & k_{23} & H_3^2 \end{vmatrix},$$

$$D'=\frac{1}{V^{\frac{1}{2}}}\begin{vmatrix} s-k_{231} & k_{12} & k_{13} \\ H_2H_{23} & H_2^2 & k_{23} \\ H_3H_{32} & k_{23} & H_3^2 \end{vmatrix},$$

$$D''=\frac{1}{V^{\frac{1}{2}}}\begin{vmatrix} k_{133}-H_3H_{31} & k_{12} & k_{13} \\ k_{233}-H_3H_{32} & H_2^2 & k_{23} \\ H_3H_{33} & k_{23} & H_3^2 \end{vmatrix}.$$

And $E=H_2^2$, $F=k_{23}$, $G=H_3^2$. These values substituted in (G_1) give the Gauss equation.

The equations (C) of course must be equivalent to the equations given by Codazzi, *Ann. di Mat.*, Ser. 3, Vol. II, 1869, p. 285. The connection can be obtained without very much reckoning by using the form of the Codazzi relations standing in Darboux, *Surfaces*, Vol. III, p. 248, (24) and (25). It is thus seen that this equation (24) taken for the surface $\rho_1=$const. is the combination of (G_1) and (C_2), and the equation (25) of (G_1) and (C_3).

BELTON, MONTANA, *August* 19, 1912.

*Transformations and Invariants Connected with Linear Homogeneous Difference Equations and Other Functional Equations.**

By Cora B. Hennel.

Introduction.

The subject of Difference Equations has had an extended development during the last few years. Contributions† of importance have been made by Guichard, Nörlund, Galbrun, Carmichael, Birkhoff, Horn, Ford, Perron, Bôcher, and others.

The object of the present paper is to discuss for the difference equation primarily (Part I), and also for a general type of functional equation (Part II), the question of functions that remain invariant for a certain broad type of transformations. The results assume a very simple and elegant form.

In § 1, it is shown that the most general point transformation that changes every linear homogeneous difference equation into a difference equation that is linear, homogeneous, and of the same order as the given equation, is of the form

$$x = u(\xi), \quad y = \lambda(\xi)\,\eta_\xi, \tag{I}$$

where $u(\xi)$ satisfies one of the two relations

$$u(\xi + 1) - u(\xi) = \pm 1.$$

In § 2, the group character of the transformations (I) is verified and the existence of a certain subgroup is shown.

In § 3, a list of definitions of terms used in the remaining part of the paper is given.

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* Presented before the American Mathematical Society, December, 1912.

† See: Guichard, *Ann. de l'Éc. Norm.*, ser. (3), Vol. IV (1887), pp. 361–380; Nörlund, *Mém. Acad. Roy. Sc. et Let. de Danemark*, ser. (7), Vol. VI (1911), pp. 309–326; Galbrun, Thesis, Upsala, 1912, pp. 1–68; Carmichael, *Trans. Am. Math. Soc.*, Vol. XII (1911), pp. 99–134; Birkhoff, *Trans. Am. Math. Soc.*, Vol. XII (1911), pp. 243–284; Horn, *Crelle's Journal*, Vol. CXXXVIII (1910), pp. 159–191; Ford, *Trans. Am. Math. Soc.*, Vol. X (1909), pp. 319–336; Perron, *Acta Math.*, Vol. XXXIV (1911), pp. 109–137; Bôcher, *Ann. of Math.*, Vol. XVIII (1911), pp. 71–88. Further references will be found in the papers here referred to.

In §§ 4–7, certain fundamental sets of seminvariants, invariants, semi-covariants and covariants are determined.

In §§ 8–9, a certain general type of functional equation is dealt with, the procedure being along the same lines as that in the discussion of the difference equation in the preceding articles. In § 8, the most general point transformation that changes every linear homogeneous functional equation of the type under consideration into another of the same type and order is determined, the group character of these transformations is verified, and the existence of a certain subgroup is shown. In § 9, fundamental sets of seminvariants, invariants, semi-covariants and covariants are determined.

I wish to express my gratitude to Professor R. D. Carmichael for his aid and encouragement in the preparation of the paper.

PART I. TRANSFORMATIONS AND INVARIANTS CONNECTED WITH LINEAR HOMOGENEOUS DIFFERENCE EQUATIONS.

§ 1. *Determination of Point Transformations Leaving Certain Properties of Linear Homogeneous Difference Equations Invariant.*

We consider the totality of linear homogeneous difference equations. As a type of such equations, we use*

$$y_{x+n}+p_1(x)y_{x+n-1}+p_2(x)y_{x+n-2}+\ldots.+p_{n-1}(x)y_{x+1}+p_n(x)y_x=0, \quad \text{(A)}$$

where $p_1(x), \ldots, p_n(x)$ are functions of x, arbitrary except that $p_n(x)$ is not identically zero, a restriction made necessary by the fact that, if $p_n(x)$ is not different from zero, the equation reduces at once to an equation of order less than n.

If x and y are regarded, for the moment, as coördinates of a point in two-dimensional space, then the most general so-called point transformation is of the form

$$x=\bar{u}(\xi,\eta_\xi), \quad y=\bar{v}(\xi,\eta_\xi).$$

We confine our attention to transformations of this type and equations of type (A), and determine first the most general transformation that transforms every equation (A) into another of the same type and order. That is, we determine the most general point transformation which leaves the following properties of equation (A) invariant:

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* The two forms of functional notation are used to distinguish the dependent variable as a function of the independent variable from other functions of the independent variable that enter into the equation. This notation is used in general throughout the paper.

1) Its character of being a difference equation with argument difference unity;
2) Its degree;
3) Its linearity;
4) Its homogeneity.

For the sake of simplicity, let us consider first an equation of order one:

$$y_{x+1} + p_1(x)\, y_x = 0. \tag{1}$$

By the substitution

$$\left.\begin{aligned} x &= \bar{u}(\xi, \eta_\xi), \\ y_x &= \bar{v}(\xi, \eta_\xi), \end{aligned}\right\} \tag{2}$$

this equation becomes

$$\bar{v}[\phi(\xi), \eta_{\phi(\xi)}] + p_1[\bar{u}(\xi, \eta_\xi)]\, \bar{v}(\xi, \eta_\xi) = 0, \tag{3}$$

where $\phi(\xi)$ denotes what ξ becomes when x is replaced by $x+1$. We proceed to find what restrictions must be made on (2) in order that equation (3) may be a linear homogeneous difference equation of order one.

In the first place, equation (3) is to be linear in η_ξ. To meet this requirement, two conditions must be fulfilled. First, $\bar{u}(\xi, \eta_\xi)$ must be a function of ξ alone; for otherwise, since $p_1(x)$ is arbitrary, $p_1(x)$ can be chosen so that $p_1[\bar{u}(\xi, \eta_\xi)]$ shall be non-linear in η_ξ. Secondly, $\bar{v}(\xi, \eta_\xi)$ must be linear in η_ξ. The transformation, then, must be of the form

$$\begin{aligned} x &= u(\xi), \\ y_x &= \lambda(\xi)\, \eta_\xi + \theta(\xi), \end{aligned}$$

and equation (3) takes the form

$$\lambda[\phi(\xi)]\, \eta_{\phi(\xi)} + \theta[\phi(\xi)] + p_1[u(\xi)]\, \lambda(\xi)\, \eta_\xi + p_1[u(\xi)]\, \theta(\xi) = 0.$$

Now this equation is to be homogeneous. Obviously, a necessary and sufficient condition for this is that $\theta(\xi)$ shall satisfy the equation

$$\theta[\phi(\xi)] + p_1[u(\xi)]\, \theta(\xi) = 0;$$

since p_1 is arbitrary, it follows that $\theta(\xi)$ must be zero.

Finally, the transformed equation, which now has the form

$$\lambda[\phi(\xi)]\, \eta_{\phi(\xi)} + p_1[u(\xi)]\, \lambda(\xi)\, \eta_\xi = 0,$$

is to be a difference equation of order one. That this condition may be fulfilled, one of the two equations

$$\phi(\xi) = \xi \pm 1 \tag{4}$$

must hold.

This last restriction says that as x is changed into $x+1$, ξ becomes either $\xi+1$ or $\xi-1$; it also imposes a restriction on $u(\xi)$, which we shall now determine.

From the relation $x=u(\xi)$, existing between x and ξ, we obtain, at once,

$$\xi = u^{-1}(x).$$

Let us now replace x by $x+1$, and then substitute for x its value $u(\xi)$; thus:

$$\phi(\xi) = u^{-1}(x+1) = u^{-1}[u(\xi)+1].$$

Using the latter value for $\phi(\xi)$ in equations (4), we obtain

$$u^{-1}[u(\xi)+1] = \xi \pm 1,$$

which is equivalent to the two equations

$$u(\xi)+1 = u(\xi+1),$$
$$u(\xi)+1 = u(\xi-1).$$

From the first of these two, we obtain, by transposing,

$$u(\xi+1) - u(\xi) = 1;$$

from the second, we obtain, by replacing ξ by $\xi+1$, and transposing,

$$u(\xi+1) - u(\xi) = -1.$$

That is, $u(\xi)$ must satisfy one of the two equations

$$u(\xi+1) - u(\xi) = \pm 1.$$

We have shown, then, that the most general point transformation that changes every linear homogeneous difference equation of the first order into another of the same type and order, is of the form

$$x = u(\xi),$$
$$y_x = \lambda(\xi)\eta_\xi,$$

where $\lambda(\xi)$ is arbitrary, and $u(\xi)$ satisfies one of the relations

$$u(\xi+1) - u(\xi) = \pm 1,$$

but is otherwise arbitrary.

Consider now the n-th order equation (A). Since the transformation sought must leave invariant properties 1) to 4) of equations of form (A) of any degree, the restrictions on the transformation found necessary for an equation of degree one are necessary also in the case of equations of degree n. That these restrictions are sufficient, as well, is readily seen.

By a transformation of the form

$$x = u(\xi), \quad y_x = \lambda(\xi)\eta_\xi,$$

equation (A) becomes

$$\eta_{\xi+n}+p_1[u(\xi)]\frac{\lambda(\xi+n-1)}{\lambda(\xi+n)}\eta_{\xi+n-1}+\ldots.+p_n[u(\xi)]\frac{\lambda(\xi)}{\lambda(\xi+n)}\eta_\xi=0$$

or

$$\eta_{\xi-n}+p_1[u(\xi)]\frac{\lambda(\xi-\overline{n-1})}{\lambda(\xi-n)}\eta_{\xi-\overline{n-1}}+\ldots.+p_n[u(\xi)]\frac{\lambda(\xi)}{\lambda(\xi-n)}\eta_\xi=0,$$

according as $u(\xi)$ satisfies the relation

$$u(\xi+1)-u(\xi)=1$$

or

$$u(\xi+1)-u(\xi)=-1.$$

Obviously, in either case the transformed equation is a linear homogeneous equation of order n.

This completes the proof of the following theorem:

THEOREM I. *The most general transformation of the form*

$$x=\bar{u}(\xi,\eta_\xi),$$
$$y_x=\bar{v}(\xi,\eta_\xi)$$

that transforms every linear homogeneous difference equation of the form

$$y_{x+n}+p_1(x)\,y_{x+n-1}+\ldots.+p_n(x)\,y_x=0 \qquad \text{(A)}$$

into a difference equation that is linear, homogeneous, and of the same order as the given equation, is

$$\left.\begin{aligned} x&=u(\xi),\\ y_x&=\lambda(\xi)\,\eta_\xi,\end{aligned}\right\} \text{(I)}^*$$

where $u(\xi)$ satisfies one of the two relations

$$u(\xi+1)-u(\xi)=\pm 1,$$

and $\lambda(\xi)$ is arbitrary.

§ 2. *On a Subgroup of the Group of Transformations.*

We have proved that any transformation of type (I) transforms every equation (A) into a difference equation that is linear, homogeneous and of order n, and that any point transformation that does this is of type (I). Obviously, then, the transformations (I) form a group. For we see at once — or can readily show — that any two transformations of type (I) can be replaced by a single transformation that gives the same resulting equation; this latter

* Throughout the discussion we refer to the transformation

$$x=u(\xi),\quad y_x=\lambda(\xi)\,\eta_\xi$$

as (I′) or (I″), according as $u(\xi)$ satisfies the relation $u(\xi+1)-u(\xi)=1$ or $u(\xi+1)-u(\xi)=-1$. If $u(\xi)$ satisfies merely one or the other, indiscriminately, we refer to the transformation as (I).

transformation, since it changes (A) into a difference equation that is linear, homogeneous and of order n, must be of type (I).

The transformations (I′) form a subgroup of the total group (I), while the transformations (I″) do not, as we shall now show.

Consider two transformations of type (I′):

$$x = u'(\xi'), \qquad u'(\xi'+1) - u'(\xi') = 1,$$
$$y_x = \lambda'(\xi')\,\eta'_{\xi'},$$

and

$$\xi' = u''(\xi''), \qquad u''(\xi''+1) - u''(\xi'') = 1,$$
$$\eta'_{\xi'} = \lambda''(\xi'')\,\eta''_{\xi''}.$$

Combining these transformations, we have

$$\left.\begin{aligned} x &= u'[u''(\xi'')], \\ y_x &= \lambda'[u''(\xi'')]\,\lambda''(\xi'')\,\eta''_{\xi''}, \end{aligned}\right\} \quad (5a)$$

with the restrictions

$$u'[u''(\xi'')+1] - u'[u''(\xi'')] = 1,$$
$$u''(\xi''+1) - u''(\xi'') = 1.$$

The first of these restrictions comes from that on $u'(\xi')$ in the first transformation, taken with the relation between ξ and ξ''; the second is obviously that on $u''(\xi'')$ in the second transformation.

Replacing $u''(\xi'')+1$ in the first of these equations by its equivalent $u''(\xi''+1)$ obtained from the second, we have

$$u'[u''(\xi''+1)] - u'[u''(\xi'')] = 1.$$

Now this is precisely the restriction that $u'[u''(\xi'')]$ must satisfy in order that transformation (5a) may be of type (I′). Hence the transformations (I′) form a subgroup of (I).

To show that the transformations (I″) do not form a subgroup, let us use the same two transformations, except that we shall now take as the restrictions on $u'(\xi')$ and $u''(\xi'')$ the following:

$$u'(\xi'+1) - u'(\xi') = -1,$$
$$u''(\xi''+1) - u''(\xi'') = -1.$$

Combining the transformations, we get

$$\left.\begin{aligned} x &= u'[u''(\xi'')], \\ y_x &= \lambda'[u''(\xi'')]\,\lambda''(\xi'')\,\eta''_{\xi''}, \end{aligned}\right\} \quad (5b)$$

with the restrictions

$$u'[u''(\xi'')+1] - u'[u''(\xi'')] = -1,$$
$$u''(\xi''+1) - u''(\xi'') = -1.$$

If in the first of these restrictions we substitute for $u''(\xi'')+1$ its equivalent obtained from the second by replacing ξ by $\xi-1$ and transposing, we obtain:

$$u'[u''(\xi''-1)]-u'[u''(\xi'')]=-1;$$

whence, putting $\xi''+1$ for ξ'', and transposing:

$$u'[u''(\xi''+1)]-u'[u''(\xi'')]=+1.$$

This result shows that while transformation (5b) is indeed of type (I), it is of type (I′) and not (I″). Hence the transformations (I″) do not form a subgroup.

We have shown, incidentally, that the successive operation of two transformations of form (I′) or of form (I″) is equivalent to a single transformation of type (I). Similarly, we might show that the combination of any transformation of form (I′) with one of (I″) is also equivalent to a single transformation of form (I), thus completely verifying the fact that the transformations (I) do indeed form a group.

We sum up these results in the form of a theorem, as follows:

THEOREM II. *The transformations* (I) *form a group, of which the transformations* (I′) *form a subgroup. The transformations* (I″) *do not form a subgroup.*

§ 3. *Definitions.*

We state now definitions of certain terms used in the sections that follow. Terms used in the field of differential equations are given analogous meanings here.

Any equation which can be obtained from equation (A) by a transformation of form (I) is said to be *equivalent* to (A).

An *absolute invariant*, or more briefly, an *invariant*, is any function of the p's and their differences,

$$F(p_1, p_2, \ldots, p_n;\ \Delta p_1, \ldots, \Delta p_n;\ \Delta^2 p_1, \ldots;\ \ldots),$$

which is equal to the same function formed for an equivalent equation.

A *seminvariant* is any function of the p's and their differences which is equal to the same function formed for an equation derived from (A) by a transformation of the form

$$x=x,\quad y_x=\lambda(x)\,\eta_x.$$

A number of invariants [seminvariants], $\alpha_1, \alpha_2, \ldots$, is said to be a *fundamental set* of invariants [seminvariants] if every invariant [seminvariant]

can be expressed as a function of the α's and their differences, and no α can be expressed as a function of the remaining α's and their differences.

A *covariant* is any function of the p's and their differences, and y and its differences, which is equal to the same function formed for an equivalent equation.

A *semi-covariant* is any function of the p's and their differences, and y and its differences, which is equal to the same function formed for an equation derived from (A) by a transformation of the form

$$x = x, \quad y_x = \lambda(x)\,\eta_x.$$

A number of covariants [semi-covariants], $\beta_1, \beta_2, \ldots$, is said to be a *fundamental set* of covariants [semi-covariants] if every covariant [semi-covariant] can be expressed as a function of the β's and their differences, and invariants [seminvariants] and their differences, and no β can be expressed in terms of the remaining β's and their differences and invariants [seminvariants] and their differences.

Let

$$\bar{p}_0(\xi)\eta_{\xi\pm n} + \bar{p}_1(\xi)\eta_{\xi\pm\overline{n-1}} + \ldots + \bar{p}_{n-1}(\xi)\eta_{\xi\pm 1} + \bar{p}_n(\xi)\eta_\xi = 0$$

be any equation equivalent to (A). This equation is said to be a normal form of (A) provided either of the following conditions holds:

$$1) \quad \bar{p}_0(\xi) \equiv \bar{p}_1(\xi) \equiv 1,$$
$$2) \quad \bar{p}_{n-1}(\xi) \equiv \bar{p}_n(\xi) \equiv 1.$$

§ 4. *Fundamental Sets of Seminvariants.*

We determine now certain fundamental sets of seminvariants.

If we make the substitution

$$x = x, \quad y_x = \lambda(x)\,\eta_x \tag{6}$$

in equation (A),

$$y_{x+n} + p_1(x)\,y_{x+n-1} + \ldots + p_n(x)\,y_x = 0, \tag{A}$$

we obtain, after dividing through by $\lambda(x+n)$,

$$\eta_{x+n} + p_1(x)\frac{\lambda(x+n-1)}{\lambda(x+n)}\eta_{x+n-1} + p_2(x)\frac{\lambda(x+n-2)}{\lambda(x+n)}\eta_{x+n-2} + \ldots + p_n(x)\frac{\lambda(x)}{\lambda(x+n)}\eta_x = 0.$$

Let us reduce this to a normal form, by choosing λ so that the coefficient of η_{x+n-1} shall be unity:

$$\frac{\lambda(x+n-1)}{\lambda(x+n)}\,p_1(x) = 1.$$

Obviously it is possible to choose λ in this way, for the equation determining λ is a first-order difference equation which has a solution, provided $p_1(x)$ is not identically zero. We assume, in the determination of a first fundamental set of seminvariants, that $p_1(x)$ is not identically zero.

Dividing both sides of the last equation by $p_1(x)$, we have

$$\frac{\lambda(x+n-1)}{\lambda(x+n)} = \frac{1}{p_1(x)}. \tag{7}$$

Without actually solving this equation, we find what the coefficients

$$p_2(x)\frac{\lambda(x+n-2)}{\lambda(x+n)}, \ \ldots, \ p_n(x)\frac{\lambda(x)}{\lambda(x+n)}$$

become when λ satisfies the equation. Replacing x by $x-1$, in (7), and multiplying the resulting equation by (7), member by member, we obtain

$$\frac{\lambda(x+n-2)}{\lambda(x+n)} = \frac{1}{p_1(x)\,p_1(x-1)}.$$

Replacing x by $x-1$ in this equation, and multiplying the resulting equation by (7), we obtain

$$\frac{\lambda(x+n-3)}{\lambda(x+n)} = \frac{1}{p_1(x)\,p_1(x-1)\,p_1(x-2)}.$$

Obviously, a continuation of this process gives the result

$$\frac{\lambda(x+n-r)}{\lambda(x+n)} = \frac{1}{p_1(x)\,p_1(x-1)\ldots p_1(x-r-1)}, \quad (r=2, \ldots, n).$$

Hence a normal form of (A) is

$$\eta_{x+n} + \eta_{x+n-1} + P_2(x)\,\eta_{x+n-2} + \ldots + P_n(x)\,\eta_x = 0, \tag{8}$$

where

$$P_r(x) = \frac{p_r(x)}{p_1(x)\,p_1(x-1)\ldots p_1(x-r-1)}.$$

The functions $P_r(x)$, $r=2, \ldots, n$, are themselves seminvariants, as we shall now show.

If we apply the transformation

$$x = x, \quad y_x = \lambda'(x)\,\eta'_x \tag{9}$$

to equation (A), we obtain

$$\eta'_{x+n} + p'_1(x)\,\eta'_{x+n-1} + \ldots + p'_n(x)\,\eta'_x = 0, \tag{10}$$

where

$$p'_r(x) = p_r(x)\frac{\lambda'(x+n-r)}{\lambda'(x+n)}, \quad (r=1, \ldots, n).$$

We can reduce this to a normal form by making the further transformation

$$x = x, \quad \eta'_x = \lambda''(x)\,\eta_x, \tag{11}$$

where $\lambda''(x)$ satisfies the relation

$$\frac{\lambda''(x+n-1)}{\lambda''(x+n)} = \frac{1}{p'_1(x)}. \tag{12}$$

Let us take as the value of $\lambda''(x)$, $\frac{\lambda(x)}{\lambda'(x)}$. That this value of λ'' satisfies relation (12) is readily seen, by substituting it for λ'' in (12), making use of the value of $p'_r(x)$ given in (10):

$$\frac{\lambda(x+n-1)}{\lambda(x+n)} \cdot \frac{\lambda'(x+n)}{\lambda'(x+n-1)} = \frac{1}{p_1(x)} \cdot \frac{\lambda'(x+n)}{\lambda'(x+n-1)},$$

whence

$$\frac{\lambda(x+n-1)}{\lambda(x+n)} = \frac{1}{p_1(x)},$$

which is an identity. By means of transformation (11), equation (10) becomes

$$\eta_{x+n} + \eta_{x+n-1} + P'_2(x)\,\eta_{x+n-2} + \ldots + P'_n(x)\,\eta_x = 0, \tag{13}$$

where

$$P'_r(x) = \frac{p'_r(x)}{p'_1(x)\,p'_1(x-1)\ldots p'_1(x-r+1)}, \quad (r = 2, \ldots, n).$$

Now the transformation obtained by combining (9) and (11) is exactly the same as (6); hence equations (13) and (8) are identical. Consequently for any particular r, $P'_r(x)$ and $P_r(x)$ are the same. But $P'_r(x)$ is a function of the coefficients $p'_1, \ldots, p'_n$ of the equation derived from (A) by a transformation of form (6); since it is the same function of the coefficients p' that $P_r(x)$ is of the p's, $P_r(x)$ must be a seminvariant.

We have as seminvariants, then, the functions $P_2(x), \ldots, P_n(x)$. Obviously no P can be expressed in terms of the remaining P's and their differences, and any function of these seminvariants and their differences is a seminvariant. We show, finally, that every seminvariant can be expressed in terms of the P's and their differences.

Let

$$F_1(p_1, p_2, \ldots, p_n;\ \Delta p_1, \ldots;\ \ldots)$$

be a seminvariant. This function must by definition be equal to the same function formed for any equation derived from (A) by a transformation of type (6); it must in particular, therefore, be equal to that formed for the normal form (8):

$$F_1(1, P_2, \ldots, P_n;\ \Delta P_2, \ldots;\ \ldots);$$

that is, it must be a function of the seminvariants $P_2, \ldots, P_n$ and their differences. Hence the P's form a fundamental set of seminvariants.

We have now determined one fundamental set of seminvariants, on the assumption that $p_1(x)$ is not identically zero. On the assumption that $p_{n-1}(x)$ is not identically zero, we can readily determine another, by a method differing from that just employed only in that the second of the two normal forms of (A), instead of the first, is used.

To obtain the second normal form of (A), we first divide the equation through by $p_n(x)$ in order that the coefficient of y_x may be unity, denoting the resulting coefficients by $\pi_n(x), \ldots, \pi_1(x)$ respectively. We then choose λ so that the coefficient of η_{x+1} in the transformed equation (after the equation has been divided through by the coefficient of η_x) shall be unity. Using then, as we have already indicated, a method analogous to that employed in determining the first fundamental set of seminvariants, we find, as a second set, the functions $\Pi_r(x)$, $r = n, \ldots, 2$, where

$$\Pi_r(x) = \frac{\pi_r(x)}{\pi_1(x)\,\pi_1(x+1)\,\ldots\,\pi_1(x+r-1)}.$$

We have yet to consider the exceptional case in which both $p_1(x)$ and $p_{n-1}(x)$ are identically zero. We include this case in the following discussion, which covers every other case as well. Inasmuch as the two cases already discussed are but particular cases of the two general cases which we now consider, we might have omitted the separate discussion of them, taking up at once the general problem. On account of the importance of the two particular cases, however, and because the treatment of them first simplifies the proof for the general cases, it has seemed advisable to adopt the order here followed.

Let us suppose that $p_s(x)$ is the first of the coefficients in (A) (after the coefficient of η_{x+n}, which is unity) that is not identically zero. The equation then has the form

$$y_{x+n} + p_s(x)\,y_{x+n-s} + p_{s+1}(x)\,y_{x+n-s+1} + \ldots + p_{n-1}(x)\,y_{x+1} + p_n(x)\,y_x = 0. \qquad (A_1)$$

By means of the transformation

$$x = x, \quad y_x = \lambda(x)\,\eta_x,$$

where $\lambda(x)$ satisfies the relation

$$\frac{\lambda(x+n-s)}{\lambda(x+n)} = \frac{1}{p_s(x)},$$

we change (A_1) into a form in which the coefficient η_{x+n-s} is unity:

$$\eta_{x+n} + \eta_{x+n-s} + Q_{s+1}(x)\,\eta_{x+n-\overline{s+1}} + \ldots + Q_n(x)\,\eta_x = 0, \qquad (14)$$

where

$$Q_r(x) = p_r(x)\frac{\lambda(x+n-r)}{\lambda(x+n)}, \quad (r = s+1, \ldots, n).$$

In the particular case in which s is unity (which we have already considered), the normal form (14) is unique and the coefficients $Q_r(x)$ are readily expressible in terms of the p's; for any value of s different from unity, the normal form (14) is not unique and the Q's can not be expressed as simple functions of the p's; hence a modification of the method used in the case $s=1$ is necessary.

Although the Q's themselves are not readily expressible in terms of the p's, there are simple combinations of them that are, and we now form a set of such combinations constituting a fundamental set of seminvariants. These functions, which we denote by $S_r(x)$, $r = s+1, \ldots, n$, are:*

$$S_r(x) = Q_r(x)\,Q_r(x-r)\,\ldots\,Q_r(x-\overline{s-1}\,r).$$

Let us express these in terms of the p's by means of relation (12):

$$\begin{aligned}
S_r(x) &= p_r(x)\,p_r(x-r)\ldots p_r(x-\overline{s-1}\,r)\cdot\frac{\lambda(x+n-r)}{\lambda(x+n)} \\
&\qquad\cdot\frac{\lambda(x+n-2r)}{\lambda(x+n-r)}\cdots\frac{\lambda(x+n-sr)}{\lambda(x+n-\overline{r-1}\,s)} \\
&= p_r(x)\ldots p_r(x-\overline{s-1}\,r)\cdot\frac{\lambda(x+n-rs)}{\lambda(x+n)} \\
&= p_r(x)\ldots p_r(x-\overline{s-1}\,r)\cdot\frac{\lambda(x+n-s)}{\lambda(x+n)}\cdot\frac{\lambda(x+n-2s)}{\lambda(x+n-s)}\cdots\frac{\lambda(x+n-rs)}{\lambda(x+n-\overline{r-1}\,s)} \\
&= \frac{p_r(x)\,p_r(x-r)\ldots p_r(x-\overline{s-1}\,r)}{p_s(x)\,p_s(x-s)\ldots p_s(x-\overline{r-1}\,s)}.
\end{aligned}$$

That the S's are seminvariants can be readily verified; that no S can be expressed in terms of the remaining S's and their differences is easily seen. To prove, then, that they form a fundamental set of seminvariants, we need merely show that any seminvariant can be expressed in terms of the S's and their differences. For this purpose let us consider the function

$$F[p_s(x),\ p_{s+1}(x),\ \ldots;\ \Delta p_s(x),\ \ldots;\ \ldots]$$

and show that a function G can be found satisfying the following relation:

* The form of these functions was suggested by the formal process of obtaining from the seminvariants $P_2, \ldots, P_n$ functions independent of $p_1(x)$.

$$F[p_s(x), p_{s+1}(x), \ldots; \ldots] = G[p_s(x)\, p_s(x-s) \ldots p_s(x-\overline{s-1}\, s), \\ p_{s+1}(x) \ldots p_{s+1}(x-\overline{s-1}\, \overline{s+1}), \ldots; \ldots]. \quad (15)$$

A sufficient condition that a G exist, satisfying this equation, is that there exist functions H_r, $r = s, \ldots, n$, satisfying the relations

$$H_r[p_r(x)\, p_r(x-r) \ldots . p_r(x-\overline{s-1}\, r)] = p_r(x). \quad (16)$$

If we write

$$p_r(x)\, p_r(x-r) \ldots . p_r(x-\overline{s-1}\, r) = t,$$

whence we have

$$x = f(t),$$

the existence of such functions is obvious:

$$p_r(x) = p_r[f(t)] = H_r(t).$$

Now let F_1 be any invariant. Since F_1 is an invariant, it is equal to the same function formed for the normal form (14); that is, we have

$$F_1[p_s(x), p_{s+1}(x), \ldots; \ldots] = F_1[1, Q_{s+1}(x), \ldots].$$

But by (15), we have

$$F_1[1, Q_{s+1}(x), \ldots] = G_1[1, Q_{s+1}(x) \ldots Q_{s+1}(x-sr), \ldots] \\ = G_1[1, S_{s+1}(x), \ldots].$$

Hence any seminvariant F_1 can be expressed in terms of the functions $S_r(x)$ and their differences, and therefore the functions $S_r(x)$ form a fundamental set of seminvariants.

Similarly, on the supposition that all coefficients in (A) following $p_{n-m}(x)$ (except, of course, that of y_x, which can not be zero) are identically zero, we obtain as another fundamental set of seminvariants the following:

$$\Sigma_r(x) = \frac{\pi_r(x)\, \pi_r(x+r) \ldots . \pi_r(x+\overline{m-1}\, r)}{\pi_m(x)\, \pi_m(x+m) \ldots . \pi_m(x+\overline{r-1}\, m)}, \quad (r = n, \ldots, m+1).$$

We state these results in the form of a theorem as follows:

THEOREM III. *If all the coefficients in* (A) *preceding* $p_s(x)$ (*except, of course, that of* y_{x+n}, *which is unity*) *are identically zero, the functions* $S_r(x)$, $r = s+1, \ldots, n$, *where*

$$S_r(x) = \frac{p_r(x)\, p_r(x-r) \ldots . p_r(x-\overline{s-1}\, r)}{p_s(x)\, p_s(x-s) \ldots . p_s(x-\overline{r-1}\, s)},$$

form a fundamental set of seminvariants.

If all the coefficients in (A) *following* $p_{n-m}(x)$ *(except, of course, that of* y_x*, which is different from zero) are identically zero, the functions* $\Sigma_r(x)$*,* $r=n, \ldots, m+1$*, where*

$$\Sigma_r(x) = \frac{\pi_r(x)\,\pi_r(x+r) \ldots . \pi_r(x+\overline{m-1}\,r)}{\pi_m(x)\,\pi_m(x+m) \ldots . \pi_m(x+\overline{r-1}\,m)}$$

and

$$\pi_i(x) = \frac{p_{n-i}(x)}{p_n(x)}, \quad \pi_0(x) = \frac{1}{p_n(x)},$$

form a fundamental set of seminvariants.

In the important cases in which $s=1$ *and* $m=1$*, the functions* $S_r(x)$ *and* $\Sigma_r(x)$ *reduce to* $P_r(x)$ *and* $\Pi_r(x)$ *respectively, where*

$$P_r(x) = \frac{p_r(x)}{p_1(x)\,p_1(x-1) \ldots . p_1(x-\overline{r-1})}$$

and

$$\Pi_r(x) = \frac{\pi_r(x)}{\pi_1(x)\,\pi_1(x+1) \ldots . \pi_1(x+\overline{r-1})}.$$

§ 5. *Fundamental Sets of Invariants.*

By a method analogous to that used in the discussion of seminvariants, we now determine certain fundamental sets of invariants. In dealing with transformations of type (I), we must remember that transformations (I′) and (I″) give the transformed equation in different forms. (See page 435.) However, the methods for determining the invariants for the two transformations differ very little, and the results turn out to be the same. Accordingly we shall discuss only the transformation (I′) and assume the result for (I″).

Let us consider first the case in which $p_1(x)$ is not identically zero. We can reduce (A) to a normal form by means of the transformation

$$\left.\begin{aligned} x &= u(\xi), \qquad u(\xi+1)-u(\xi)=1, \\ y_x &= \lambda(\xi)\eta_\xi, \end{aligned}\right\} \quad (17)$$

where

$$\frac{\lambda(\xi+n-1)}{\lambda(\xi+n)} = \frac{1}{p_1[u(\xi)]}.$$

We obtain thus

$$\eta_{\xi+n} + \eta_{\xi+n-1} + P_2[u(\xi)]\,\eta_{\xi+n-2} + \ldots . + P_n[u(\xi)]\,\eta_\xi = 0, \qquad (18)$$

where

$$P_r[u(\xi)] = \frac{p_r[u(\xi)]}{p_1[u(\xi)]\,p_1[u(\xi-1)] \ldots . p_1[u(\xi-r+1)]}.$$

The functions P_r form a fundamental set of invariants, as we shall now show.

Going back to equation (A) and transforming it by means of the transformation

$$\left.\begin{aligned} x &= u'(\xi'), \qquad u'(\xi'+1)-u'(\xi')=1, \\ y_x &= \lambda'(\xi')\eta'_{\xi'}, \end{aligned}\right\} \quad (19)$$

we obtain

$$\eta'_{\xi'+n}+p'_1[u'(\xi')]\eta'_{\xi'+n-1}+\ldots.+p'_n[u'(\xi')]\eta'_{\xi'}=0, \qquad (20)$$

where

$$p'_r[u'(\xi')]=p_r[u'(\xi')]\frac{\lambda'(\xi'+n-r)}{\lambda'(\xi'+n)}, \quad (r=1, \ldots., n).$$

We change this equation to a form in which the coefficient of $\eta_{\xi''+n-1}$ is unity by means of the further transformation

$$\left.\begin{aligned} \xi' &= u''(\xi''), \qquad u''(\xi''+1)-u''(\xi'')=1, \\ r'_{\xi'} &= \lambda''(\xi'')\eta_{\xi''}, \end{aligned}\right\} \quad (21)$$

where

$$\frac{\lambda''(\xi''+n-1)}{\lambda''(\xi''+n)}=\frac{1}{p'_1[u''(\xi'')]}$$

and

$$u'[u''(\xi'')]=u(\xi).$$

This restriction is placed on $u''(\xi)$ in order that the combination of transformations (19) and (21) may change x exactly as the single transformation (I′) does.

By means of (21) we get

$$\eta_{\xi''+n}+\eta_{\xi''+n-1}+P'_2\{u'[u''(\xi'')]\}\eta_{\xi''+n-2}+\ldots+P'_n\{u'[u''(\xi'')]\}\eta_{\xi''}=0, \quad (22)$$

where

$$P'_r\{u'[u''(\xi'')]\}=P'_r[u(\xi)]=\frac{p'_r\{u'[u''(\xi'')]\}}{p'_1\{u'[u''(\xi'')]\}\ldots p'_1\{u'[u''(\xi''-r+1)]\}}.$$

Obviously, the functions P'_r are precisely the same functions of $p'_1, \ldots., p'_n$ as the functions P_r are of $p_1, \ldots., p_n$. But equations (18) and (22) are identical, as we shall now show, and hence

$$P'_r[u(\xi)]=P_r[u(\xi)];$$

that is, the functions $P_r[u(\xi)]$ are invariants.

To show that equations (18) and (22) are identical, we need merely consider the combination of transformations used to get (22):

$$\left.\begin{aligned} x &= u'[u''(\xi'')], \qquad u'[u''(\xi''+1)]-u'[u''(\xi'')]=1, \\ y_x &= \lambda'[u''(\xi'')]\lambda''(\xi'')\eta_{\xi''}, \end{aligned}\right\} \quad (23)$$

and notice the following relations:

$$\frac{\lambda''(\xi''+n-1)}{\lambda''(\xi''+n)} = \frac{1}{p_1'[u'(\xi')]} = \frac{\lambda'(\xi''+n)}{\lambda'(\xi''+n-1)} \cdot \frac{1}{p_1'\{u'[u''(\xi'')]\}},$$

whence

$$\frac{\lambda''(\xi''+n-1)}{\lambda''(\xi''+n)} \cdot \frac{\lambda'(\xi''+n-1)}{\lambda'(\xi''+n)} = \frac{1}{p_1[u(\xi)]} = \frac{\lambda(\xi+n-1)}{\lambda(\xi+n)}.$$

This last relation shows that $\lambda'[u''(\xi'')]\,\lambda''(\xi'')$ can differ from $\lambda(\xi)$ only by a periodic multiplier, and hence transformations (17) and (23) give the same resulting equation. Hence P_r' and P_r are identical, for any particular r. But P_r' is a function of the coefficients $p_1', \ldots, p_n'$ of any equation derived from (A) by a transformation of type (I'); since it is the same function of the coefficients p' that P_r is of the p's, P_r must be an invariant.

We have as invariants, then, the functions $P_2[u(\xi)]$, $P_3[u(\xi)]$, ..., $P_n[u(\xi)]$. Obviously, no P can be expressed in terms of the remaining P's and their differences, and any function of these invariants and their differences is an invariant. We show, finally, that every invariant can be expressed in terms of the P's and their differences.

Let

$$F_1(p_1, p_2, \ldots, p_n;\ \Delta p_1, \ldots;\ \ldots)$$

be an invariant. This function must by definition be equal to the same function formed for any equation derived from (A) by a transformation of type (I'); it must, in particular, therefore, be equal to that formed for the normal form (18):

$$F_1(1, P_2, \ldots, P_n;\ \Delta P_2, \ldots;\ \ldots);$$

that is, it must be a function of the P's and their differences. Hence the functions $P_r[u(\xi)]$, or, expressed in terms of x, $P_r(x)$, $r = 2, \ldots, n$, where

$$P_r(x) = \frac{p_r(x)}{p_1(x)\,p_1(x-1)\ldots p_1(x-r+1)},$$

form a fundamental set of invariants.

On the assumption that $p_{n-1}(x)$ is not identically zero, we readily obtain as a second fundamental set of invariants the functions $\Sigma_r(x)$, $r = n, \ldots, n-2$, where

$$\Sigma_r(x) = \frac{\pi_r(x)}{\pi_1(x)\,\pi_1(x+1)\ldots\pi_1(x+r-1)}.$$

The case in which all the coefficients preceding $p_s(x)$ (except the coefficient of y_{x+n}) and that in which all the coefficients following $p_{n-m}(x)$ (except the coefficient of y_x) are identically zero, lead to the following results:

THEOREM IV. *If all the coefficients in* (A) *preceding* $p_s(x)$ *(except, of course, that of* y_{x+n}*, which is unity) are identically zero, the functions* $S_r(x)$*,* $r=s+1, \ldots, n$*, where*

$$S_r(x) = \frac{p_r(x)\, p_r(x-r) \ldots . p_r(x-\overline{s-1}\, r)}{p_s(x)\, p_s(x-s) \ldots . p_s(x-\overline{r-1}\, s)},$$

form a fundamental set of invariants.

If all the coefficients in (A) *following* $p_{n-m}(x)$ *(except, of course, that of* y_x*, which is different from zero) are identically zero, the functions* $\Sigma_r(x)$*,* $r=n, \ldots, m+1$*, where*

$$\Sigma_r(x) = \frac{\pi_r(x)\, \pi_r(x+r) \ldots . \pi_r(x+\overline{m-1}\, r)}{\pi_m(x)\, \pi_m(x+m) \ldots . \pi_m(x+\overline{r-1}\, m)},$$

form a fundamental set of invariants.

In the important cases in which $s=1$ *and* $m=1$ *the functions* $S_r(x)$ *and* $\Sigma_r(x)$ *reduce to* $P_r(x)$ *and* $\Pi_r(x)$ *respectively, where*

$$P_r(x) = \frac{p_r(x)}{p_1(x)\, p_1(x-1) \ldots . p_1(x-\overline{r-1})}$$

and

$$\Pi_r(x) = \frac{\pi_r(x)}{\pi_1(x)\, \pi_1(x+1) \ldots . \pi_1(x+\overline{r-1})}.$$

§ 6. *Fundamental Sets of Semi-Covariants.*

We determine now certain fundamental sets of semi-covariants of (A).

Let us consider, for the moment, a particular transformation of the form in question, namely, one in which $\lambda(x)$ is a constant λ_1:

$$x = x, \quad y_x = \lambda_1 \eta_x.$$

From the relation between y_x and η_x, we have

$$y_x = \lambda_1 \eta_x;\ y_{x+1} = \lambda_1 \eta_{x+1};\ \ldots .;\ y_{x+n} = \lambda_1 \eta_{x+n}. \tag{24}$$

Now any function that is a semi-covariant must remain the same for every constant value of λ_1; hence it must be independent of λ_1. Obviously, the only way in which λ_1 can be eliminated from equations (24) is by the setting up of a function that is homogeneous and of degree zero in the y's; hence every semi-covariant is expressible in terms of the ratios

$$\frac{y_{x+n-1}}{y_{x+n}},\ \frac{y_{x+n-2}}{y_{n+x}},\ \ldots .,\ \frac{y_x}{y_{x+n}}$$

and their differences, and the p's and their differences.

Among these ratios there exists one relation, namely, equation (A); hence

but $n-1$ of them are independent. Let us take as the independent ratios

$$\frac{y_{x+n-r}}{y_{x+n}} = a_r, \quad r = 1, \ldots, n, \quad r \neq i,$$

where i is any integer between one and n. Furthermore, all the differences of the ratios a_r can be expressed in terms of the a's themselves, or in terms of the a's and the p's by aid of equation (A). Hence every semi-covariant must be of the form

$$F(a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n; p_1, \ldots; \Delta p_1, \ldots; \ldots).$$

Some very simple functions of this form are the following:

$$p_1(x)\frac{y_{x+n-1}}{y_{x+n}}, \quad p_2(x)\frac{y_{x+n-2}}{y_{x+n}}, \quad \ldots, \quad p_n(x)\frac{y_x}{y_{x+n}},$$

and these we readily show to be semi-covariants. Transforming (A) by the transformation

$$x = x, \quad y_x = \lambda(x)\eta_x,$$

we obtain

$$\eta_{x+n} + p_1'(x)\eta_{x+n-1} + \ldots + p_n'(x)\eta_x = 0,$$

where

$$p_r'(x) = p_r(x)\frac{\lambda(x+n-r)}{\lambda(x+n)}, \quad (r = 1, \ldots, n).$$

From the relation between $p_r(x)$ and $p_r'(x)$, and that between y_x and η_x, we obtain

$$p_r(x)\frac{y_{x+n-r}}{y_{x+n}} = p_r'(x)\frac{\lambda(x+n)}{\lambda(x+n-r)}\cdot\frac{y_{x+n-r}}{y_{x+n}} = p_r'(x)\frac{\eta_{x+n-r}}{\eta_{x+n}};$$

hence the functions

$$p_r(x)\frac{y_{x+n-r}}{y_{x+n}} = c_r, \quad (r = 1, \ldots, n)$$

are semi-covariants.

Obviously, no one of the semi-covariants c_r, $r = 1, \ldots, n$, $r \neq i$ can be expressed in terms of the remaining c's and their differences. Moreover, every semi-covariant can be expressed in terms of the c's and their differences, and seminvariants and their differences, as we shall now show, and hence the c's form a fundamental set.

We have already proved that every semi-covariant is expressible in the form

$$F(a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n; p_1, \ldots; \ldots).$$

By means of the relation

$$\frac{y_{x+n-r}}{y_{x+n}} = \frac{c_r}{p_r(x)},$$

we may write F in the form

$$F_1(c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n; p_1, \ldots; \ldots).$$

Now let F_1 be any semi-covariant. It must, by definition, be equal to the same function formed for any equation derived from (A) by a transformation of the form under consideration; it must, in particular, be equal to that set up for the normal form (7):

$$F_1(c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n; P_2, \ldots, P_n; \ldots).$$

Hence every semi-covariant is expressible in terms of the functions c_r, $r = 1, \ldots, n$, $r \neq i$, and seminvariants and their differences.

We state our result in the form of a theorem, as follows:

THEOREM V. *Any $n-1$ of the functions c_r, $r = 1, \ldots, n$, where*

$$c_r = p_r(x) \frac{y_{x+n-r}}{y_{x+n}},$$

form a fundamental set of semi-covariants.

§ 7. *Fundamental Sets of Covariants.*

By a method differing so little from that used in the determination of semi-covariants that it seems unnecessary to state it, it can be proved that the fundamental sets of semi-covariants determined in § 6 are also fundamental sets of covariants. We have then the theorem:

THEOREM VI. *Any $n-1$ of the functions c_r, $r = 1, \ldots, n$, where*

$$c_r = p_r(x) \frac{y_{x+n-r}}{y_{x+n}}$$

form a fundamental set of covariants.

PART II. TRANSFORMATIONS AND INVARIANTS CONNECTED WITH LINEAR HOMOGENEOUS FUNCTIONAL EQUATIONS.

§ 8. *Determination of the Group of Transformations.*

We turn now to a study of the general linear homogeneous functional equation

$$y_{\theta^{(n)}(x)} + p_1(x)\, y_{\theta^{(n-1)}(x)} + \ldots + p_{n-1}(x)\, y_{\theta(x)} + p_n(x)\, y_x = 0, \qquad \text{(B)}$$

where

$$\theta^{(n)}(x) = \theta[\theta^{(n-1)}(x)]; \quad \theta^{(1)}(x) \equiv \theta(x),$$

$\theta(x)$ being a known function.* The procedure is similar to that in the case of the difference equation.

We determine first the most general transformation of the type

$$x = \bar{u}(\xi, \eta_\xi), \quad y_x = \bar{v}(\xi, \eta_\xi) \qquad (25)$$

* It will be noticed that for $\theta(x) = x \pm 1$, we have the difference equation. The difference equation was treated separately and with greater detail because of the importance of the equation and the greater simplicity of the treatment.

that transforms every linear homogeneous functional equation (B) of order n into a functional equation of the same type and order.

For the sake of simplicity we consider first an equation of order one:

$$y_{\theta(x)} + p_1(x)\, y_x = 0.$$

By means of transformation (25), this becomes

$$\bar{v}\,[\phi(\xi), \eta_{\phi(\xi)}] + p_1[\bar{u}(\xi, \eta_\xi)]\,\bar{v}(\xi, \eta_\xi) = 0,$$

where $\phi(\xi)$ denotes what ξ becomes when x is changed to $x+1$.

We readily see that in order that this equation may be linear in η_ξ, $\bar{u}(\xi, \eta_\xi)$ must be a function of ξ alone and $\bar{v}(\xi, \eta_\xi)$ must be linear in η_ξ; in order that it may be homogeneous, $\bar{v}(\xi, \eta_\xi)$ must be of the form $\lambda(\xi)\eta_\xi$. With these restrictions on $\bar{u}$ and $\bar{v}$, the equation becomes

$$\lambda[\phi(\xi)]\,\eta_{\phi(\xi)} + p_1[u(\xi)]\,\eta_\xi = 0, \tag{26}$$

and the transformation takes the form

$$x = u(\xi), \quad y_x = \lambda(\xi)\,\eta_\xi.$$

In order that equation (26) may be a functional equation of the given type, of order unity, it is evident that one of the two relations

$$\left.\begin{aligned} \phi(\xi) &= \theta(\xi), \\ \phi(\xi) &= \theta^{-1}(\xi)^* \end{aligned}\right\} \tag{27}$$

must exist. This restriction says that when x is changed to $\theta(x)$, ξ becomes either $\theta(\xi)$ or $\theta^{-1}(\xi)$; it also imposes a restriction on $u(\xi)$ which we now determine. From the relation $x = u(\xi)$ we obtain at once

$$\xi = u^{-1}(x).$$

Let us now replace x by $\theta(x)$ and then substitute for x its equivalent $u(\xi)$:

$$\phi(\xi) = u^{-1}[\theta(x)] = u^{-1}\{\theta[u(\xi)]\}.$$

Using this latter value for $\phi(\xi)$ in the first of equations (27), we have

$$u^{-1}\{\theta[u(\xi)]\} = \theta(\xi),$$

whence

$$\theta[u(\xi)] = u[\theta(\xi)]. \tag{28a}$$

Using the same value for $\phi(\xi)$ in the second of equations (27), we have

$$u^{-1}\{\theta[u(\xi)]\} = \theta^{-1}(\xi),$$

whence

$$\theta[u(\xi)] = u[\theta^{-1}(\xi)]. \tag{28b}$$

Consider now the n-th order equation (B). Since the transformation sought must change every linear homogeneous functional equation into another of the

* If $\theta(\xi)$ has more than one inverse, any inverse can be taken for $\theta^{-1}(\xi)$.

same type and order, the restrictions on the transformation (25) found necessary for the case $n=1$ are necessary also for the general case. That they are sufficient as well is readily verified.

The group character of these transformations and the existence of a subgroup composed of those transformations in which $u(\xi)$ satisfies relation (28a) can be readily shown by the method used in § 2.

We have, then, the following theorems:

THEOREM VII. *The most general point transformation that transforms every linear homogeneous functional equation of the form*

$$y_{\theta^{(n)}(x)} + p_1(x)\, y_{\theta^{(n-1)}(x)} + \ldots + p_{n-1}(x)\, y_{\theta(x)} + p_n(x)\, y_x = 0$$

into a functional equation of the same type and order is of the form

$$\left.\begin{aligned} x &= u(\xi), \\ y_x &= \lambda(\xi)\eta_\xi, \end{aligned}\right\} \quad \text{(II)}$$

where $u(\xi)$ satisfies one of the two relations

$$\theta[u(\xi)] = u[\theta(\xi)],$$
$$\theta[u(\xi)] = u[\theta^{-1}(\xi)].$$

THEOREM VIII. *The transformations of type* (II) *in which $u(\xi)$ satisfies the relation*

$$\theta[u(\xi)] = u[\theta(\xi)]$$

form a subgroup of the total group, while those in which $u(\xi)$ satisfies the relation

$$\theta[u(\xi)] = u[\theta^{-1}(\xi)]$$

do not.

§ 9. *Fundamental Sets of Seminvariants, Invariants, Semi-Covariants and Covariants.*

Fundamental sets of seminvariants, invariants, semi-covariants and covariants of the functional equation (B) can be determined by methods similar to those used in determining the corresponding functions in connection with the difference equation (A). Indeed the methods to be used here differ so little from those employed in the case of the difference equation that it seems unnecessary to do more than merely write out the results. We state as proved, then, the following theorems:

THEOREM IX. *If every coefficient in* (B) *preceding $p_s(x)$ (except, of course, the coefficient of $y_{\theta^{(n)}(x)}$, which is unity) is identically zero, the functions*

$I_r(x)$, $r = s+1, \ldots, n$, *where*

$$I_r(x) = \frac{p_r(x)\, p_r[\theta^{(-r)}(x)] \ldots . p_r[\theta^{(-\overline{s-1}\, r)}(x)]}{p_s(x)\, p_s[\theta^{(-s)}(x)] \ldots . p_s[\theta^{(-\overline{r-1}\, s)}(x)]}$$

and

$$\theta^{(-i)}(x) = \theta^{-1}[\theta^{(-\overline{i-1})}(x)],\ \theta^{(-1)}(x) \equiv \theta^{-1}(x),$$

form a fundamental set of seminvariants and also a fundamental set of invariants.

If every coefficient in (B) *following* $p_{n-m}(x)$ (*except, of course, that of* y_x, *which must be different from zero*) *is identically zero, the functions* $\Omega_r(x)$, $r = n, \ldots, m+1$, *where*

$$\Omega_r(x) = \frac{\pi_r(x)\, \pi_r[\theta^{(r)}(x)] \ldots . \pi_r[\theta^{(\overline{m-1}\, r)}(x)]}{\pi_m(x)\, \pi_m[\theta^{(m)}(x)] \ldots . \pi_m[\theta^{(\overline{r-1}\, m)}(x)]}$$

and

$$\theta^{(i)}(x) = \theta[\theta^{(i-1)}(x)],\ \theta^{(1)}(x) \equiv \theta(x),$$

form a fundamental set of seminvariants and also a fundamental set of invariants.

THEOREM X. *Any* $n-1$ *of the functions* $c_r(x)$, $r = 1, \ldots, n$, *where*

$$c_r(x) = p_r(x)\, \frac{y_{\theta^{(n-r)}}(x)}{y_{\theta^{(n)}}(x)},$$

form a fundamental set of semi-covariants and also a fundamental set of covariants.

INDIANA UNIVERSITY, *June*, 1912.

Projective Differential Geometry of Rational Cubic Curves.

By Joseph A. Nyberg.

Introduction.

In his "Projective Differential Geometry of Curves and Ruled Surfaces" Wilczynski has shown that the general projective theory of plane curves may be based upon the consideration of a linear differential equation of the third order

$$y''' + 3p_1 y'' + 3p_2 y' + p_3 y = 0. \tag{1}$$

That this can be done depends essentially upon the fact that the coefficients p_1, p_2, p_3 have the same significance for all integral curves of (1), and that the most general integral curve of (1) is obtained from any particular one by a projective transformation. The invariants of a linear homogeneous differential equation therefore characterize the projective properties of its integral curves.

The object of this paper is to apply to certain particular rational cubic curves the interpretations which Wilczynski has given to the invariants and covariants of equation (1). Before stating the problem more in detail it is necessary to summarize briefly those theories which are to be applied.

The most general transformation which converts (1) into another equation of the same form and order is:

$$x = f(\xi), \quad y = \lambda(\xi)\zeta, \tag{2}$$

where $f(\xi)$ and $\lambda(\xi)$ are arbitrary functions of ξ. All transformations of the form (2) form an infinite continuous group which we shall denote by "G." Any differential equation obtained from (1) by a transformation of the form (2) is said to be equivalent to (1). Any function of p_1, p_2, p_3 and of their derivatives which has the same value as that same function formed from the coefficients of an equivalent equation is called an *absolute invariant.* If such a function contains y or any of its derivatives it is called a *covariant.* An irreducible rational integral expression

$$f(y, y', y'', \ldots, p_1, p_2, p_3, p_1', p_2', p_3', \ldots)$$

is called a *relative* covariant if the equation $f = 0$ has as its consequence the same equation in the new variables, *i. e.*,

$$f(\zeta, \zeta', \zeta'', \ldots) = 0.$$

The transformation (2) may be decomposed into two others, a transformation of the dependent variable of the form

$$y = \lambda(x)\,\zeta \tag{3}$$

followed by a transformation of the independent variable

$$x = f(\xi). \tag{4}$$

All transformations of the form (3) form a subgroup of G. If we make a transformation of this form, equation (1) becomes

$$\zeta''' + 3\pi_1\zeta'' + 3\pi_2\zeta' + \pi_3\zeta = 0, \tag{5}$$

where

$$\left.\begin{aligned} \pi_1 &= \frac{1}{\lambda}(\lambda' + p_1\lambda), \\ \pi_2 &= \frac{1}{\lambda}(\lambda'' + 2p_1\lambda' + p_2\lambda), \\ \pi_3 &= \frac{1}{\lambda}(\lambda''' + 3p_1\lambda'' + 3p_2\lambda' + p_3\lambda). \end{aligned}\right\} \tag{6}$$

Those functions of p_1, p_2, p_3 and of their derivatives which remain invariant under all transformations of this form are called *seminvariants* and *semi-covariants.* The fundamental seminvariants are:

$$P_2 = p_2 - p_1^2 - p_1', \quad P_3 = p_3 - 3p_1p_2 + 2p_1^3 - p_1''. \tag{7}$$

The fundamental semi-covariants are y itself and

$$z = y' + p_1 y, \quad \rho = y'' + 2p_1y' + p_2y. \tag{8}$$

Under the transformation (4) equation (1) becomes

$$y''' + 3\bar{p}_1y'' + 3\bar{p}_2y' + \bar{p}_3y = 0, \tag{9}$$

where

$$\bar{p}_1 = \frac{1}{\xi'}(p_1 + \eta), \quad \bar{p}_2 = \left(\frac{1}{\xi'}\right)^2(p_2 + \eta p_1 + \frac{1}{3}\mu + \frac{1}{2}\eta^2), \quad \bar{p}_3 = \left(\frac{1}{\xi'}\right)^3 p_3, \tag{10}$$

the following abbreviations being also used:

$$\xi' = \frac{d\xi}{dx}, \quad \eta = \frac{\xi''}{\xi'}, \quad \mu = \eta' - \frac{1}{2}\eta^2.$$

The most important relative invariants for our purpose are

$$\theta_3 = P_3 - \frac{3}{2}P_2', \quad \theta_8 = 6\theta_3\theta_3'' - 7(\theta_3')^2 + 27P_2\theta_3^2. \tag{11}$$

It is convenient for some purposes to reduce the equation (1) to certain standard forms: the semi-canonical form, the Laguerre-Forsyth and the Halphen canonical forms. The first is characterized by the absence of the term involving $\frac{d^2y}{dx^2}$. The second is characterized by the absence of the terms involving $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$. The Halphen canonical form is characterized by the conditions $\theta_3 = 1$ and $p_1 = 0$.

As the letter x will be needed to represent a cartesian coordinate, we shall hereafter denote the independent variable by t. Let y_1, y_2, y_3 be the solutions of equation (1). We shall interpret them as the homogeneous coordinates of a point P_y in a plane, the triangle of reference being the x axis, the y axis and the line at infinity, and the unit point being so chosen that $\frac{y_2}{y_1} = x$ and $\frac{y_3}{y_1} = y$ are cartesian coordinates. As t varies, P_y describes a plane curve and the two other points P_z and P_ρ, determined by the semi-covariants (8), describe curves C_z and C_ρ which are semi-covariantly connected with C_y. P_z is a point on the tangent to C_y at P_y, and P_ρ is some point of the plane, not on the tangent if P_y is not a point of inflection. These three points may therefore be taken as the vertices of a triangle of reference. More specifically we define the coordinates of a point with respect to the triangle of reference $P_y P_z P_\rho$ as follows: Any expression of the form

$$\sigma = x_1 y + x_2 z + x_3 \rho$$

assumes three values $\sigma_1, \sigma_2, \sigma_3$ if we substitute for y in succession the three solutions y_1, y_2, y_3. Then $\sigma_1, \sigma_2, \sigma_3$ will be the homogeneous rectangular coordinates of a point P_σ whose homogeneous coordinates with respect to the triangle $P_y P_z P_\rho$ will be proportional to x_1, x_2, x_3, the unit point of the latter system being chosen in an appropriate fashion. The cartesian coordinates of the same point will be $x = \frac{\sigma_2}{\sigma_1}$, $y = \frac{\sigma_3}{\sigma_1}$.

Since a change of the independent variable displaces P_z along the tangent, it follows that unless the independent variable be chosen in some special way, the curve C_z may be thought of as arbitrary. If, however, the differential equation is in the Halphen canonical form, P_z is the intersection of the tangent with the line upon which are situated the three points of inflection of the eight-pointic nodal cubic of P_y. The rectangular homogeneous coordinates of this point, P_c, are given by the covariant expressions*

$$\theta_3' y_i + 3\theta_3 z_i, \qquad (i = 1, 2, 3), \tag{12}$$

*Wilczynski, "Projective Differential Geometry of Curves and Ruled Surfaces," p. 85.

which, in accordance with the above statement, reduce to $3z_i$ for $\theta_3 = 1$. Its locus will be designated as the curve C_c.

At any point of the curve C_y the equation of the osculating conic, referred to the triangle of reference $P_y P_z P_\rho$, is

$$x_2^2 - 2x_1 x_3 + P_2 x_3^2 = 0. \tag{13}$$

The eight-pointic cubics at any point P_y have a ninth point in common which is called the Halphen point of P_y. It will be denoted by P_h. Its coordinates referred to the triangle $P_y P_z P_\rho$ are given by the expressions*

$$\begin{aligned} x_1 &= 7(5\theta_3'\theta_8 - 756\theta_3^4)^2 + 25\theta_8^3 + 1575\theta_3^2\theta_8^2 P_2, \\ x_2 &= 210\theta_3\theta_8(5\theta_3'\theta_8 - 756\theta_3^4), \\ x_3 &= 3150\theta_3^2\theta_8^2. \end{aligned}$$

The object of this paper may now be stated as follows: Given a rational cubic curve in cartesian coordinates; to set up a linear differential equation of the third order determined by it; to determine the nature of the semi-covariant curves C_z and C_ρ; to find the equation of the osculating conic in cartesian coordinates, study its nature and the locus of its centers; to determine the locus of the Halphen points and the curve C_c.

1. *The Cubical Parabola.*

The cubic equation $y = ax^3$ may be written in the homogeneous form

$$ay_2^3 - y_3 y_1^2 = 0, \tag{1}$$

and, therefore, the curve may be represented parametrically thus:

$$y_1 = 1, \quad y_2 = t, \quad y_3 = at^3, \tag{2}$$

the cartesian coordinates being defined by the equations

$$x = \frac{y_2}{y_1}, \quad y = \frac{y_3}{y_1}.$$

To determine the differential equation

$$y''' + 3p_1 y'' + 3p_2 y' + p_3 y = 0,$$

whose fundamental solutions are y_1, y_2, y_3, we need to solve the following system for p_1, p_2, p_3:

$$y_i''' + 3p_1 y_i'' + 3p_2 y_i' + 3p_3 y_i = 0, \qquad (i = 1, 2, 3). \tag{3}$$

We find

$$p_1 = -\frac{1}{3t}, \quad p_2 = 0, \quad p_3 = 0. \tag{4}$$

---

* Wilczynski, *ibid.*, p. 68.

Therefore the theory of the cubic curve $y = ax^3$ may be based upon the equation

$$y''' - \frac{1}{t}y'' = 0. \tag{5}$$

The parametric equations of the curve C_z are

$$z_i = y_i' + p_1 y_i, \qquad (i = 1, 2, 3),$$

or

$$z_1 = -\frac{1}{3t}, \quad z_2 = \frac{2}{3}, \quad z_3 = \frac{8}{3}at^2. \tag{6}$$

Eliminating t, the equation becomes

$$a z_2^3 - z_3 z_1^2 = 0,$$

which is the original curve itself, although P_y and P_z do not coincide. *The point P_z which corresponds to any point P_y of the cubic is found by drawing the tangent to the cubic at P_y. The second point of intersection of this tangent with the curve is P_z.*

The parametric equations of the curve C_ρ are

$$\rho_i = y_i'' + 2p_1 y_i' + p_2 y_i, \qquad (i = 1, 2, 3),$$

or

$$\rho_1 = 0, \quad \rho_2 = -\frac{2}{3t}, \quad \rho_3 = 4at. \tag{7}$$

Hence the curve C_ρ is the line at infinity. P_ρ being one of the vertices of the triangle of reference, and being at infinity, it follows that two sides of the triangle of reference $P_y P_z P_\rho$ are always parallel. The slope of the parallel sides is $y : x$ or $\rho_3 : \rho_2 = -6at^2$.

The parametric equations of the curve C_σ (which is the locus of P_z when equation (1) is in the Halphen canonical form) are

$$\sigma_i = \theta_3' y_i + 3\theta_3 z_i, \qquad (i = 1, 2, 3),$$

or

$$\sigma_1 = \frac{60}{27t^3} \cdot \frac{4}{3t}, \quad \sigma_2 = \frac{60}{27t^3} \cdot \frac{1}{3}, \quad \sigma_3 = -\frac{5}{3} \cdot \frac{60}{27t^3} \cdot at^2.$$

Its cartesian equation, obtained by eliminating t, is

$$y = -80ax^3. \tag{8}$$

It is again a cubical parabola.

The equation of the osculating conic at P_y, referred to the particular triangle of reference at that point, is

$$x_2^2 - 2x_1 x_3 + P_2 x_3^2 = 0.$$

Since

$$P_2 = p_2 - p_1^2 - p_1' = -\frac{4}{9t^2},$$

the equation becomes

$$x_2^2 - 2\,x_1 x_3 - \frac{4}{9\,t^2}\,x_3^2 = 0. \tag{9}$$

In order to find the equation of this conic in cartesian coordinates, we form the three expressions

$$\sigma_i = x_1 y_i + x_2 z_i + x_3 \rho_i, \qquad (i = 1, 2, 3),$$

solve them for x_1, x_2, x_3 and then substitute these values in equation (9).

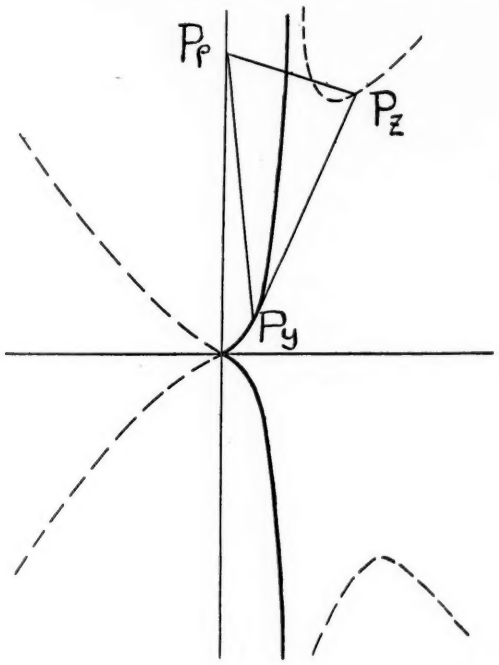


FIG. 1.

Putting $\frac{\sigma_2}{\sigma_1} = x$, $\frac{\sigma_3}{\sigma_1} = y$, we have

$$\frac{x_2}{x_1} = 3 \cdot \frac{6\,at^2x + y - 7\,at^3}{20\,at^2 + 6\,atx + y/t}, \quad \frac{x_3}{x_1} = \frac{9\,t}{2} \cdot \frac{y + 4\,at^3 - 3\,axt^2}{20\,at^2 + 6\,atx + y/t}.$$

The equation of the conic in cartesian coordinates is therefore

$$45\,a^2t^4x^2 - y^2 + 15\,at^2xy - 24\,a^2t^5x - 40\,at^3y + 5\,a^2t^6 = 0. \tag{12}$$

On examining this equation we can see that *the osculating conic never degenerates except at the origin, where it reduces to a pair of coincident lines. Everywhere else it is a hyperbola.*

The centers of the osculating hyperbolas are given by the equations

$$x = \frac{8}{5}t, \quad y = -8at^3.$$

They are therefore on the curve

$$y = a\left(-\frac{5x}{4}\right)^3, \tag{13}$$

which is again a cubical parabola. (See Fig. 1.) (In the figures, C_y is represented by the unbroken line ———; C_x, by — — —; C_ρ, by ------; centers of conics, by — - — - — .)

To find the Halphen point we must first compute the following quantities:

$$7\,(5\,\theta_3'\theta_8 - 756\,\theta_3^4)^2 + 25\,\theta_8^3 + 1575\,\theta_3^2\theta_8^2 P^2 = \frac{1}{27^6}\cdot\frac{1}{t^{24}}\cdot(219520)\cdot 10^8,$$

$$210\,\theta_3\theta_8\,(5\,\theta_3'\theta_8 - 756\,\theta_3^4) = \frac{1}{27^6}\cdot\frac{1}{t^{22}}\cdot(691488)\cdot 10^8,$$

$$3150\,\theta_3^2\theta_8^2 = \frac{1}{27^6}\cdot\frac{1}{t^{22}}\,(889056)\cdot 10^8.$$

Its coordinates are therefore

$$x_1 = \frac{\alpha}{t^2}, \quad x_2 = \frac{\beta}{t}, \quad x_3 = \gamma,$$

where

$$\alpha = 219520, \quad \beta = 691488, \quad \gamma = 889056.$$

The rectangular homogeneous coordinates are

$$\sigma_1 = \frac{1}{3t^2}\,(3\alpha - \beta), \quad \sigma_2 = \frac{1}{3t}\,(3\alpha + 2\beta - 2\gamma), \quad \sigma_3 = \frac{t}{3}\,(3\alpha + 8\beta + 12\gamma),$$

so that

$$x = -8\,t, \quad y = -512\,a t^3. \tag{14}$$

These two equations show that P_h is again a point of the cubic, as the general notion of an Halphen point enables us to predict. From the signs of the coordinates it follows that P_h is always in the opposite quadrant from P_y. The class of points consisting of successive Halphen points is not a closed class, since the x coordinate of each P_h is 64 times as great as the second preceding one, and so the original point can never be reached.

2. *The Semi-cubical Parabola.*

The curve $y^2 = \frac{1}{a}x^3$ or $a\,y_3^2 y_1 - y_2^3 = 0$ can be represented parametrically by the equations

$$y_1 = 1, \quad y_2 = at^2, \quad y_3 = at^3. \tag{1}$$

The differential equation whose solutions are y_1, y_2, y_3, is

$$y''' - \frac{2}{t}y'' + \frac{2}{t^2}y' = 0. \tag{2}$$

Using the methods of the previous paragraph, we find the equations of the curves C_z and C_ρ to be respectively

$$y^2 = -\frac{49}{32a}x^3, \tag{3}$$

$$x = 0. \tag{4}$$

The equation of C_c becomes

$$y^2 = -\frac{80}{a}x^3, \tag{5}$$

which is again a semi-cubical parabola.

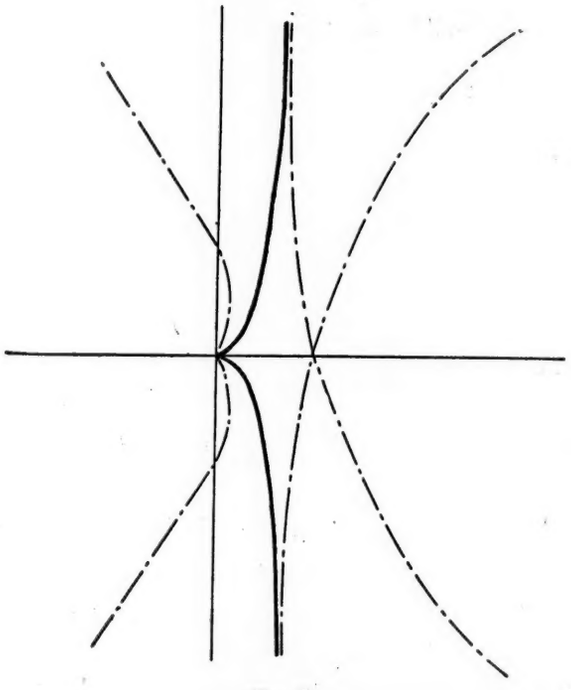


FIG. 2.

The equation of the osculating conic is

$$45\,t^2x^2 + 5\,y^2 - 24\,txy + 15\,t^4ax - 40\,at^3y - a^2t^6 = 0, \tag{6}$$

which is an ellipse except at the origin, where it degenerates into a pair of coincident straight lines.

The centers of these conics lie on a semi-cubical parabola

$$y^2 = \frac{32}{5a}x^3. \quad \text{(See Fig. 2.)} \tag{7}$$

The cartesian coordinates of the Halphen point are

$$x = a\left(\frac{t}{8}\right)^2, \quad y = a\left(\frac{-t}{8}\right)^3.$$

From the choice of parameters in equation (1), at^2 is the x coordinate of P_y. The x of P_h is then 1/64 of the x of P_y. Hence the class of successive Halphen points of P_y is not closed, as the original point is never reached again. As in the case of the cubic $y = ax^3$, P_h always lies in a different quadrant from P_y.

3. *The Cissoid of Diocles.*

The equations of the cissoid of Diocles in cartesian and homogeneous coordinates are, respectively,

$$y^2 = \frac{1}{b} \cdot \frac{x^3}{2a - x}, \quad (a > 0,\ b > 0),$$

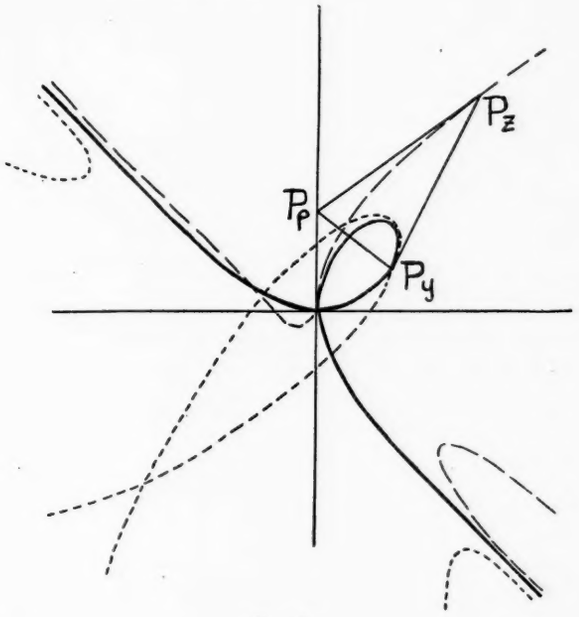


FIG. 3.

and

$$2ab\,y_3^2 y_1 - b\,y_3^2 y_2 - y_2^3 = 0;$$

so that we may put

$$y_1 = 1 + bt^2, \quad y_2 = 2abt^2, \quad y_3 = 2abt^3. \tag{1}$$

The differential equation whose fundamental solutions are y_1, y_2, y_3 is

$$y''' - \frac{2}{t} y'' + \frac{2}{t^2} y' = 0. \tag{2}$$

The equations of the curves C_z and C_p are

$$y^2 = -\frac{49}{32b} \cdot \frac{x^3}{2a - x}, \tag{3}$$

$$x = 0. \tag{4}$$

The equation of C_c becomes

$$y^2 = -\frac{80}{b} \cdot \frac{x^3}{2a + x}, \tag{5}$$

and this is again a cissoid. (See Fig. 3.)

The equation of the osculating conic is

$$x^2t^2(b^2t^4+15bt^2-45)-5y^2-8txy(5bt^2-3)-2xct^4(2bt^2+15)$$
$$+80cyt^3+4c^2t^6=0,\quad (c=ab). \qquad (6)$$

At the origin the osculating conic degenerates, and for $x=a\frac{1+\sqrt{5}}{3}$ *it is a parabola. For* $0<x<a\frac{1+\sqrt{5}}{3}$ *the conics are ellipses, and for* $x>a\frac{1+\sqrt{5}}{3}$ *they are hyperbolas.*

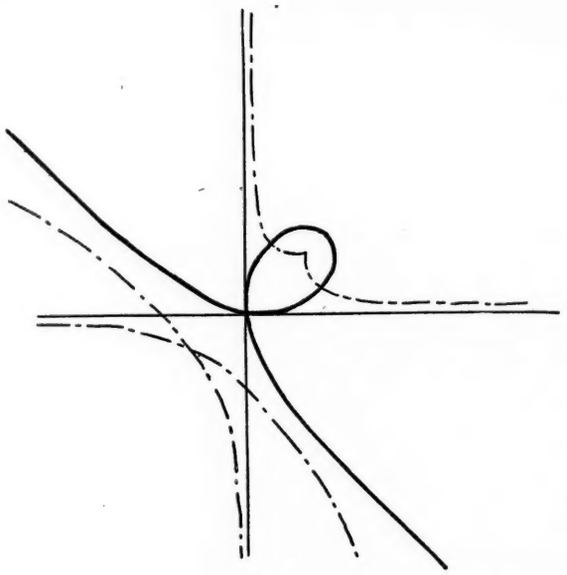


FIG. 4.

The center of the osculating conic has the coordinates

$$x=5ct^2\frac{2bt^2-1}{5b^2t^4-5bt^2-1},\quad y=4ct^3\cdot\frac{5+bt^2}{5b^2t^4-5bt^2-1}.$$

Its locus is therefore a unicursal quintic. (See Fig. 4.)

The cartesian coordinates of P_h are

$$x=abt^2\frac{6\alpha+8\beta}{3\alpha(1+bt^2)-2\beta(1-2bt^2)+2\gamma},$$

$$y=abt^3\frac{6\alpha+14\beta+16\gamma}{3\alpha(1+bt^2)-2\beta(1-2bt^2)+2\gamma},$$

where

$$\alpha=3380608,\quad \beta=-2469600,\quad \gamma=889056.$$

Between these there is the relation

$$\frac{y}{x}=t\cdot\frac{6\alpha+14\beta+16\gamma}{6\alpha+8\beta}=-\frac{t}{8}.$$

Consequently, to find the Halphen point of P_y, draw the line through the origin with the slope $-t/8$. Its second intersection with the given curve C_y is the desired point P_h. We may note that here, as in the previous two cases, P_h never lies in the same quadrant as P_y, and that the class of successive Halphen points of P_y is not closed. The latter remark also follows from the fact that these three curves are projectively equivalent to each other.

4. *The Folium of Descartes.*

We choose the parametric representation

$$y_1 = 1 + t^3, \quad y_2 = at, \quad y_3 = at^2$$

to represent the folium whose cartesian equation is

$$x^3 + y^3 - axy = 0.$$

The parametric equations of C_z are

$$x = \frac{a}{2t^2} \cdot \frac{1}{1+t^3}, \quad y = \frac{a}{2t} \cdot \frac{2+t^3}{1+t^3}.$$

The locus of C_ρ is, as in the previous case,

$$x = 0.$$

The parametric equations of C_c are

$$x = \frac{a}{1+t^3} \cdot \frac{-2t^6+5t^3-1}{2t^2}, \quad y = \frac{a}{1+t^3} \cdot \frac{-t^6+5t^3-2}{2t},$$

a unicursal sextic.

The equation of the osculating conic at P_y is

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0,$$

where

$$A = 1+10t^3+5t^6+5t^9, \quad B = t(5+5t^3+10t^6+t^9), \quad H = -t^2(5+t^3+t^6),$$

$$G = -\frac{at^4}{2}(5+t^6), \qquad F = -\frac{a}{2}(1+5t^6), \qquad C = a^2t^5.$$

The conic is degenerate only for $t = -1$, *i. e.*, at infinity. *For all points of the loop the osculating conic is an ellipse; at the origin it is a parabola; and for all other points it is a hyperbola.*

The locus of the centers of the conics is given by the equations

$$x = \frac{at}{10} \cdot \frac{5+t^3}{1+t^3}, \quad y = \frac{a}{10t} \cdot \frac{1+5t^3}{1+t^3}.$$

It is a unicursal quartic.

The cartesian coordinates of P_h are found to be

$$x = \frac{at}{1+t^3} \cdot \frac{t^{15}}{(1-t^3)(t^{18}+t^{12}+t^6+1)}, \quad y = \frac{at^2}{1+t^3} \cdot \frac{-t^6}{(1-t^3)(t^{18}+t^{12}+t^6+1)},$$

which satisfy the equation $x^3+y^3-axy=0$, as they should. From the above equation follows the relation

$$x = -t^8 y.$$

Consequently, for any value of t, the signs of x and y are opposite. *Therefore P_h can never lie in the first quadrant.*

CHICAGO, ILL.

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